Uniform Global Asymptotic Stability for Nonlinear Systems under
Input Delays and Sampling of the Controls

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Abstract—We study nonlinear time varying continuous time systems that are globally asymptotically stabilizable by state feedbacks. We study their stability in closed loop with controls that are corrupted by both sampling and time delays. We prove uniform global asymptotic stability using a new type of Lyapunov approach.

I. INTRODUCTION

Sampling in controllers is a well known problem, having been studied in several contributions [9], [18], [19], [20], [21], [25]. Time delay problems have also been studied for many years, and the last two decades have seen much research on nonlinear systems with delay [11], [12], [15], [22], [23], [24], [27]. While sampling and delay occur simultaneously in practice in a wide variety of applications, many works consider systems that have both delay and sampling in controllers. Four notable exceptions are [2], [6], [16], [17]. Even more rare are works on nonlinear systems with delay and sampling; [7] appears to be the only general result on this problem, but it uses a prediction strategy that requires knowing the delay and the sampling interval. See also [8] for sampling for feedforward systems without delays.

Given a time varying nonlinear system with a uniformly globally asymptotically stabilizing time varying undelayed continuous time state feedback controller, it is natural to look for conditions under which the closed loop system remains uniformly globally asymptotically stable (UGAS) when we introduce delays and sampling into the controller. To the best of our knowledge, this problem has never been addressed. On the other hand, implementing controls with measurement delays frequently involves sampling of the control with delay. Hence, we consider a nonlinear system

$$\dot{x}(t) = f(t,x(t)) + g(t,x(t))u$$ (1)

where the state $x$ and the control $u$ are valued in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively for any dimensions $n$ and $p$, and $f$ and $g$ are locally Lipschitz in $x$ and piecewise continuous in $t$. We assume that (1) is rendered UGAS by a $C^1$ controller $u_0(t,x)$. We give conditions under which the UGAS property is maintained when the input has sampling and delays, in which case the control value $u$ is $u_0(t_i-\tau,x(t_i-\tau))$ for all $t \in [t_i,t_{i+1})$ and $i = 0, 1, 2, \ldots$, where $\tau > 0$ is the given positive pointwise delay and $\{t_i\}$ is a given sequence of sample times. We give upper bounds for the admissible delay and for the admissible lengths of the sampling intervals.

The work [26] seems to be the one closest to ours, but it differs from ours in several ways. In [26], (i) there is an offset and a restriction on the initial states, so its conclusions are weaker than UGAS in the zero disturbance case unless special conditions are satisfied such as global exponential stability of the undelayed unsampled system, (ii) only time-invariant systems are studied, (iii) its main result is shown using the Razumikhin theorem, (iv) the maximum allowable sampling interval is given by a condition on the gains, while ours is expressed directly in terms of a controller, the dynamics, and a Lyapunov function, and (v) [26] shows ISS.

Our result in this paper cannot be proven by adapting the proofs from [4] or [14]. We show through examples that we establish our main result under assumptions that do not imply those of [14], including cases where the unsampled undelayed system is not locally exponentially stabilizable. To help overcome this obstacle, we use a new functional of Lyapunov type, which is reminiscent of the one used in [3] to study time invariant linear systems and [11] for neutral time delay systems. For simplicity, we only consider control affine systems, but we conjecture that extensions to systems that are not control affine can also be shown. We illustrate our work through several examples with sampled inputs and input delays, including a tracking problem for a model from [5] related to wheeled mobile robots. This note is a summary of [13]; see [13] for proofs of all results to follow.

II. NOTATION AND DEFINITIONS

We use the standard classes of comparison function $\mathcal{K}_\infty$ and $\mathcal{KL}$ [13]. Given any function $\phi : \mathbb{I} \to \mathbb{R}^p$ defined on any interval $\mathbb{I}$, we let $\|\phi\|_\mathbb{I}$ denote its (essential) supremum over $\mathbb{I}$. We let $| \cdot |$ denote the Euclidean norm (or the induced matrix norm, depending on the context). Given any continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ and any $t \geq 0$, the function $\varphi_t$ is defined by $\varphi_t(\theta) = \varphi(t + \theta)$ for all $\theta \in [-r,0]$, where the constant $r > 0$ will depend on the context. A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is called positive definite provided $\alpha(0) = 0$ and $\alpha(r) > 0$ for all $r > 0$. More generally, a continuous function $W : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is called positive definite provided there are positive definite functions $\underline{\alpha}$ and $\bar{\alpha}$ such that $\underline{\alpha}(|x|) \leq W(t,x) \leq \bar{\alpha}(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$; if, in addition, $\underline{\alpha} \in \mathcal{K}_\infty$, then we also say that $W$ is radially unbounded. A function $\varphi(t,x)$ is called uniformly bounded with respect to $t$ provided there is a function $\rho$ of class $\mathcal{K}_\infty$ such that $|\varphi(t,x)| \leq \rho(1 + |x|)$. 

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for all \((t, x)\) in the domain of \(\varphi\). Throughout this paper, we assume that all of our time varying functions are uniformly bounded with respect to \(t\). We set \(\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}\).

### III. Assumptions and Main Result

Consider the nonlinear system (1) and let \(\{t_i\}\) be a sequence in \([0, \infty)\) such that \(t_0 = 0\) and such that there are two constants \(\nu > 0\) and \(\delta > \nu\) such that

\[
t_i + 1 - t_i \in [\nu, \delta] \quad \forall i \in \mathbb{Z}_{\geq 0}.
\]

(2)

Our first assumption is:

**Assumption 1:** There exist a \(C^1\) function \(u_s(t, x)\), a \(C^1\) positive definite and radially unbounded function \(V\), and a continuous positive definite function \(W\) such that

\[
W_0(t, x) = -\left[\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)(f(t, x) + g(t, x)u_s(t, x))\right]
\]

(3)

satisfies

\[
W_n(t, x) \geq W(x)
\]

(4)

for all \(t \geq 0\) and \(x \in \mathbb{R}^n\), and \(u_s(t, 0) = 0\) for all \(t \in \mathbb{R}\).

Hence, \(V\) is a strict Lyapunov function for \(\dot{x} = f(t, x) + g(t, x)u_s(t, x)\), and we can find class \(\mathcal{K}_\infty\) functions \(\alpha_1\) and \(\alpha_2\) such that \(\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|)\) for all \(t \geq 0\) and \(x \in \mathbb{R}^n\). Define the function \(h\) by

\[
h(t, x) = \frac{\partial u_s}{\partial t}(t, x) + \frac{\partial u_s}{\partial x}(t, x)f(t, x)
+ \frac{\partial u_s}{\partial x}(t, x)g(t, x)u_s(t, x).
\]

(5)

Our second assumption is:

**Assumption 2:** There are constants \(c_i \geq 0\) for \(i = 1, 2, 3, 4\) such that

\[
\left|\frac{\partial u_s}{\partial t}(t, x)g(t, x)\right|^2 \leq c_1,
\]

(6)

\[
\left|\frac{\partial u_s}{\partial x}(t, x)g(t, x)\right|^2 \leq c_2W(x),
\]

(7)

\[
|h(t, x)|^2 \leq c_3W(x), \quad \text{and}
\]

(8)

\[
\left|\frac{\partial u_s}{\partial x}(t, x)g(t, x)u_s(t, x)\right| \leq c_4|V(t, x)| + 1
\]

(9)

for all \(t \geq 0\) and \(x \in \mathbb{R}^n\).

We prove the following result in [13]:

**Theorem 1:** Let the system (1) satisfy Assumptions 1 and 2. If \(\delta\) and \(\tau_s\) are any two positive constants such that

\[
\delta + \tau_s \leq \frac{1}{\sqrt{4c_1 + 8c_2c_3}}
\]

(10)

and if \(\tau \in (0, \tau_s]\), then the system (1) in closed loop with

\[
u(t) = u_s(t_i - \tau, x(t_i - \tau)\) when \(t \in [t_i, t_{i+1})
\]

(11)

with the sequence \(\{t_i\}\) from (2) is UGAS.

**Remark 1:** By UGAS of the closed loop system, we mean that there is a function \(\beta \in \mathcal{KL}\) such that for all initial functions \(x_0\), all initial times \(t_0 \geq 0\), and all \(t \geq t_0\), we have \(|x(t)| \leq \beta(|x_0|_{[t_0 - \tau_i, t_{i+1])}, t - t_0)\). Assumption 1 implies that the origin of (1) in closed loop with \(u_s(t, x)\) without delay and sampling is UGAS. Assumptions 1 and 2 do not imply that \(f\) and \(g\) are globally Lipschitz with respect to \(x\) or that (1) is locally exponentially stabilizable, and allows cases where \(W\) may not be radially unbounded; see below.

**Remark 2:** Requirement (9) is often satisfied. We can use (4), (7) and (9) to show that the finite escape phenomenon does not occur for the closed loop system [13].

**Remark 3:** Theorem 1 applies to systems that are not necessarily globally Lipschitz or locally exponentially stabilizable by continuous feedback. For example, take \(n = 1\) and \(\dot{x} = \frac{\tau}{1+\tau}u\), with \(u_s(x) = -x\) and \(V(x) = \frac{x^2}{1+\tau}\). Using the notation from above with the time dependency omitted, we have

\[
f(x) = 0, \quad g(x) = \frac{x^2}{1+\tau}, \quad \frac{\partial u_s}{\partial x}(x)g(x) = -\frac{x^2}{1+\tau}, \quad h(x) = \frac{x^2}{1+\tau}, \quad L_2V(x) = \frac{x^2}{1+\tau}, \quad \text{and} \quad W(x) = \frac{x^2}{1+\tau}.
\]

Then Assumptions 1-2 hold with \(c_1 = c_2 = c_3 = 1\), so Theorem 1 ensures that the corresponding input delayed sampled system is UGAS if \(\delta + \tau < 1/(2\sqrt{3})\).

### IV. Sketch of Proof of Theorem 1

We only give part of the UGAS proof, which gives the main ideas. See [13] for the complete proof. Throughout the proof, all time derivatives are over all trajectories of

\[
\dot{x}(t) = f(t, x(t)) + g(t, x(t))u_s(t_i - \tau, x(t_i - \tau)).
\]

(12)

In what follows, all equalities and inequalities should be understood to hold for all \(t \in [t_i, t_{i+1})\) and \(i \in \mathbb{Z}_{\geq 0}\). Set

\[
\Delta u_s(t_i) = u_s(t_i - \tau, x(t_i - \tau)) - u_s(x(t_i)).
\]

(13)

We introduce an operator \(\Gamma\) such that

\[
\Gamma(t, x_i) = \int_{t_i - \delta - \tau_i}^{t_i} \int_{t_i - \tau_i}^{\tau_i} \left[\psi(m, m_n)\right]^2 dm d\ell
\]

(14)

along the trajectories of (1), where \(\epsilon > 0\) is a constant to be selected later and

\[
\psi(t, x_i) = \frac{\partial u_s}{\partial t}(t, x) + \frac{\partial u_s}{\partial x}(t, x)\dot{x}(t).
\]

(15)

Since \(\dot{x}\) is a piecewise continuous function of \(t\), the function \(\Gamma\) is piecewise differentiable and satisfies

\[
\dot{\Gamma}(t) = \epsilon \left|\psi(t, x_i)\right|^2
- \int_{t_i - \delta - \tau_i}^{t_i} \int_{t_i - \tau_i}^{\tau_i} \left|\psi(m, m_n)\right|^2 dm.
\]

(16)

From the definition of \(W_0\) in (3), (16), the expression for \(\dot{\Gamma}\) and the definition of \(h\) in (5), it follows that the function

\[
U(t, x_i) = V(t, x(t)) + \Gamma(t, x_i)
\]

satisfies

\[
\dot{U} = -W_0(t, x(t)) + \frac{\partial V}{\partial x}(t, x)g(t, x)\Delta u_s(t)
- \frac{\partial u_s}{\partial x}(t, x)\dot{\Gamma}(t)
+ \frac{\partial u_s}{\partial x}(t, x)\dot{x}(t)\Delta u_s(t)^2
\]

(17)

along all trajectories of the delayed sampled dynamics (1). Using the inequality \((a + b)^2 \leq 2a^2 + 2b^2\) which is valid for all \(a \in \mathbb{R}\) and \(b \in \mathbb{R}\), (4), and Assumption 2, we get

\[
\dot{U} \leq -W(x(t))
- \frac{\partial u_s}{\partial x}(t, x)\dot{\Gamma}(t)
+ \frac{\partial u_s}{\partial x}(t, x)\dot{x}(t)\Delta u_s(t)^2
+ 2\epsilon |h(t, x(t))|^2
+ |\frac{\partial u_s}{\partial x}(t, x)g(t, x)|^2 |\Delta u_s(t)|^2
+ 2\epsilon |\frac{\partial u_s}{\partial x}(t, x)g(t, x)|^2 |\Delta u_s(t)|^2
\]

(18)
so
\[
\dot{U} \leq -W(x(t)) - \frac{c_1}{4c_3} \int_{t-\tau_*}^t |\psi(m,x_m)|^2 \, dm + 2c_6 \left(2c_1 + c_2\right) \frac{\delta + \tau_*}{2} \times \int_{t-\tau_*}^t |\psi(m,x_m)|^2 \, dm.
\]
From the triangle inequality, we deduce that
\[
\sqrt{c_2 W(x(t))} |\Delta u_s(t)| \leq \frac{1}{2} W(x(t)) + c_2 |\Delta u_s(t)|^2.
\]
Using (19), (20), and Jensen’s inequality, we get
\[
\dot{U} \leq \left(-\frac{3}{4} + 2c_6 \right) W(x(t)) - \frac{c_1}{4c_3} \int_{t-\tau_*}^t |\psi(m,x_m)|^2 \, dm + 2c_6 \left(2c_1 + c_2\right) \delta \times \int_{t-\tau_*}^t |\psi(m,x_m)|^2 \, dm.
\]
By grouping terms and using the fact that \(\tau_* - \tau \geq t - \tau_* - \delta\) when \(t \in [t_i, t_{i+1})\) to upper bound the second integral in (21) by the first integral, and then taking \(\epsilon = \frac{1}{4c_3}\), we get
\[
\dot{U} \leq -\frac{1}{4} W(x(t)) + \frac{c_1}{4c_3} + \left(\frac{c_6}{c_3} + c_2\right) \frac{\delta + \tau_*}{2} \times \int_{t-\tau_*}^t |\psi(m,x_m)|^2 \, dm.
\]
where the last inequality used our bound (10) on \(\delta + \tau_*\).
Then we can find \(C^1\) functions \(\kappa\) and \(\gamma\) of class \(K_\infty\) such that the function \(U = \kappa(U)\) satisfies \(U \leq -\gamma(U)\) along all trajectories of (1).
Finally, we can use suitable upper and lower bounds on \(U\) to get the desired UGAS and locally exponentially stable by
\[
\dot{u}_s(x) = -\frac{\xi x}{\sqrt{1 + x^2}},
\]
where \(\xi\) is any positive constant. Then, with the notation of Section III, we have \(f(x) = 0\) and \(g(x) = 1\). We choose the positive definite radially unbounded function \(V(x) = \frac{1}{2} x^2 - 1\). Then Assumption 1 is satisfied with \(W(x) = \xi x^2/(1 + x^2)\).

V. EXAMPLES
A. Two-Dimensional Nonlinear Example
Take the \(n = 2\) dimensional system
\[
\begin{align*}
\dot{x}_1 &= -x_1 - x_1^2 + x_2 \\
\dot{x}_2 &= u + x_1
\end{align*}
\]
and the stabilizing feedback \(u_s(x) = -x_1 - x_2\). Then our assumptions hold with \(V(x) = \frac{1}{10} x_1^2 + \frac{1}{2} x_2^2 + 2x_2^2\) and
\[
W(x) = 2x_1^2 + x_1 x_2 + x_1^2 - x_1 x_2 + 2x_2^2.
\]
In this case, we have \(\frac{\partial u_s}{\partial x}(x) = 2\), \(L_g V(x) = 1\), \(L_g V(x) = 16x_2^2\), and \(\dot{h}(x) = x_1^2 + x_1\), which has the upper bound \(\dot{h}(x) \leq 2x_1^2 + 2x_2^2\).

B. Saturating Controller
Our work [14] used Lyapunov-Krasovskii functionals to prove robustness of closed loop control affine systems \(\dot{x}(t) = f(t, x(t)) + g(t, x(t)) u_s(t, x(t)) + h_1(t)\) with respect to small enough input delays. Setting \(F(t, x, u_s) = f(t, x) + g(t, x) u_s(t, x)\), the assumptions from [14] are that \(f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n\), \(g : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}\) and \(u_s : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m\) satisfy:

H The function \(u_s\) is \(C^1\) and \(u_s(t, 0) = 0\), and \(f\) and \(g\) are locally Lipschitz. Also, there exist a \(\sigma \in K_\infty\) for which \(\sigma(r) \leq r\) for all \(r \geq 0\), a \(C^1\) positive definite radially unbounded function \(V : [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty)\); positive constants \(L\) and \(K_1\), and constants \(K_1 \geq 0\) \((i = 2, 3, 4)\) such that for all \(x \in \mathbb{R}^n\), \(q \in \mathbb{R}^n\), and \(i \geq 0\), we have:

\[
\begin{align*}
V_i(l, x) + V_i(l, x) F_i(l, x, u_s(l, x)) &\leq -\sigma(\sqrt{n}(x)) \\
|f(l, x)| &\leq K_1 \sigma(|x|) \quad \text{and} \quad \frac{\partial f}{\partial x}(l, x) \leq L \\
\|f(l, x)\|^2 &\leq K_2 \sigma^2(|x|) \quad \text{and} \quad \|g(l, x)\|^2 \leq K_3 \sigma^2(|x|) + 1 \\
\|g(l, x) (|u_s(l, q)|)^2 \leq K_4 [\sigma^2(|x|) + \sigma^2(|q|)]
\end{align*}
\]

so is covered by Theorem 1, but does not satisfy [14, Assumption H]. It will be key to the higher dimensional tracking dynamics in the next subsection. Take \(\dot{x} = u\), where the state \(x\) and input \(u\) are one dimensional. This is rendered UGAS and locally exponentially stable by
\[
\dot{u}_s(x) = -\frac{\xi x}{\sqrt{1 + x^2}},
\]
where \(\xi\) is any positive constant. Then, with the notation of Section III, we have \(f(x) = 0\) and \(g(x) = 1\). We choose the positive definite radially unbounded function \(V(x) = \frac{\xi}{1 + x^2} - \frac{\xi}{1 + x^2} \leq \frac{\xi}{1 + x^2} \leq \xi\).

C. Tracking Example
Take the dynamics
\[
\begin{align*}
\dot{x}_1 &= \omega x_2 \\
\dot{x}_2 &= -\omega x_1 + \lambda \\
\dot{x}_3 &= \omega,
\end{align*}
\]
where \(\lambda\) and \(\omega\) are controls. This is obtained from the kinematics of a wheeled mobile robot using a change of coordinates [5]. Choose any constant \(\zeta > 0\).

Case 1: Our goal is to track the periodic trajectory \((0, 0, -\cos(\zeta t))^T\), using delayed sampled feedback. To this end, notice that the change of coordinates \(x = x_3 + \cos(\zeta t)\) gives the tracking system
\[
\begin{align*}
\dot{x}_1 &= \omega x_2 \\
\dot{x}_2 &= -\omega x_1 + \lambda \\
\dot{x}_3 &= -\zeta \sin(\zeta t) + \omega
\end{align*}
\]
The change of feedback $\omega = \zeta \sin(\zeta t) + \mu$ produces
\[
\begin{array}{ll}
\dot{x}_1 &= (\zeta \sin(\zeta t) + \mu)x_2 \\
\dot{x}_2 &= -(\zeta \sin(\zeta t) + \mu)x_1 + \lambda \\
\dot{z} &= \mu.
\end{array}
\tag{29}
\]

The $z$ subsystem of (29) can be stabilized by
\[
\mu(z(t_i - \tau)) = -\zeta_0 z(t_i - \tau) \sqrt{1 + z^2(t_i - \tau)},
\tag{30}
\]
for any positive constant $\zeta_0$. (See Case 2 below for the case where the full feedback $\omega$ has sampling.) In fact, our discussion from Section V-B shows that
\[
\dot{z}(t) = -\frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}}
\tag{31}
\]
is UGAS if $\delta + \tau_* < 1/(2\sqrt{3}\zeta_0)$. Assume that $\zeta_0 \leq \frac{1}{30}$.

Using the control $\lambda(x(t)) = -\zeta x_2(t)$ gives
\[
\begin{array}{l}
\dot{x}_1 = \zeta \left( \sin(\zeta t) - \frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right) x_2 \\
\dot{x}_2 = -\zeta \left( \sin(\zeta t) - \frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right) x_1 + \lambda \\
\dot{z} = -\zeta x_2(t_i - \tau)
\end{array}
\tag{32}
\]

We need the following lemma from [13]:

**Lemma 2:** Let $\zeta > 0$ be any constant. Then for any piecewise continuous function $N$ satisfying $|N(t)| \leq 1/30$ for all $t \geq 0$, the time derivative of
\[
Q_\zeta(t, x) = \frac{1}{4} |x|^2 + 2 \sin(\zeta t)x_1 x_2 - \sin(\zeta t) \cos(\zeta t) x_1^2
\tag{33}
\]
along all trajectories of
\[
\begin{array}{l}
\dot{x}_1 = \zeta \sin(\zeta t)x_2(t) + \zeta N(x_2(t)) \\
\dot{x}_2 = -\zeta \sin(\zeta t)x_1(t) - \zeta x_2(t) \\
\dot{z} = -\zeta N(x_1(t))
\end{array}
\tag{34}
\]
satisfies $\dot{Q}_\zeta(t, x) \leq -\frac{\zeta}{4} |x|^2$. Also, $5|x|^2 \geq Q_\zeta(t, x) \geq \frac{1}{4} |x|^2$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$.

By Lemma 2, it follows that the time derivative of the time varying positive definite and proper quadratic function $Q_\zeta(t, x)$ given in (33) along all trajectories of (32) satisfies
\[
\dot{Q}_\zeta \leq -\frac{\zeta}{4} |x|^2 + \frac{\zeta_0}{2} |x_2|^2
\tag{35}
\]
We now replace the control $\lambda$ by the delayed sampled controller $\lambda(x(t_i - \tau)) = -\zeta x_2(t_i - \tau)$, which gives
\[
\begin{array}{l}
\dot{x}_1 = \zeta \left( \sin(\zeta t) - \frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right) x_2 \\
\dot{x}_2 = -\zeta \left( \sin(\zeta t) - \frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right) x_1 \\
\dot{z} = -\zeta x_2(t_i - \tau)
\end{array}
\tag{36}
\]
\[
\begin{array}{l}
\dot{x}_1 = \zeta \sin(\zeta t + \gamma(t))x_2 \\
\dot{x}_2 = -\zeta \sin(\zeta t + \gamma(t))x_1 + \lambda, \text{ where}
\end{array}
\tag{37}
\]
\[
\gamma(t) = -\frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}},
\tag{38}
\]
Then, we apply Theorem 1 to (37), with $\lambda$ acting as $u$ in (1), so $\lambda(x_2) = -\zeta x_2$ plays the role of $u_s$.

With the notation of (1) we choose
\[
f(t, x) = \begin{pmatrix} \zeta \sin(\zeta t + \gamma(t))x_2 \\ -\zeta \sin(\zeta t + \gamma(t))x_1 \end{pmatrix}, \quad g(t, x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
\[
V(t, x) = Q_\zeta(t, x), \quad u_s(t, x) = -\zeta x_2,
\]
so $h(t, x) = \zeta^2 (\sin(\zeta t + \gamma(t))x_2 + x_1)$. Theorem 1 applies to (37) [13], and it gives an upper bound on $\delta + \tau$, that is independent of the choice of the solution $z$. Combining this with our result on (31) gives [13]:

**Corollary 1:** The system (36) is UGAS when
\[
\delta + \tau_* < \frac{1}{2\zeta} \min \left\{ \frac{1}{\sqrt{30^2}}, \frac{1}{53(3 + 1)} \right\},
\tag{39}
\]
is satisfied.

The preceding corollary is shown by taking the minimum of the upper bounds for $\delta + \tau$, for the $x$ and $z$ subsystems.

**Case 2:** Next take the case where there is also sampling in the sin part of the change of feedback, i.e., $\omega = \zeta \sin(\zeta(t_i - \tau)) + \mu$, which we substitute into (27) to get
\[
\begin{array}{l}
\dot{x}_1 = [\zeta \sin(\zeta(t_i - \tau)) + \mu]x_2 \\
\dot{x}_2 = -[\zeta \sin(\zeta(t_i - \tau)) + \mu]x_1 + \lambda \\
\dot{x}_3 = \zeta \sin(\zeta(t_i - \tau)) + \mu.
\end{array}
\tag{40}
\]
We assume that $t_i = i\delta$, where $\delta = \pi/(\zeta L)$ for some positive integer $L$, and we set $\varphi(t) = t_i - \tau$ for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_{\geq 0}$. Let
\[
z = x_3 - \int_0^t \sin(\zeta \varphi(m)) dm.
\tag{41}
\]
This gives the dynamics
\[
\begin{array}{l}
\dot{x}_1 = \zeta \sin(\zeta(t_i - \tau)) + \mu x_2 \\
\dot{x}_2 = -[\zeta \sin(\zeta(t_i - \tau)) + \mu]x_1 + \lambda \\
\dot{z} = \mu.
\end{array}
\tag{42}
\]
We next use this lemma from [13]:

**Lemma 3:** Let $t_i = i\delta$, where $\delta = \pi/(\zeta L)$, $L$ is any positive integer, and $\zeta > 0$ is any constant. Define $\varphi$ as follows: $\varphi(t) = t_i - \tau$ for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_{\geq 0}$. Then $\sin(\zeta \varphi(t + \pi/\zeta)) = -\sin(\zeta \varphi(t))$ for all $t \in \mathbb{R}$, $\int_{0}^{2\pi/\zeta} \sin(\zeta \varphi(m)) dm = 0$, and $\sin(\zeta \varphi(t))$ is periodic of period $2\pi/\zeta$.

Lemma 3 implies that (0, 0, 0, 0) $\zeta \sin(\zeta \varphi(m)) dm$ is a bounded trajectory, and this is the new trajectory we wish to track. To this end, choose
\[
\mu(z(t_i - \tau)) = -\frac{\zeta_0 z(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}},
\tag{43}
\]
where $\mathcal{U}$ is such that $0 \leq \frac{1}{60}$. This gives

$$
\begin{align*}
\dot{x}_1 &= \zeta \left[ \sin(\zeta(t_i - \tau)) - \frac{U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right] x_2 \\
\dot{x}_2 &= -\zeta \left[ \sin(\zeta(t_i - \tau)) - \frac{U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right] x_1 + \lambda \\
\dot{z} &= -\frac{\zeta U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}}.
\end{align*}
$$

(44)

We rewrite the $(x_1, x_2)$ subsystem as

$$
\begin{align*}
\dot{x}_1 &= \zeta \left[ \sin(\zeta t + \omega(t)) \right] x_2 \\
\dot{x}_2 &= -\zeta \left[ \sin(\zeta t + \omega(t)) \right] x_1 + \lambda, \text{ where}
\end{align*}
$$

(45)

$$
\omega(t) = \sin(\zeta(t - \tau)) - \sin(\zeta t) - \frac{U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}}.
$$

(46)

We have $|\omega(t)| \leq \zeta (\delta + \tau_*) + \mathcal{U}$. Therefore, if $\delta + \tau_* \leq \frac{1}{60\zeta}$, then $|\omega(t)| \leq \frac{1}{30}$. Hence, our analysis of (37) applies to (45) with the sampled feedback $\lambda = -\zeta x_2(t_i - \tau)$ to give a bound on the admissible values of $\tau_* + \delta$. This gives:

**Corollary 2:** For any constant $\zeta > 0$, the system

$$
\begin{align*}
\dot{x}_1 &= \zeta \left[ \sin(\zeta (t_i - \tau)) - \frac{U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right] x_2 \\
\dot{x}_2 &= -\zeta \left[ \sin(\zeta (t_i - \tau)) - \frac{U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}} \right] x_1 \nonumber \\
\dot{z} &= -\frac{\zeta U_2(t_i - \tau)}{\sqrt{1 + z^2(t_i - \tau)}}
\end{align*}
$$

is UGAS when

$$
\delta + \tau_* < \frac{1}{2\zeta} \min \left\{ \frac{1}{\sqrt{30}}, \frac{1}{53(\mathcal{U} + 1)} \right\}
$$

(48)

is satisfied.

**VI. CONCLUSIONS**

Delays and sampling in controllers produce considerable challenges. We studied nonlinear control affine systems whose feedbacks are subjected to delay and sampling. We found conditions on the length of the delay and the maximal sampling interval that imply uniform global asymptotic stability, using a new Lyapunov approach. We applied our method to a tracking problem, in which the bound on the sampling interval and delay can be made arbitrarily large. We conjecture that our approach can be adapted to dynamics that are not control affine, systems that can be locally but not globally asymptotically stabilized, and dynamics with controller uncertainty where one wishes to show input-to-state stability under disturbances added to the controller.

**REFERENCES**


