Feedback Stabilization of Non-uniform Spatial Pattern in Reaction-Diffusion Systems

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Abstract—In this paper, we formulate and solve feedback stabilization problem of unstable non-uniform spatial pattern in reaction-diffusion systems. By considering spatial spectrum dynamics, we obtain a finite dimensional approximation that takes over the semi-passivity of the original partial differential equation. By virtue of this property, we can show the diffusive coupling in the spatial frequency domain achieves the desired pattern formation.

I. INTRODUCTION

The goal of this paper is to formulate and solve feedback stabilization problem of unstable non-uniform spatial pattern in reaction-diffusion systems [1], [2], [3], [4], [5]. See also [6], [7], [8], [9], [10], [11] for existing results related to control theory.

We begin with a brief introduction of pattern generation mechanism in reaction-diffusion systems. Let spatio-temporal variable \( z(t, \xi) \) where the second variable \( \xi \in \Omega \) is a spatial variable. Then, we consider the following partial differential equation

\[
\frac{\partial}{\partial t} z(t, \xi) = f(z(t, \xi)) + D \Delta z(t, \xi), \quad \xi \in \Omega
\]

(1)

(with suitable boundary condition) with non-negative diagonal matrix \( D \). We show an illustrative example in which \( z = [u, v]^T \) and \( \Omega \) is a two-dimensional rectangular domain equipped with a periodic boundary condition. The origin of the reaction term (represented by \( f \)) is globally asymptotically stable; see the following sections for more detail. Figure 1 is snapshots of \( u(t, \xi) \) of (1), where an initial state close to \( z_{eq} \equiv 0 \) was randomly generated. The spatio-temporal pattern gradually goes away from the trivial equilibrium pattern, and finally converges to a roll pattern.

On the other hand, we can see transient, but long lasting enough to observe, other specific patterns. In view of this, we attempt stabilize such patterns that cannot exist stably by spatially distributed feedback control. In this paper, this problem is formulated suitably, and then solved based on diffusive coupling in the spatial frequency domain.

We briefly compare the contribution of this paper with existing results in controls community. Distributed parameter system theory has investigated control of semilinear systems, including reaction-diffusion systems, from various aspects e.g., [12], [13], [14], [15]. However, the control objective is usually the stabilization of the trivial equilibrium \( z_{eq} \). This shows a clear contrast to our case where the exact profile of target pattern is unavailable. Though Arcak investigated non-uniform spatial pattern [6], [7], detailed discussion on the resulting pattern and control system design are not fully explored.

The organization of this paper is as follows. Section II gives prerequisite for analysis of reaction-diffusion systems. In Section III, we formulate feedback control problem of spatial pattern formation. In Section IV, we reformulate and solve the problem based on spatial spectrum dynamics. Numerical examples are given in Section V to show the effectiveness of the obtained result. Section VI makes some concluding remarks.

II. PRELIMINARY

A. Reaction-diffusion systems

In this paper, we consider the following form of reaction-diffusion system: the spatial domain is

\[
\Omega := [0, L_x] \times [0, L_y].
\]

(2)

and spatio-temporal state variable is two-dimensional

\[
z(t, x, y) = \begin{bmatrix} u(t, x, y) \\ v(t, x, y) \end{bmatrix} \in \mathbb{R}^2.
\]

Then, the spatio-temporal dynamics is given by

\[
\frac{\partial u}{\partial t} = a_{11} u - a_{12} v - u^3 + d_u \Delta u
\]

(3)

\[
\frac{\partial v}{\partial t} = a_{21} u - a_{22} v + d_v \Delta v
\]

(4)

for all \((x, y) \in \Omega\) with the periodic boundary condition

\[
\begin{align*}
{z(t, 0, y) &= z(t, L_x, y),} \\
{z(t, x, 0) &= z(t, x, L_y),}
\end{align*}
\]

(5)

This reaction-diffusion system has a trivial equilibrium pattern

\[
z_{eq}(x, y) \equiv 0 \text{ on } \Omega.
\]

(6)

Even if the reaction term is globally stable, this trivial equilibrium of reaction-diffusion system (3), (4) is not necessarily stable.

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B. Spatial spectral analysis of uniform pattern instability

In this section, the diffusion-induced local instability of the trivial equilibrium pattern is analyzed based on spatial spectral analysis. Consider the spatial Fourier transform

\[ z_m(t) := \begin{bmatrix} u_m(t) \\ v_m(t) \end{bmatrix} := \int_{\Omega} z(t, x, y) p_m(x, y) e^{ix + i j y} dx dy \in \mathbb{C}^2 \]  

for wave number \( m = (m_x, m_y) \in \mathbb{Z}^2 \) where

\[ p_m(x, y) := e^{2\pi i (m_x x + m_y y)} \]

and \( p_m^* \) is its complex conjugate. This satisfies

\[ z(t, x, y) = \sum_{m \in \mathbb{Z}^2} z_m(t) e^{2\pi i (m_x x + m_y y)}. \]

Note that

\[ z_m = z_{-m}^* \quad \text{for all } m \in \mathbb{Z}^2 \]  

since the dynamics of our interest is real-valued. Then, it is useful to investigate the dynamics of each spatial component \( \{z_m(t)\}_{m \in \mathbb{Z}^2} \) instead of \( \{z(t, x, y)\}_{(x,y) \in \Omega} \).

Let us consider the localized dynamics around the trivial pattern \( z_{eq} \). It should be emphasized that each wave number has its decoupled local dynamics:

\[ \frac{d}{dt} z_m(t) = A_m z_m(t), \]  

\[ A_m := A - s_m D =: \begin{bmatrix} \bar{a}_{11} & -a_{12} \\ a_{21} & -\bar{a}_{22} \end{bmatrix}, \]  

\[ A := \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix}, \]  

\[ D := \begin{bmatrix} d_u & 0 \\ 0 & d_v \end{bmatrix}, \]  

\[ s_m := \left( \frac{2\pi m_x}{L_x} \right)^2 + \left( \frac{2\pi m_y}{L_y} \right)^2. \]

It should be emphasized that when \( d_u \neq d_v \), stability of \( A \) does not necessarily guarantee stability of \( A_m \). In such a case, the corresponding spatial wave \( p_m \) grows around the \( z_{eq} \). Further discussion on the pattern formation needs to consider the effect of nonlinearity.

III. PROBLEM FORMULATION

A. Motivating example

We take

\[ A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}, \]  

\[ d_u = 0.2, \]  

\[ d_v = 1.5. \]

and

\[ L_x = \frac{8\pi}{\sqrt{1.8}}, \quad L_y = \frac{L_x}{\sqrt{3}}. \]

For these parameters, \( A_m \) has one (real) unstable eigenvalue for \( m = \pm m_1 \)

\[ m_1 = (4, 0), \quad m_2 = (2, 2), \quad m_3 = (2, -2), \]

and \( A_m \) stable otherwise.

The simulation in Fig. 1 was executed under these parameter settings. The initial patterns are randomly generated but sufficiently close to \( z_{eq} \). We can expect this reaction-diffusion system can generate three roll patterns corresponding to \( m_i \)'s in (17). Actually, all of observed patterns (including transient ones) look like superpositions of these roll patterns.

B. Stabilization of unknown target pattern

Having the example in the previous section in our mind, let us formulate stabilization problem of unstable spatial patterns.

We define the set of wave numbers for which the local dynamics is unstable.

**Assumption 1:** The finite set \( \mathcal{M} \subset \mathbb{Z}^2 \) satisfies

1) \( A_m \) has at least one eigenvalue in \( \mathbb{C}_+ \) if \( m \in \mathcal{M} \),

2) \( A_m \) is stable if \( m \notin \pm \mathcal{M} := \{ \pm m : m \in \mathcal{M} \} \), and

3) if \( m \in \mathcal{M} \), then \( -m \notin \mathcal{M} \).

Because \( A_m = A_{-m} \) and (8), we imposed the condition 3) in order to avoid redundancy. It should be emphasized that \( s_{m_1} = s_{m_2} \) can hold for \( m_1 \neq m_2 \).

Next, for feedback control problem, we assume that we can observe and also manipulate \( u \) in a spatially distributed
manner. Thus, the controlled reaction-diffusion system is given by
\[
\frac{\partial}{\partial t} z = f(z) + D \Delta z + w(t, x, y) = W(u(t, x, y))
\]  
(19)

**Problem 1:** Under definitions above and Assumption 1, find a feedback control law \( W(\cdot) \) such that
1) \( z \) does not diverge,
2) \( z_m \) for \( m \notin \pm M \) asymptotically vanishes,
3) \( z_m \) for \( m \in M \) converges to the same nonzero value, and
4) \( w(t, x, y) \) asymptotically vanishes.

**IV. FEEDBACK CONTROL OF CENTER MANIFOLD DYNAMICS**

**A. Reformulation on center manifold dynamics**

In view of Assumption 1-2), let us assume that \( z_m \) is negligible for \( m \notin \pm M \) to avoid infinite-dimensionality. We analyze the dynamics that \( z_m \) should obey under the following technical assumption:

**Assumption 2:** If \( n_1, n_2, n_3 \in M \) satisfy \( n_1 + n_2 + n_3 = m \in \pm M \), then at least one of \( n_i \)'s is equal to \( m \).

By putting \( z_m \approx 0 \) for \( m \notin \pm M \),
\[
u^3 = \sum_{n_1, n_2, n_3 \in \pm M} u_{n_1} u_{n_2} u_{n_3} p_{n_1 + n_2 + n_3}(x, y).
\]

It follows from Assumption 2 and the orthogonality of \( \{p_m\}_{m \in \mathbb{Z}^2} \) that the wave number \( m \) component appears only from the combination
\[(n_1, n_2, n_3) = (m, n, -n)\]

and its permutation, where \( n \in \pm M \) is arbitrary. Therefore, we obtain the following approximation in the spatial frequency domain:
\[
\frac{d}{dt} \begin{bmatrix} u_m \\ v_m \\ 0 \end{bmatrix} = A_m \begin{bmatrix} u_m \\ v_m \\ 0 \end{bmatrix} - u_m \left( 3|u_m|^2 + \sum_{n \neq m} |u_n|^2 \right) \left( \begin{array}{c} w_m \\ 0 \end{array} \right)
\]

where
\[w_m(t) := \int_{\Omega} w(t, x, y)p_m^*(x, y) dx dy.
\]

It should be emphasized that, in the rest of this paper, \( z_m = \begin{bmatrix} u_m \\ v_m \end{bmatrix} \) represents the solution of finite dimensionally approximated system (20), instead of the spatial spectral component of the solution \( u, v \) to the original partial differential equation (3), (4). Now, Problem 1 is naturally rephrased in this domain:

**Problem 2:** Consider the complex-valued system (20). Then, find a feedback control law \( w_m(t) = W_m(u_M(t)) \) such that,
1) \( z_m \) is **ultimately bounded**, that is, there exists (initial state independent) \( R > 0 \) such that
\[
\limsup_{t \to \infty} \|z_m\| < R \text{ for all } m \in M,
\]

3'-a)
\[
\lim_{t \to \infty} |z_m(t) - z_n(t)| = 0 \text{ for all } m, n \in M
\]

3'-b) the origin is locally unstable, and
4')
\[
\lim_{t \to \infty} w_m(t) = 0 \text{ for all } m \in M.
\]

For this problem, we can simply obtain a desired control law by considering the specific structure of Problem 2.

**Theorem 1:** For Problem 2, consider the following diffusive coupling on \( M \)
\[
w_m(t) = \sigma \sum_{n \in M, n \neq m} \gamma_{mn}(u_n(t) - u_m(t))
\]

where \( \gamma_{mn} \geq 0 \). If \( \gamma_{mn} \) is associated to a strongly connected graph on \( M \), then there exists \( \sigma > 0 \) such that the control law (24) with any strength \( \sigma > 2 \) satisfies the following:

- the conditions 1’) in Problem 2 hold.
- if \( s_m = \delta \) (see (12) for the definition) for all \( m \in M \), then 3’-a), 3’-b), 4’) are also satisfied.

**B. Proof of Theorem 1**

The proof is similar to that of [16], [17]. However, we need to pay attention to the complex-valued dynamics and also the extra interaction term arising from \( u^3 \). For notational simplicity, denote
\[
z_M = (z_m)_{m \in M}, u_M = (u_m)_{m \in M}, \quad v_M = (v_m)_{m \in M}
\]
\[
f^x_m(u_M, v_M) = a_{11} u_{m} - a_{12} v_{m} - u_m \beta_m(u_M),
\]
\[
f^v_m(u_M, v_M) = a_{21} v_{m} - a_{22} v_{m},
\]
\[
\beta_m(u_M) := 3|u_m|^2 + \sum_{n \neq m} |u_n|^2 \geq 0.
\]

1) **Ultimate boundedness:** In this section, we prove the first claim. First, we begin with semi-passivity like property for subsystems.

**Lemma 1:** Define the real-valued functions defined on \( C^2 \)
\[
V(u, v) = \frac{1}{2} \left( |u|^2 + \frac{a_{12}}{a_{21}} |v|^2 \right)
\]

and
\[
H(u, v) = |u|^2 (3|u|^2 - a_{11}) + \frac{a_{12} a_{22}}{a_{21}} |v|^2.
\]

Then,
\[
\frac{d}{dt} V(u_m(t), v_m(t)) \leq \text{Re}[w_m^* u_m] - H(u_m, v_m)
\]

holds for all \( m \in M \).

3767
Proof: By direct computation
\[
\frac{d}{dt} V(u_m(t), v_m(t)) = \text{Re} \left( \begin{bmatrix} a_{12} u_m^* + w_m^* & f_m^* \\ a_{21} f_m + w_m & v_m \end{bmatrix} \begin{bmatrix} u_m & v_m \end{bmatrix} \right) \
\leq \text{Re} [w_m^* u_m] - H(u_m, v_m).
\]

Lemma 2: Take positive numbers \( \nu_m \) such that
\[
\sum_{n \in \mathcal{M}} \nu_n \gamma_{n,m} = 0 \quad \text{for all } m
\]
with
\[
\gamma_{mm} := - \sum_{n \neq m} \gamma_{mn}.
\]
Then,
\[
\sum_{m \in \mathcal{M}} \nu_m \text{Re}[u_m^* w_m] \leq 0 \quad \text{for all } m.
\]

Proof: The existence of \( \nu_m \) follows from the Perron-Frobenius theorem. We have
\[
\sum_m \nu_m (u_m^* w_m + w_m^* u_m) \leq \sum_m \nu_m \left( \sum_{n \neq m} \gamma_{mn} (|u_n|^2 - |u_m|^2) \right)
\]
\[
= \sum_m \left( \sum_{n \in \mathcal{M}} \nu_n \gamma_{nm} \right) |u_m|^2 = 0
\]
where we used
\[
|u_m|^2 + |u_n|^2 - u_m^* u_n - u_n^* u_m = |u_m - u_n|^2 \geq 0.
\]
Hence, (30) holds.

We are ready to show the boundedness. Define
\[
W(z_\mathcal{M}) := \sum_{m \in \mathcal{M}} \nu_m V(u_m, v_m).
\]
Then, by Lemma 1 and 2,
\[
\frac{d}{dt} W(z_\mathcal{M}) \leq - \sum_{m \in \mathcal{M}} \nu_m H(z_m).
\]
(31)

This inequality combined with the positivity of \( \nu_m \) indicate that we can take \( c \) such that \( \frac{d}{dt} W(z_\mathcal{M}) \) is strictly negative outside of
\[
\mathcal{B}_2 := \{ z_\mathcal{M} : W(z_\mathcal{M}) \leq c \}.
\]
This means that \( \mathcal{B}_2 \) is positively invariant, and all solutions enter this compact set in finite time. This completes the proof of the ultimate boundedness.

2) Consensus: Next, we show the second claim. Define
\[
\tilde{z}_i = z_m - z_m, \quad i = 2, \ldots, |\mathcal{M}|,
\]
(32)
\[
\tilde{z} = [\tilde{z}_2, \ldots, \tilde{z}_{|\mathcal{M}|}]
\]
(33)
(33)
and also \( \tilde{u}_i, \tilde{v}_i \) and \( \tilde{u}, \tilde{v}, \tilde{w} \), similarly. It suffices to show \( \tilde{z} \) converges to the origin.

For \( \bullet = u, v \), define
\[
\tilde{f}_i^u(u_m, v_m) := f_i^u(u_m, v_m), \quad i = 1, \ldots, |\mathcal{M}|
\]
(34)
\[
\tilde{f}_i^v(u_m, v_m) := f_i^v(u_m, v_m) - \sum_{m \in \mathcal{M}} u_m^2 - u_m \] (35)

Lemma 3: For any matrix \( P \in \mathbb{R}^{(|\mathcal{M}| - 1) \times (|\mathcal{M}| - 1)} \) such that \( \|P\| \leq 1 \), there exists a positive constant \( c_3 \) such that
\[
\text{Re}(\tilde{u}^* P \tilde{f}_i^u(u_m, v_m)) \leq a_{12} \|u\| \|\tilde{v}\| + c_3 \|\tilde{u}\|^2 (1 + \|u\|^2).
\]
(36)

Proof: We have
\[
\tilde{f}_i^u = \tilde{u}_i (1 - a_{12} \tilde{v}_i - u_m^2 \beta_i \beta_m (u_M) + u_m \beta_m (u_M))
\]
\[
= \tilde{u}_i |1 - a_{12} \tilde{v}_i| - a_{12} \tilde{v}_i - 3 \tilde{u}_i \] (37)
\[
= 3 \tilde{u}_i \sum_{m \in \mathcal{M}} |u_m|^2 + 3 (u_m^2 u_m^2 - u_m^2 |u_m|^2).
\]
For the last term,
\[
\tilde{u}_i (|u_m|^2 + |u_m|^2) + \tilde{u}_i u_m^2 - u_m^2 |u_m|^2
\]
\[
= u_m^2 |u_m|^2 - u_m^2 |u_m|^2.
\]
Therefore, there exists \( c_3 > 0 \) such that
\[
\|\tilde{f}_i^u\| \leq c_3 \|u\| \|\tilde{v}\| (1 + \|u\|^2), \quad i = 2, \ldots, |\mathcal{M}|
\]
and consequently
\[
\|\tilde{f}_i^u\| \leq a_{12} \|\tilde{v}\| + c_3 \|\tilde{u}\| (1 + \|u\|^2).
\]
This readily gives the desired result.

We are in the position to prove the consensus formation, that is, the convergence of \( \tilde{z} \) to 0.

By definition,
\[
\frac{d}{dt} \tilde{u} = \tilde{f}_i^u(u_m, v_m) + \tilde{u}.
\]
\[
\frac{d}{dt} \tilde{v} = \tilde{f}_i^v(u_m, v_m).
\]
We can always take Hurwitz matrix \( L \) such that
\[
\tilde{w} = \sigma L \tilde{u}.
\]
Take \( \mu > 0 \) and a positive definite matrix \( P \) such that \( \|P\| \leq 1 \) satisfying the Lyapunov equation
\[
LP + PL' = -2\mu I
\]
so that
\[
\text{Re}(\tilde{u}^* P \tilde{L} \tilde{u}) = -\mu \|\tilde{u}\|^2.
\]
Let us define
\[
V_3(\tilde{u}, \tilde{v}) := \frac{1}{2} (\tilde{u}^* P \tilde{u} + \|\tilde{v}\|^2).
\]
By direct computation, we have
\[ \hat{v}_i^*(f^*(u_{m_i}, v_{m_i}) - f^*(u_{m_1}, v_{m_1})) = -a_{22}\|\hat{v}_i\|^2 + a_{21}\hat{v}_i^*\hat{u}_i. \]
Therefore, by the ultimate boundedness, there exists \( c_4 > 0 \) such that
\[ \frac{d}{dt} V_3(\hat{u}(t), \hat{v}(t)) = \text{Re} \left( \hat{u}^* P(\hat{f}^*(\hat{u}, \hat{v}) \mu \hat{u} \right) + \sum_{i \neq 1} (-a_{22}\|\hat{v}_i\|^2 + a_{21}\hat{v}_i^*\hat{u}_i) \]
\[ = \text{Re} \left( \hat{u}^* P(\hat{f}^*(\hat{u}, \hat{v}) + a_{21}\hat{v}_i^*\hat{u}_i) - a_{22}\|\hat{v}_i\|^2 - a_{21}\|\hat{u}_i\|^2 \right) \]
\[ \leq a_{12}\|\hat{u}_i\|\|\hat{v}_i\| + c_4\|\hat{u}_i\|^2 - \text{Re} \right( a_{12}\|\hat{u}_i\|^2 - a_{22}\|\hat{v}_i\|^2 + a_{21}\|\hat{u}_i\|^2 \right) \]
\[ \leq (c_4 - \sigma\mu)\|\hat{u}_i\|^2 + (a_{12} + a_{21})\|\hat{u}_i\|\|\hat{v}_i\| - a_{22}\|\hat{v}_i\|^2. \]
The right hand side is negative definite if
\[ c_4 - \sigma\mu < 0, \quad -a_{22}(c_4 - \sigma\mu) > (a_{12} + a_{21})^2. \]
This means if
\[ \sigma > \frac{1}{\mu} \left( c_4 + \frac{(a_{12} + a_{21})^2}{4a_{22}} \right) \]
then \( V_3(\hat{u}, \hat{v}) \) converge to 0 as \( t \to \infty \). Thus, we have 3’-a).

Let \( \hat{A} := A_{x}. \) Then, the linearized dynamics of \( z_{M_i} \) around the origin is given by
\[ \frac{d}{dt} z_M = (I \otimes \hat{A} + \Gamma \otimes D_1)z_M, \]
\[ D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ (\Gamma)_{i,j} = \begin{cases} \gamma_{m_i, m_j} & i \neq j \\ \gamma_{m_i, m_i} & i = j \end{cases}, \]
\[ (39) \]
Note \( \Gamma \) has 0 as an eigenvalue. This readily means
\[ \text{eig}(\hat{A}) \subset \text{eig}(I \otimes \hat{A} + \Gamma \otimes D_1). \]
Recall that \( \hat{A} \) is unstable by Assumption 1-2). This completes the proof of 3’-b).

Trivially, we have 4’) since \( w(t) \) also converges to 0 by definition.

V. NUMERICAL EXAMPLE
We consider system (20) arising from the motivating example in Section III. The initial state is generated from the spatial spectrum of the initial pattern in Fig. 1. Fig. 2 shows the time response without any control input \( u_m \equiv 0 \), where
- real- and imaginary-parts are shown by solid and dashed lines, and
- unstable wave numbers \( m = (4, 0), (2, 2), (2, -2) \) are shown by blue, red, and magenta, respectively. As expected from the convergence to the roll pattern in Fig. 1, two spatial spectrum asymptotically vanishes.

On the other hand, Figs. 3 and 4 show the time response for the controlled case with \( \sigma = 0.1 \). We can see all \( u_m (m \in M) \) converges to the same nonzero value, which corresponds to the \textit{hexagonal} pattern in the original reaction-diffusion system.

Note that we cannot generate square pattern by \textit{partial} diffusive coupling of the corresponding spatial spectrum, for example, by taking
\[ \Gamma = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
In this case, the coupled spatial spectrum achieve consensus, however, possibly converges to 0, as shown in Fig. 5. Then, we observe a roll pattern as in Fig. 1.

VI. CONCLUSION
In this paper, we formulated a feedback stabilization problem of unstable non-uniform spatial pattern in the reaction-diffusion systems. This problem was solved in the finite dimensionally approximated system. The proposed law, which is a diffusive coupling in the spatial spectrum, suitably achieves spectrum consensus and instability preservation.

As a next step, we need to embed the results for the proposed control law onto the original partial differential
equation. The corresponding input pattern should be given as

$$w(t, x, y) := 2\sigma \sum_{m \in M} \text{Re} \left( p_m \left( \sum_{n \neq m, \ n \in M} \gamma_{mn} (u_m - u_n) \right) \right), \quad (40)$$

where $u_m$ for $m \in M$ is defined by (7). We have already verified numerically that this control law achieves the expected pattern formation.

Ongoing works contain
- theoretical evaluation of the control effect in the original reaction-diffusion systems,
- generalization of generated pattern, e.g., square pattern for the example above, and
- generalization of the dynamics such as chemotaxis equations, Rayleigh-Bénard convection.

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