A Note on the Input-Output Structure of Linear Periodic Continuous-Time Systems with Real-Valued Coefficients

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Abstract—In this paper, a Kalman canonical decomposition of finite-dimensional linear periodic continuous-time systems is considered. This paper firstly investigates the invariance properties of the controllable subspace and the observable subspace. This paper then illustrates a counterexample to the existence of the periodic Kalman canonical decomposition in a typical setting, where the coefficients are restricted to be real-valued and the period of the transformed system is restricted to be the same as the given system. Motivated by this counterexample, this paper gives a self-contained exposition of the periodic Kalman canonical decomposition in real-valued coefficients.

I. INTRODUCTION

In this paper, we revisit the Kalman canonical decomposition of linear periodic continuous-time systems for the \((C, A, B)\) triplet. This is an extension of the periodic Kalman canonical decomposition for the \((A, B)\) pair in [1] and is inspired by the satellite control problem in [2] and the stabilizability analysis in [3]. We firstly investigate the invariance properties of the controllable subspace and the observable subspace. We then present a counterexample to the existence of the periodic Kalman canonical decomposition in a typical setting, where the coefficients are restricted to be real-valued and the period of the transformed system is restricted to be the same as the given system. Motivated by this counterexample, we obtain a self-contained exposition of the periodic Kalman canonical decomposition in real-valued coefficients. We obtain the necessary and sufficient condition for the existence of the periodic Kalman canonical decomposition with the same period of the given system. We also prove that, by relaxing a class of coordinate transformation, it is always possible to construct a periodic coordinate transformation with the octuple period of the given periodic system. We notice here that proofs in this paper is are similar to the ones in the Kalman canonical decomposition for linear time-varying systems [4, 5]; however, a key technical tool of block diagonalization method relying on Dolezal [6] is not directly applicable in the periodic decomposition of periodic matrix-valued functions. Hence we introduce an alternative block diagonalization method relying on Sibuya [7]. In this way, the discussion becomes unified and straightforward.

We use the following notations. \(X := Y\) and \(Y := X\) denote that \(X\) is defined by \(Y\). \(\mathbb{R}\) (respectively, \(C, Z, N\)) denotes the set of all real numbers (respectively, complex numbers, integers, natural numbers). \(\mathbb{R}^n\) denotes the set of all vectors whose elements consist of \(\mathbb{R}\) with \(n\)-rows. \(\mathbb{R}^{n \times m}\) denotes the set of all matrices whose elements consist of \(\mathbb{R}\) with \(n\)-rows and \(m\)-columns. \(0_{n \times m} \in \mathbb{R}^{n \times m}\) denotes the zero matrix. \(I_n \in \mathbb{R}^{n \times n}\) denotes the identity matrix. If the sizes are clear from the context, \(0_{n \times m}\) and \(I_n\) are simply denoted by 0 and \(I\), respectively. In the case of block matrices, zero matrix components might be omitted for notational simplicity. \(X^T\) denotes the transpose of \(X \in \mathbb{R}^{n \times m}\). \(\det X\) (respectively, \(X^{-1}\)) denotes the determinant (respectively, inverse) of a matrix \(X \in \mathbb{R}^{n \times n}\). \(X^{-1} := (X^{-1})^T\) denotes the transpose of a matrix \(X^{-1}\). \(\text{Im} X := \{X\xi : \xi \in \mathbb{R}^n\}\) (respectively, \(\text{Ker} X = \{\xi \in \mathbb{R}^n : X\xi = 0\}\)) denotes the image (respectively, kernel) of a matrix \(X \in \mathbb{R}^{n \times m}\). \(\text{dim} V\) denotes a dimension of a subspace \(V \subseteq \mathbb{R}^n\), \(\text{i.e.}\), a number of linearly independent vectors in \(V\). \(XV := \{Xv : v \in V\}\) denotes a subspace for a matrix \(X \in \mathbb{R}^{n \times m}\) and a subspace \(V \subseteq \mathbb{R}^m\). \(\mathbb{R}^n := \{u : u^T v = 0, \forall v \in V\}\) denotes the annihilator of a subspace \(V \subseteq \mathbb{R}^n\). \(C^k(\mathbb{R}, \mathbb{R}^{n \times m})\) denotes the set of all \(C^k\)-functions, \(i.e., k\)-times continuously differentiable functions, from \(\mathbb{R}\) to \(\mathbb{R}^{n \times m}\). \(C^k_{\text{inv}}(\mathbb{R}, \mathbb{R}^{n \times m})\) denotes the set of all invertible functions in \(C^k(\mathbb{R}, \mathbb{R}^{n \times m})\). If the function \(P(t)\) is periodic with a period \(T > 0\), \(i.e., P(t + T) = P(t)\) for \(t \in \mathbb{R}\), it is called \(T\)-periodic. The set of all \(T\)-periodic functions in \(C^k(\mathbb{R}, \mathbb{R}^{n \times m})\) is denoted by \(C^k_T(\mathbb{R}, \mathbb{R}^{n \times m})\). The set of all invertible \(T\)-periodic functions in \(C^k_T(\mathbb{R}, \mathbb{R}^{n \times m})\) is denoted by \(C^k_{T, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times m})\).

In this paper, all proofs are omitted due to page limitations.

II. PROBLEM FORMULATION

Consider a linear \(T\)-periodic continuous-time systems

\[
\dot{x} = A(t)x + B(t)u, \quad \dot{x} := \frac{dx}{dt} \tag{1}
\]
\[
y = C(t)x, \tag{2}
\]

where \(t \in \mathbb{R}\) is a time, \(x(t) \in \mathbb{R}^n\) is a state vector, \(u(t) \in \mathbb{R}^m\) is an input, \(y(t) \in \mathbb{R}^p\) is an output, and \(A \in C^0_T(\mathbb{R}, \mathbb{R}^{n \times n}), B(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n \times m}), C(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n \times p})\) denote the coefficient matrices for certain nonnegative integers \(n, m, p\). Let \(\Phi\) denote the state transition matrix of (1) with \(u = 0\) for \(t \in \mathbb{R}\), \(i.e.,\), a unique solution of the following equations:

\[
\frac{\partial}{\partial s} \Phi(s, t) = A(s)\Phi(s, t), \Phi(t, t) = I_n.
\]

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Φ satisfies the following properties:

\[ \Phi(s, t) = \Phi(s, \tau) \Phi(\tau, t) \quad (3) \]
\[ \Phi(s, t)^{-1} = \Phi(t, s) \quad (4) \]
\[ \det(\Phi(s, t)) \neq 0 \quad (5) \]
\[ \Phi(s, t) = \Phi(s + kT, t + kT) \quad (6) \]

for \( s, \tau, t \in \mathbb{R}, \ k \in \mathbb{Z} \). We remark that (3)-(5) are valid for linear time-varying systems and that (6) is peculiar to linear periodic systems.

Consider a coordinate transformation

\[ \xi = Z(t)x \quad (7) \]

where \( Z(t) \in C^1_{\text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n}) \) is called the coordinate transformation matrix. The system in (1)-(2) is transformed to

\[ \dot{\xi} = F(t)\xi + G(t)u \quad (8) \]
\[ y = H(t)\xi \quad (9) \]

where

\[ F(t) := (\dot{Z}(t) + Z(t)A(t))Z(t)^{-1} \quad (10) \]
\[ G(t) := Z(t)B(t) \quad (11) \]
\[ H(t) := C(t)Z(t)^{-1} \quad (12) \]

The problem is to find a periodic coordinate transformation matrix \( Z(t) \) such that the triplet \( (H, F, G) \) has a certain block structure.

**Problem 1:** Consider the \( T \)-periodic system (1)-(2) with the \( \mathbb{R} \)-valued coefficients \( A \in C^1_T(\mathbb{R}, \mathbb{R}^{n \times n}), B(t) \in C^1_T(\mathbb{R}, \mathbb{R}^{n \times m}), C(t) \in C^1_T(\mathbb{R}, \mathbb{R}^{p \times n}) \). Find a \( kT \)-periodic coordinate transformation matrix \( Z \in C^1_{kT, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n}) \), where \( k \in \mathbb{N} \), satisfying the following three conditions (i)-(iii):

The first condition (i): the triplet \( (H, F, G) \) in (10)-(12) takes on the form

\[ F(t) = \begin{bmatrix}
F_{11}(t) & F_{12}(t) & F_{13}(t) & F_{14}(t) \\
0 & F_{22}(t) & 0 & F_{24}(t) \\
0 & 0 & F_{33}(t) & F_{34}(t) \\
0 & 0 & 0 & F_{44}(t)
\end{bmatrix} \quad (13) \]
\[ G(t) = \begin{bmatrix}
G_1(t) \\
G_2(t) \\
0 \\
0
\end{bmatrix} \quad (14) \]
\[ H(t) = \begin{bmatrix}
0 & H_2(t) & 0 & H_4(t)
\end{bmatrix} \quad (15) \]

at each \( t \in \mathbb{R} \), where the block diagonal components of (13) are square with appropriate dimensions.

The second condition (ii): The pair \( (F_c, G_c) \) is controllable at each \( t \in \mathbb{R} \), where \( F_c(t) \) and \( G_c(t) \) are defined by

\[ F_c(t) := \begin{bmatrix}
F_{13}(t) & F_{12}(t) \\
0 & F_{22}(t)
\end{bmatrix}, \quad (16) \]
\[ G_c(t) := \begin{bmatrix}
G_1(t) \\
G_2(t)
\end{bmatrix}, \quad (17) \]

where the submatrices in the right hand sides are composed of (13)-(14).

The condition (iii): The pair \( (H_o, F_o) \) is observable at each \( t \in \mathbb{R} \), where \( H_o(t) \) and \( F_o(t) \) are defined by

\[ H_o(t) := \begin{bmatrix}
H_2(t) & H_4(t)
\end{bmatrix}, \quad (18) \]
\[ F_o(t) := \begin{bmatrix}
F_{22}(t) & F_{24}(t) \\
0 & F_{44}(t)
\end{bmatrix}, \quad (19) \]

where the submatrices in the right hand sides are composed of (13) and (15).

If such coordinate transformation matrix \( Z(t) \) exists, the transformed system (8)-(9) or the triplet \( (H, F, G) \) in (13)-(15) is called the \( kT \)-periodic Kalman canonical decomposition of the system (1)-(2) in \( \mathbb{R} \).

We remark here that the phrase “in \( \mathbb{R} \)” in “the \( kT \)-periodic Kalman canonical decomposition of the system (1)-(2) in \( \mathbb{R} \)” indicates that the coordinate transformation matrix \( Z(t) \) is real-valued. We have added this phrase because it is possible to extend the class of \( Z(t) \) to be complex-valued even if the triplet \( (A, B) \) is real-valued.

### III. Structure of Subspaces

In this section, we study the structure of subspaces resulting from the controllability and the observability.

#### A. Controllable Subspace

Let us recall the definition of the controllable subspace for linear time-varying systems.

**Definition 2:** A state \( x_0 \in \mathbb{R}^n \) of the system (1) is said to be controllable at time \( t \) if there exist a finite \( s \geq t \), depending on \( x_0 \) and \( t \), and a piecewise continuous function \( u \in \mathbb{U} \), depending on \( x_0, t \) and \( s \) and defined on the closed interval \([t, s]\), which satisfy the integral equation

\[ \Phi(s, t)x_0 + \int_t^s \Phi(s, \tau)B(\tau)u(\tau)d\tau = 0. \]

The system (1) is said to be controllable at time \( t \), or the pair \( (A, B) \) is said to be controllable at time \( t \), if all states in \( \mathbb{R}^n \) are controllable at time \( t \).

**Definition 3:** The controllability Gramian is defined by

\[ W_c(t, s) := \int_t^s \Phi(t, \tau)B(\tau)B^T(t, \tau) \Phi(t, \tau)^T d\tau. \]

**Definition 4:** The set of states controllable at time \( t \) is denoted by

\[ \mathcal{C}(t) := \bigcup_{p \in [t, \infty)} \left\{ \int_t^p \Phi(t, \tau)B(\tau)u(\tau) d\tau : u \in \mathbb{U} \right\}, \]

which is said to be a **controllable subspace** at time \( t \).

Since linear periodic systems are time-varying, Definitions 2-4 are also valid for linear periodic systems.

Let us recall the properties of \( \mathcal{C}(t) \) for linear time-varying systems.
Theorem 5 ([4]): Consider the system (1) which is not necessarily periodic. Then $C(t)$ satisfies the following conditions. (i) 

$$C(t) = \bigcup_{p\in[t,\infty)} \text{Im} \ W_c(t,p).$$

(ii) There exists a positive scalar function $p_c(t)$ such that 

$$C(t) = \text{Im} \ W_c(t,t+p_c(t)).$$

(iii) $C(t)$ is backward $\Phi$-invariant, i.e., 

$$C(t) \supset \Phi(t,s)C(s), \quad t \leq s.$$ 

(iv) The pair $(A, B)$ is controllable at time $t$ if and only if 

$$\dim C(t) = n$$ 

at time $t$.

The backward $\Phi$-invariance in Theorem 5 (iii) is a bottleneck for constructing the Kalman canonical decomposition which is globally valid in time for linear time-varying systems. Hence, the authors have introduced the concept of interval-wise $\Phi$-invariance and have studied the various Kalman canonical decompositions depending on the variants of the controllabilities and the observabilities in [5].

In the case of linear periodic systems, we have more stronger properties.

Proposition 6 ([11]): Consider the $T$-periodic system (1). Then $C(t)$ satisfies the following conditions. (i) 

$$C(t) = \text{Im} \ W_c(t,t+nT), \quad t \in \mathbb{R}.$$ 

(ii) $C(t)$ is $T$-periodic, i.e., 

$$C(t) = C(t+T), \quad t \in \mathbb{R}.$$ 

(iii) $C(t)$ is $\Phi$-invariant, i.e., 

$$C(t) = \Phi(t,s)C(s), \quad t, s \in \mathbb{R}.$$ 

We note that similar arguments are valid if the controllability is replaced by the reachability or the influenceability.

B. Observable Subspace

Let us recall the definition and the properties of the observable subspace for linear time-varying systems.

Definition 9: The set of states observable at time $t$ is denoted by 

$$\mathcal{O}(t) := \bigcup_{p\in[t,\infty)} \left\{ \int_t^p \Phi(t,\tau)^T C(\tau)^T \bar{u}(\tau)d\tau : \bar{u} \in \bar{U} \right\},$$

which is said to be an observable subspace at time $t$.

Theorem 10 ([4]): Consider the system (1)-(2) which is not necessarily periodic. Then $\mathcal{O}(t)$ satisfies the following properties. (i) 

$$\mathcal{O}(t) = \bigcup_{p\in[t,\infty)} \text{Im} \ W_o(t,p).$$

(ii) There exist a positive scalar function $o_c(t)$ such that 

$$\mathcal{O}(t) = \text{Im} \ W_o(t,t+o_c(t)).$$

(iii) $\mathcal{O}(t)$ is backward $\Phi^T$-invariant, i.e., 

$$\mathcal{O}(t) \supset \Phi(s,t)^T \mathcal{O}(s), \quad t \leq s.$$ 

(iv) The pair $(C, A)$ is observable at time $t$ if and only if 

$$\dim \mathcal{O}(t) = n$$ 

at time $t$.

The backward $\Phi^T$-invariance in Theorem 10 (iii) is not convenient for constructing the Kalman canonical decomposition. Hence, the authors have shown that the annihilator $\mathcal{O}^-(t)$ is forward $\Phi$-invariant for linear time-varying systems in [5].

In the case of linear periodic systems, we have more stronger properties.

Proposition 11: Consider the $T$-periodic system (1)-(2). Then $\mathcal{O}(t)$ satisfies the following conditions. (i) 

$$\mathcal{O}^-(t) = \text{Ker} \ W_o(t,t+nT), \quad t \in \mathbb{R}.$$ 

(ii) $\mathcal{O}(t)$ is $T$-periodic, i.e., 

$$\mathcal{O}^-(t) = \mathcal{O}^-(t+T), \quad t \in \mathbb{R}.$$ 

(iii) $\mathcal{O}^-(t)$ is $\Phi$-invariant, i.e., 

$$\Phi(s,t) \mathcal{O}(t)^- = \mathcal{O}(s)^-, \quad t, s \in \mathbb{R}.$$ 

We note that similar arguments are valid if the observability is replaced by the determinability or the visibility.

C. Intersection Subspace

The $\Phi$-invariance properties of $C(t)$ and $\mathcal{O}^-(t)$ take over to their intersection subspace $C(t) \cap \mathcal{O}(t)^-$. 

Proposition 12: Consider the $T$-periodic system (1)-(2). Then $C(t) \cap \mathcal{O}(t)^-$ satisfies the following conditions. (i) 

$$C(t) \cap \mathcal{O}(t)^- = \mathcal{O}^-(t)$$ 

is $T$-periodic, i.e., 

$$C(t) \cap \mathcal{O}(t)^- = C(t+T) \cap \mathcal{O}(t+T)^-, \quad t \in \mathbb{R}.$$ 

(ii) $C(t) \cap \mathcal{O}(t)^-$ is $\Phi$-invariant, i.e., 

$$\Phi(s,t)\{C(t) \cap \mathcal{O}(t)^-\} = \{C(s) \cap \mathcal{O}(s)^-\}, \quad s, t \in \mathbb{R}.$$ 

Definition 8: The observability Gramian is defined by 

$$W_o(t,s) := \int_t^s \Phi(\tau,t)^T C(\tau)^T \Phi(\tau,t) d\tau.$$ 

(21)
The \( \Phi \)-invariance properties of \( C(t), O, C(t) \cap O(t) \) in Proposition 6 (iii), Proposition 11 (iii), Proposition 12 (ii) do not imply the \( \Phi \)-invariance properties of the other intersection subspaces \( C(t) \cap O(t), C(t) \cap O(t)^\perp, C(t)^\perp \cap O(t) \). But their dimensions are shown to be constant.

**Proposition 13:** Consider the \( T \)-periodic system (1)-(2). Then the intersection subspaces satisfies the following conditions

\[
\dim \{ C(t) \cap O(t) \} = \text{const. } = n_1 \quad (22)
\]

\[
\dim \{ C(t) \cap O(t) \} = \text{const. } = n_2 \quad (23)
\]

\[
\dim \{ C(t)^\perp \cap O(t) \} = \text{const. } = n_3 \quad (24)
\]

\[
\dim \{ C(t)^\perp \cap O(t) \} = \text{const. } = n_4 \quad (25)
\]

for \( t \in \mathbb{R} \).

The conditions (22)–(25) are equivalent to the existence of the Kalman canonical decomposition which is valid global in time. It follows that, for any given \( T \)-periodic system (1)-(2), it is always possible to construct a coordinate transformation matrix \( Z \in C_{\text{inv}}^1(\mathbb{R}, \mathbb{R}^{n \times n}) \) such that the transformed system (8)-(9) takes on the Kalman canonical decomposition. However, it is not clear whether it is possible to find a \( kT \)-periodic transformation matrix \( Z \in C_{kT, \text{inv}}^1(\mathbb{R}, \mathbb{R}^{n \times n}) \).

**IV. Periodical Kalman Canonical Decomposition in \( \mathbb{R} \)**

**A. Motivating Example**

Firstly we illustrate a counterexample to the existence of the \( T \)-periodical Kalman canonical decomposition in \( \mathbb{R} \).

**Example 14:** Let \( A \in C_T^0(\mathbb{R}, \mathbb{R}^{2 \times 2}), B \in C_T^0(\mathbb{R}, \mathbb{R}^{2 \times 1}), \) and \( C \in C_T^0(\mathbb{R}, \mathbb{R}^{1 \times 2}) \) be given by

\[
A(t) := \begin{bmatrix} 0 & \frac{4}{3} \\ -\frac{4}{3} & 0 \end{bmatrix},
\]

\[
B(t) := \begin{bmatrix} \sin \left( \frac{4t}{3} \right) \\ \sin \left( \frac{4t}{3} \right) \cos \left( \frac{2t}{3} \right) + \sin \left( \frac{2t}{3} \right) \end{bmatrix},
\]

\[
C(t) = 0_{1 \times 2}.
\]

Then the controllability Gramian over \( [t, t+2T] \) is given by

\[
W_c(t, t+2T) = \begin{bmatrix} T \left( 1 + \sin \left( \frac{2t}{3} \right) \right) & T \cos \left( \frac{2t}{3} \right) \\ T \cos \left( \frac{2t}{3} \right) & T \left( 1 - \sin \left( \frac{2t}{3} \right) \right) \end{bmatrix}
\]

and satisfy

\[
\text{rank } W_c(t, t+2T) = 1 < 2
\]

for all \( t \in \mathbb{R} \); therefore, the pair \( (A, B) \) is uncontrollable. It is clear that all nonzero states in \( \mathbb{R}^2 \) are not observable. Suppose that there exists a \( T \)-periodic coordinate transformation matrix \( Z \in C_{kT, \text{inv}}^1(\mathbb{R}, \mathbb{R}^{2 \times 2}) \) such that the triplet \( (C, A, B) \) is transformed to \( (H, G, F) \) of the forms (13)-(15). If such \( Z(t) \) exists, we have

\[
n_1 = 1, \ n_2 = 0, \ n_3 = 1, \ n_4 = 0.
\]

It follows that \( F(t) \) in (13) takes on the form

\[
F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}
\]

where \( F_{11}, F_{12}, F_{33} \in C_T^0(\mathbb{R}, \mathbb{R}) \). By the \( T \)-periodicity of \( Z(t) \), the monodromy matrices in \( \xi \)-coordinate and in \( \xi \)-coordinate are similar; in other words, the characteristic multipliers are invariant with respect to a coordinate transformation \( \xi = Z(t)x \).

Let us investigate the necessary and sufficient condition for the existence of the \( T \)-periodical Kalman canonical decomposition in \( \mathbb{R} \).

**Theorem 15:** Consider the \( T \)-periodic system (1)-(2) with the real-valued coefficients \( A \in C_T^0(\mathbb{R}, \mathbb{R}^{n \times n}), B(t) \in C_T^0(\mathbb{R}, \mathbb{R}^{n \times m}), \) \( C(t) \in C_T^0(\mathbb{R}, \mathbb{R}^{p \times n}) \). Then there exists a \( T \)-periodic coordinate transformation \( Z \in C_{kT, \text{inv}}^1(\mathbb{R}, \mathbb{R}^{n \times n}) \) such that the transformed system (8)-(9) takes on the \( T \)-periodical Kalman canonical decomposition in \( \mathbb{R} \) if and only if there exists a \( T \)-periodic \( Z(t) \in C_{kT, \text{inv}}^1(\mathbb{R}, \mathbb{R}^{n \times n}) \) satisfying

\[
Z(t)W_c(t, t+nT)Z(t)^T = \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{21}(t)^T & \tilde{P}_{22}(t) \end{bmatrix} 0_{n_3 \times n_3} 0_{n_4 \times n_4},
\]

\[
Z(t)^{-T}W_o(t, t+nT)Z(t)^{-1} = \begin{bmatrix} \tilde{Q}_{22}(t) \\ \tilde{Q}_{24}(t) \end{bmatrix} 0_{n_3 \times n_3} \tilde{Q}_{24}(t),
\]

for certain matrix-valued functions \( \tilde{P}_{11} \in C^1(\mathbb{R}, \mathbb{R}^{n_1 \times n_1}), \)

\( \tilde{P}_{12} \in C^1(\mathbb{R}, \mathbb{R}^{n_1 \times n_2}), \)

\( \tilde{P}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2 \times n_2}), \)

\( \tilde{Q}_{22} \in C^1(\mathbb{R}, \mathbb{R}^{n_2 \times n_4}), \)

\( \tilde{Q}_{24} \in C^1(\mathbb{R}, \mathbb{R}^{n_4 \times n_4}), \)

where the submatrices

\[
\tilde{P}_{\text{sub}} := \begin{bmatrix} \tilde{P}_{11}(t) & \tilde{P}_{12}(t) \\ \tilde{P}_{21}(t)^T & \tilde{P}_{22}(t) \end{bmatrix},
\]

\[
\tilde{Q}_{\text{sub}} := \begin{bmatrix} \tilde{Q}_{22}(t) \\ \tilde{Q}_{24}(t) \end{bmatrix} \tilde{Q}_{24}(t)^T
\]

are positive definite for \( t \in \mathbb{R} \).
In the cases of the \((A, B)\)-pair or the \((C, A)\) pair, the statement in Theorem 15 is simplified as follows.

**Corollary 16:** Consider the \(T\)-periodic system (1)-(2) with the real-valued coefficients \(A \in C^0_T(\mathbb{R}, \mathbb{R}^{n \times n})\), \(B(t) \in C^0_T(\mathbb{R}, \mathbb{R}^{n \times m})\), \(C(t) = 0 \in C^0_T(\mathbb{R}, \mathbb{R}^{p \times n})\). Then there exists a \(T\)-periodic coordinate transformation \(Z \in C^1_T,\text{inv}(\mathbb{R}, \mathbb{R}^{n \times n})\) such that the transformed system (8)-(9) takes on the \(T\)-periodic Kalman canonical decomposition in \(\mathbb{R}\). The following lemma, which relies on Sibuya [7], is fundamental for constructing the Kalman canonical decomposition in \(\mathbb{R}\).

We have revisited the problem of the Kalman canonical decomposition for linear periodic continuous-time systems. The first key idea is to explicitly mention the classes of the period and the coefficients; from this perspective, several decompositions are precisely characterized. The second key idea is to investigate the invariance properties of the time-varying subspaces resulting from the controllability and the observability. The third key idea is to investigate the periodic simultaneous block diagonalization of matrix-valued function by extending Sibuya [7]. Based on these key ideas, we have proved the existence of periodic Kalman canonical decomposition in real-valued coefficients.

**APPENDIX**

The block diagonalization of matrix-valued function, which is due to Dolezal [6] is fundamental for constructing the Kalman canonical decomposition. It was claimed in [8] that the periodic block diagonalization with the same period of the given periodic matrix-valued function is possible; however, this is not precise as indicated in Example 14.

Hence we introduce the following lemma, which relies on Sibuya [7], is fundamental for constructing the Kalman canonical decomposition in \(\mathbb{R}\).
Lemma 22 ([7]): Suppose that $P \in C^1_T(\mathbb{R}, \mathbb{R}^{n \times n})$ is symmetric and has constant rank $n_1$ for $t \in \mathbb{R}$. Then, $P(t)$ is factored as
\[
Z(t)P(t)Z(t)^T = \begin{bmatrix}
\hat{P}_{11}(t) & \hat{P}_{12}(t) \\
\hat{P}_{12}(t)^T & \hat{P}_{22}(t)
\end{bmatrix},
\]
(35)
where $\hat{P}_{11} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_1 \times n_1})$ is symmetric and $Z \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n})$.

By using Lemma 22, we can obtain the following lemma.

Lemma 23: Suppose that $P$, $Q \in C^1_T(\mathbb{R}, \mathbb{R}^{n \times n})$ are positive semidefinite for $t \in \mathbb{R}$. Then, the following statements are equivalent. (i) $P(t)$ and $Q(t)$ are factored as
\[
Z(t)P(t)Z(t)^T = \begin{bmatrix}
\hat{P}_{11}(t) & \hat{P}_{12}(t) \\
\hat{P}_{12}(t)^T & \hat{P}_{22}(t)
\end{bmatrix},
\]
(35)
where $\hat{P}_{11} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_1 \times n_1})$ is symmetric and $Z \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n})$.

(ii) $P(t)$ and $Q(t)$ are factored as
\[
Z(t)^{-T}Q(t)Z(t)^{-1} = \begin{bmatrix}
0_{n_1 \times n_1} & \hat{Q}_{22}(t) \\
\hat{Q}_{22}(t)^T & 0_{n_3 \times n_3}
\end{bmatrix},
\]
(41)
where $Z \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n})$ is a matrix-valued function, $n_1$, $n_2$, $n_3$, $n_4$ are nonnegative integers, $\hat{P}_{11} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_1 \times n_1})$, $\hat{P}_{12} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_2 \times n_1})$, $\hat{P}_{22} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_2 \times n_2})$, $\hat{Q}_{22} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_2 \times n_2})$, $\hat{Q}_{44} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_4 \times n_4})$ are positive definite for $t \in \mathbb{R}$.

(iii) $P(t)$ and $Q(t)$ are factored as
\[
Z(t)P(t)Z(t)^T = \begin{bmatrix}
\hat{P}_{11}(t) & \hat{P}_{12}(t) \\
\hat{P}_{12}(t)^T & \hat{P}_{22}(t)
\end{bmatrix},
\]
(35)
where $\hat{P}_{11} \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n_1 \times n_1})$ is symmetric and $Z \in C^1_{ST, \text{inv}}(\mathbb{R}, \mathbb{R}^{n \times n})$.

REFERENCES