Robust Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Gain-Scheduling Observer Design for Removal of NO$_x$ Sensor Ammonia Cross Sensitivity in Selective Catalytic Reduction Systems

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Abstract—This paper focuses on the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ observer design problem for removal of NO$_x$ sensor ammonia cross-sensitivity in selective catalytic reduction (SCR) systems. Due to the ammonia cross-sensitivity, the reading of the NO$_x$ sensor is contaminated severely. In order to utilize the concentration of the NO$_x$ in the downstream tailpipe, it is desired to design an observer to estimate the actual value of the NO$_x$ concentration and the time-varying cross-sensitivity factor. It is assumed that the variation of the cross-sensitivity is bounded. A design approach of the gain-scheduling robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ observer is proposed, in which the gain-scheduling strategy is adaptive to compensate for the nonlinearity and the time-varying parameters in the SCR model. Simulations and comparisons are carried out to show the efficacy and the advantages of the proposed approach over the existing extended Kalman filter on a Diesel engine powertrain.

I. INTRODUCTION

As pointed in [1], the urea-based selective catalytic reduction (SCR) is a promising technique to reduce the NO$_x$ emissions for Diesel engines in vehicle applications. In the urea-SCR systems, the concentration of the NO$_x$ is of importance in the closed-loop control and diagnosis [2], [3]. However, the commercial NO$_x$ sensors are cross-sensitive to ammonia which is inevitable in the vehicle applications with a high NO$_x$ conversion efficiency. Due to the phenomenon of this cross-sensitivity, the reading of the NO$_x$ sensor is quite different from the actual value. If the reading is directly applied to the control law, it would degrade or even deteriorate the performance of the closed-loop systems. Therefore, from the application perspective, an observer is necessary to remove the NO$_x$ sensor ammonia cross-sensitivity and estimate the cross-sensitivity factor. In [4], the authors applied the extended Kalman filter to observe the cross-sensitivity factor by assuming that the factor is a constant with a slow variation. However, the variation was not taken into account in the filter design. The performance of the observer cannot be guaranteed in the SCR systems with a time-varying cross-sensitivity factor that may change with the environmental variations.

On another research frontier, the robust filtering and control have attracted considerable attention in the past decade due to the fact that the designed filter is robust to the uncertainties or the parameter variations [5]–[11]. In robust filtering, the $\mathcal{H}_\infty$ [12] strategy is more robust than the $\mathcal{H}_2$ counterpart. But it is more sensitive to the white noise signals. In order to embrace both advantages of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ filtering, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering was proposed in [13].

In this paper, we investigate the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ gain-scheduling observer design to estimate the NO$_x$ concentration and the cross-sensitivity factor. Compared with the existing work in [4], the advantages and the contributions of the proposed method are summarized as: (1) The online computational load of the proposed method in this paper is much smaller. It is much more suitable for the real-time implementations. (2) The variation of the cross-sensitivity factor is explicitly considered into the observer design. Therefore, the performance of the designed observer can be guaranteed if the variation of the cross-sensitivity factor is within a prescribed range. (3) The transient and steady responses of the designed observer are much better in terms of the rise-time and the steady error.

II. SELECTIVE CATALYTIC REDUCTION

In this section, we will firstly introduce the operation principle of the urea-SCR aftertreatment system. Based on the main chemical reactions, the SCR system modeling will be covered. Then, the problem formulation and the design objectives will be given.

A. SCR Operation Principle

A schematic view of a urea-based SCR aftertreatment system is illustrated in Fig. 1. A NO$_x$ sensor, located before the urea injection, is used to measure the concentration of the NO$_x$ in the engine exhaust gas. If the exhaust gas is not catalyzed in a Diesel oxidation catalyst (DOC) device, the concentration of the NO can take more than 90% of the overall engine-out exhaust NO$_x$ [1], [14]. Since the upstream NO$_x$ sensor (the first one) is placed before the urea injection, the reading is not affected by the ammonia concentration. Thus, the measurement of the input NO$_x$ to the aftertreatment system is accurate.

In order to catalytically convert the NO$_x$ to N$_2$ and H$_2$O, normally, the 32.5\% aqueous urea solution (known
as AdBlue) is injected into the upstream. The reactions can be expressed in the equations as follows.

**Urea evaporation:**

\[
\text{NH}_2 - \text{CO} - \text{NH}_2 (\text{liquid}) \rightarrow \text{NH}_2 - \text{CO} - \text{NH}_2 + n\text{H}_2\text{O}. \tag{1}
\]

**Urea decomposition:**

\[
\text{NH}_2 - \text{CO} - \text{NH}_2 \rightarrow \text{NH}_3 + \text{HNCO}. \tag{2}
\]

**Isocyanic acid hydrolysation:**

\[
\text{HNCO} + \text{H}_2\text{O} \rightarrow \text{NH}_3 + \text{CO}_2. \tag{3}
\]

Here, \(\text{NH}_2 - \text{CO} - \text{NH}_2\) refers to the solid phase of the urea and \(n\) denotes the number of \(\text{H}_2\text{O}\) in the reaction. As shown in Fig. 2, the ammonia inside the SCR catalyst can be absorbed on the SCR substrate and desorbed from the SCR substrate. The absorption and the desorption can be illustrated in the following bi-directional reaction:

\[
\text{NH}_3 + \theta_{\text{free}} \longleftrightarrow \text{NH}_3^{*}, \tag{4}
\]

where \(\theta_{\text{free}}\) is the free substrate site of the SCR catalyst and \(\text{NH}_3^{*}\) denotes the ammonia adsorbed on the SCR substrate. It follows from [15] that the kinetic rates for ammonia adsorption and desorption from the catalyst surface can be expressed as:

\[
R_{\text{ads}} = k_{\text{ads}} \exp\left(\frac{-E_{\text{ads}}}{RT}\right) C_{\text{NH}_3}(1 - \theta_{\text{NH}_3}),
\]

\[
R_{\text{des}} = k_{\text{des}} \exp\left(\frac{-E_{\text{des}}}{RT}\right) \theta_{\text{NH}_3}, \tag{5}
\]

where \(R_{\text{ads}}\) and \(R_{\text{des}}\) stand for the adsorption reaction rate and the desorption reaction rate, respectively; \(k_{\text{ads}}, k_{\text{des}}, E_{\text{ads}},\) and \(E_{\text{des}}\) are constant scalars; \(T\) represents the temperature (K); \(R\) is the ideal gas constant (J/K mol), \(C_{\text{NH}_3}\) denotes the mole concentration of the ammonia in the catalyst (mol/m^3); and \(\theta_{\text{NH}_3}\) is the ammonia surface coverage ratio which is defined as:

\[
\theta_{\text{NH}_3} = \frac{M_{\text{NH}_3}^*}{\Theta}, \tag{6}
\]

Here, \(M_{\text{NH}_3}^*\) is the mole of the ammonia stored on the SCR substrate surface and \(\Theta\) is the ammonia storage capacity (mole) which is sensitive to the temperature and can be described by the following equation:

\[
\Theta = S_1 \exp(-S_2 T), \tag{7}
\]

where \(S_1\) and \(S_2\) are constant.

Though the gaseous ammonia in the catalyst can react with the \(\text{NO}_x\), most \(\text{NO}_x\) reductions are achieved by the reactions with \(\text{NH}_3\). The main catalytical reactions between the absorbed ammonia \(\text{NH}_3\) and the \(\text{NO}_x\) can be summarized as:

\[
4\text{NH}_3^* + 4\text{NO} + \text{O}_2 \rightarrow 4\text{N}_2 + 6\text{H}_2\text{O}, \tag{8}
\]

\[
2\text{NH}_3^* + \text{NO} + \text{NO}_2 \rightarrow 2\text{N}_2 + 3\text{H}_2\text{O}, \tag{9}
\]

\[
4\text{NH}_3^* + 3\text{NO}_2 \rightarrow 3.5\text{N}_2 + 6\text{H}_2\text{O}. \tag{10}
\]

The reaction in (8) is known as the ‘standard SCR’ since the reaction rate is fast and the NO dominates in the engine-out \(\text{NO}_x\). The reaction in (8) is regarded as the ‘standard SCR’ due to the fact that reaction rate is the fastest in the three reduction reactions. But it requires that the ratio of \(\text{NO}/\text{NO}_2\) is one. The \(\text{NO}_x\) is the minority of the engine-out \(\text{NO}_x\) and this reaction is not the main one. The third reaction in (10) is the ‘slow SCR’ since the reaction rate is slow.

Based on the above analysis, the reaction rate of the \(\text{NO}_x\) reduction can be described by:

\[
R_{\text{red}} = k_{\text{red}} \exp\left(\frac{-E_{\text{red}}}{RT}\right) C_{\text{NO}_x}\theta_{\text{NH}_3}, \tag{11}
\]

where \(R_{\text{red}}\) stands for the reaction rate, and \(k_{\text{red}}\) and \(E_{\text{red}}\) are two constants.

Beside the \(\text{NH}_3\) absorption/desorption and \(\text{NO}_x\) reduction, another main reaction is the \(\text{NH}_3\) oxidation which occurs at a temperature higher than 450° and can be represented as follows:

\[
4\text{NH}_3 + 3\text{O}_2 \rightarrow 2\text{N}_2 + 6\text{H}_2\text{O}. \tag{12}
\]
The corresponding reaction rate can be defined as:
\[ R_{\text{oxi}} = k_{\text{oxi}} \exp \left( -\frac{E_{\text{oxi}}}{RT} \right) \theta_{\text{NH}_3} , \]  
(13)
where \( k_{\text{oxi}} \) and \( E_{\text{oxi}} \) are two constants.

**B. Three-state SCR System Model**

The main reactions in a SCR aftertreatment system are depicted in Fig. 2 and have been introduced in details in the above subsection. By neglecting the mass transfer, the surface phase concentrations of the species, and the slow SCR reaction inside the SCR catalyst, a three-state nonlinear model was developed in [16] as:

\[
\begin{bmatrix}
\dot{C}_{\text{NO}_x} \\
\dot{\theta}_{\text{NH}_3} \\
\dot{C}_{\text{NH}_3}
\end{bmatrix} = \begin{bmatrix}
f_1(C_{\text{NO}_x}, \theta_{\text{NH}_3}, T, F) \\
f_2(C_{\text{NO}_x}, \theta_{\text{NH}_3}, T, C_{\text{NH}_3}) \\
f_3(C_{\text{NH}_3}, \theta_{\text{NH}_3}, T, F)
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\frac{F}{V}
\end{bmatrix} C_{\text{NH}_3, in} + \begin{bmatrix}
\frac{\theta}{V} \\
0 \\
0
\end{bmatrix} C_{\text{NO}_x, in},
\]  
(14)

where

\[
f_1(C_{\text{NO}_x}, \theta_{\text{NH}_3}, T, F) = -C_{\text{NO}_x} \left( \Theta_{\text{red}} \theta_{\text{NH}_3} + \frac{F}{V} \right) + r_{\text{oxi}} \Theta_{\text{NH}_3},
\]

\[
f_2(C_{\text{NO}_x}, \theta_{\text{NH}_3}, T, C_{\text{NH}_3}) = -\theta_{\text{NH}_3} \left( r_{\text{ads}} C_{\text{NH}_3} + r_{\text{des}} + r_{\text{red}} C_{\text{NO}_x} + r_{\text{oxi}} \right) + r_{\text{ads}} C_{\text{NH}_3},
\]

\[
f_3(C_{\text{NH}_3}, \theta_{\text{NH}_3}, T, F) = -C_{\text{NH}_3} \left( \Theta_{\text{ads}} (1 - \theta_{\text{NH}_3}) + \frac{F}{V} \right) + \Theta_{\text{des}} \theta_{\text{NH}_3},
\]

\[
r_x = k_x \exp \left( -\frac{E_x}{RT} \right),
\]

\( F \) is the exhaust volume flow rate through the catalyst (m³/s), \( V \) is the catalyst volume (m³), \( C_{\text{NO}_x} \) and \( C_{\text{NH}_3} \) denote the tailpipe NOₓ and NH₃ concentrations, respectively, \( C_{\text{NO}_x, in} \) is engine-out exhaust NOₓ concentration, and \( C_{\text{NH}_3, in} \) is the NH₃ concentration entering into the catalyst.

**C. Problem Formulation and Design Objectives**

As seen in Fig. 1, the NOₓ and the NH₃ are mixed downstream. Considering the cross-sensitivity, the reading of the NOₓ sensor in the downstream is contaminated by the slipped ammonia. And it can be expressed by the superposition of the NOₓ and the ammonia concentration as:

\[ C_{\text{NO}_x, re} = C_{\text{NO}_x} + K_{cs} C_{\text{NH}_3} + v \]  
(15)

where \( C_{\text{NO}_x, re} \) is the reading of the downstream NOₓ sensor, \( K_{cs} \) is the cross-sensitivity factor, \( v \) denotes the measurement noise, and \( C_{\text{NO}_x} \) and \( C_{\text{NH}_3} \) are the downstream NOₓ and NH₃ concentration, respectively.

Since the reading of the ammonia sensor is accurate, the actual NOₓ concentration can be easily obtained if the cross-sensitivity factor \( K_{cs} \) is constant and known. However, as observed in [2], [3], the cross-sensitivity factor is time-varying. The variation may be caused by the environmental changes and the sensor aging. The challenge here is that how to design a robust observer such that the actual NOₓ concentration and the cross-sensitivity factor can be precisely estimated even though the cross-sensitivity factor is time-varying.

The dynamics of the cross-sensitivity factor is modeled by the following equation:

\[ \dot{K}_{cs} = \tilde{M} \dot{N}(t) K_{cs} , \]  
(16)

where \( \tilde{M} \) is a positive constant scalar and \( \dot{N}(t) \) is time-varying satisfying \( N^2(t) \leq 1 \). Recalling the dynamics of the NOₓ concentration, we can have the following augmented system:

\[ \dot{X} = f(\theta_{\text{NH}_3}, T, F) X + g_1(T) \theta_{\text{NH}_3} + g_2(F) C_{\text{NO}_x, in} , \]  
(17)

where

\[ X = \begin{bmatrix} C_{\text{NO}_x} \\ K_{cs} \end{bmatrix} , \]

\[ f(\theta_{\text{NH}_3}, T, F) = \begin{bmatrix} - (\Theta_{\text{red}} \theta_{\text{NH}_3} + \frac{F}{V}) & 0 \\ 0 & \tilde{M} \dot{N}(t) \end{bmatrix} , \]

\[ g_1(T) = \begin{bmatrix} r_{\text{oxi}} \Theta \\ 0 \end{bmatrix} , g_2(F) = \begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix} . \]

With the augmented system, the reading of the NOₓ sensor can be rewritten as

\[ C_{\text{NO}_x, re} = \begin{bmatrix} 1 & C_{\text{NH}_3} \end{bmatrix} X + v . \]  
(18)

Note that the augmented system in (17) is a nonlinear one. But, at any specific time, the system matrices and the system inputs can be determined. The augmented system is observable at a specific time if and only if the observability matrix

\[ \text{rank} \begin{bmatrix} 1 \\ C_{\text{NH}_3} \end{bmatrix} f(\theta_{\text{NH}_3}, T, F) = 2. \]  
(19)

Evaluating the rank of the observability matrix, we can have that the rank is equal to 2 if and only if the ammonia concentration \( C_{\text{NH}_3} \) is nonzero. In order to achieve a high NOₓ conversion efficiency, the ammonia concentration \( C_{\text{NH}_3} \) at the downstream is always nonzero. If \( C_{\text{NH}_3} \) is zero, the phenomenon of cross-sensitivity does not occur and we can trust the reading of the NOₓ sensor. Therefore, the augmented system is observable and an estimator can be designed to estimate the state vector.

Suppose that the estimator has the following form:

\[ \dot{\hat{y}} = \hat{f}(\theta_{\text{NH}_3}, T, F) \eta + g_1(T) \theta_{\text{NH}_3} + g_2(F) C_{\text{NO}_x, in} + L(\theta_{\text{NH}_3}, T, F) \left( C_{\text{NO}_x, re} - \begin{bmatrix} 1 \\ C_{\text{NH}_3} \end{bmatrix} \eta \right) , \]  
(20)

where \( \hat{f}(\theta_{\text{NH}_3}, T, F) = \begin{bmatrix} - (\Theta_{\text{red}} \theta_{\text{NH}_3} + \frac{F}{V}) & 0 \\ 0 & 0 \end{bmatrix} \) and the gain \( L(\theta_{\text{NH}_3}, T, F) \) is dependent on the available time-varying \( \theta_{\text{NH}_3}, T, \) and \( F. \) Defining the estimation error \( e = \)
where the mappings $\Omega$ are continuous functions of the time-varying scheduling parameter $\sigma(t) = [\Theta_{Treq}^{T} \; \bar{F} \; C_{NH3}]^{T}$ which is dependent on the measurable signal vector $\rho = [T \; \bar{\theta}_{NH3} \; F \; C_{NH3}]^{T}$. It is obvious that $A(\sigma(t)) = [-\sigma_1(t) \; 0 \; 0 \; 0]$, and $C(\sigma(t)) = [1 \; \sigma_2(t)]$. The mapping function of the observer gain $L(\sigma(t))$ is to be designed. Thus, the observer is a gain-scheduling observer.

Suppose that the bounds of the scheduling parameters are

$$
\sigma_1(t) \in [\sigma_1, \sigma_1^*], \quad \sigma_2(t) \in [\sigma_2, \sigma_2^*].
$$

Then, the matrix set $(A(\sigma(t)), \; C(\sigma(t))) \in \mathcal{M}$, where $\mathcal{M}$ is a given convex bounded polyhedral domain described by four vertices:

$$
\mathcal{M} = \left\{ \Omega(\sigma(t)) \Big| \Omega(\sigma(t)) = \sum_{i=1}^{4} \alpha_i(t)\Omega_i \right\},
$$

$$
\Lambda = \left\{ \alpha(t) \in \mathbb{R} : \sum_{i=1}^{4} \alpha_i(t) = 1, \alpha_i(t) \geq 0, i = 1, \ldots, 4 \right\},
$$

where $\Omega_i := (A_i, C_i), \; i = 1, \ldots, 4$, represent the vertices of the polytope.

For the estimator design, the primary objective is to minimize the estimation error. The signal to be estimated is chosen as the estimation error:

$$
z = e.
$$

Note that there are two exogenous signals $v$ and $w$ in the system (21). Suppose that $w$ is energy-bounded. Both the white noise and energy-bounded noise are simultaneously involved in the filtering error system in (21). Moreover, the norm-bounded uncertainty exists. In order to tackle with these two kinds of noises and the uncertainty, we employ the strategy of mixed $H_2/H_{\infty}$ filtering in the robust estimator design in this work. The objective of the robust gain-scheduling mixed $H_2/H_{\infty}$ observer design is to design a robust gain-scheduling observer in the form of (20) such that the following requirements are achieved:

1) The filtering error system in (21) with $v = 0$ and $w = 0$ is asymptotically stable.

2) For all admissible uncertainty, time-varying parameter variation, and exogenous noises, the filtering error $e$ satisfies a prescribed $H_{\infty}$ disturbance attenuation level $\gamma$ and a prescribed $H_2$ disturbance attenuation level $\beta$, that is,

$$
\max_{\alpha(t) \in \Lambda} \| T_{zw}(s, \alpha(t)) \|_{\infty} < \gamma,
$$

$$
\max_{\alpha(t) \in \Lambda} \| T_{zw}(s, \alpha(t)) \|_2^2 < \beta.
$$

Here, $\| T_{zw}(s, \alpha(t)) \|_{\infty}$ denotes the $\infty$-norm of the transfer function from $w$ to $z$ for a specific $\alpha(t)$ and $\| T_{zw}(s, \alpha(t)) \|_2$ denotes the 2-norm of the transfer function from $v$ to $z$ for a specific $\alpha(t)$.

III. GAIN-SCHEDULING OBSERVER DESIGN

In the above section, we have introduced the SCR after-treatment system and modeled the main chemical reactions. The problem has been formulated and the design objectives have been given. In this section, we will propose the design method of the gain-scheduling observer.

Since the norm-bounded uncertainty appears in the input matrix $D$, in order to deal with it, we introduce the following lemma.

**Lemma 1:** Let $\Omega = \Omega^T$, $\bar{M}$ and $\bar{H}$ be real matrices with compatible dimensions, and $\bar{N}(t)$ be time-varying and satisfy $|\bar{N}(t)| \leq 1$, then the following condition

$$
\Omega + \bar{M} \bar{N}(t) \bar{H} + \bar{H}^T \bar{N}(t) \bar{M}^T < 0,
$$

holds if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$
\begin{bmatrix}
\Omega & \varepsilon \bar{H}^T \\
* & -\varepsilon I
\end{bmatrix} < 0
$$

is satisfied.

Suppose that the gain of the estimator is known. The mixed $H_2/H_{\infty}$ performance of the filtering error system can be guaranteed in the following proposition.

**Proposition 1.** Given a positive scalar $\gamma$ and a positive scalar $\beta$, the filtering error system in (21) is asymptotically stable with

$$
\max_{\alpha(t) \in \Lambda} \| T_{zw}(s, \alpha(t)) \|_{\infty} < \gamma
$$

and

$$
\max_{\alpha(t) \in \Lambda} \| T_{zw}(s, \alpha(t)) \|_2^2 < \beta
$$

if there exists a matrix $P = P^T > 0$ satisfying

$$
\text{tr}(-L^T(\sigma(t))P(-L(\sigma(t)))) < \beta,
$$

and

$$
P \bar{A}(\sigma(t)) + \bar{A}^T(\sigma(t))P + D^T D + \gamma^{-2} PP < 0.
$$

**Proof:** The proof can be done by setting $P$ as a constant in [17].

**Theorem 1:** Consider the filtering error system in (21). Given positive scalars $\gamma$, $\beta$, and $\delta$, if there exist matrices $N = NT > 0$, $M$, and $W(\sigma(t)) = W^T(\sigma(t)) > 0$ such that the following conditions hold

$$
\text{tr}(W(\sigma(t))) < \beta,
$$

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\[ \mathcal{N}(\sigma(t)) = \begin{bmatrix} -N & -ML(\sigma(t)) & -W(\sigma(t)) \end{bmatrix} < 0, \quad (33) \]

\[ \mathcal{M}(\sigma(t)) = \begin{bmatrix} -M - M^T & \Xi_1 & 0 & I \\ * & -\gamma I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0, \quad (34) \]

where \( \Xi_1 = \tilde{A}^T(\sigma(t))M + N + \varrho M^T, \Xi_2 = -\varrho(N + N^T) \),

then, the filtering error system is asymptotically stable with \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_\infty < \gamma \) and \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_2 < \beta \).

**Proof:** The proof is similar with the one in [18].

In Theorem 1, a slack matrix \( M \) is introduced such that the Lyapunov weighting matrix \( P \) and the system matrix \( \tilde{A}(\sigma(t)) \) in the filtering error system has been decoupled. Note that \( M \) is not required to be positive definite. Thus, the conditions in Theorem 1 are less conservative. In addition, the matrix \( D \) contains a time-varying variable \( \tilde{N}(t) \). Therefore, the conditions in Theorem 1 cannot be directly applied to design the observer. It is noted that \( D \) has the following decomposition:

\[ D = \tilde{M}\tilde{N}(t)\tilde{H} \quad (35) \]

where \( \tilde{M} = \begin{bmatrix} 0 & M \end{bmatrix} \) and \( \tilde{H} = \begin{bmatrix} 0 & I \end{bmatrix} \). With the decomposition, we have the following theorem.

**Theorem 2:** Consider the filtering error system in (21). Given positive scalars \( \varrho, \gamma, \) and \( \beta \), if there exist matrices \( N = N^T > 0, M, \varepsilon > 0, \) and \( W(\sigma(t)) = W^T(\sigma(t)) > 0 \) such that (32), (33), and the following condition hold

\[ \mathcal{L}(\sigma(t)) = \begin{bmatrix} -M - M^T & \Xi_1 & 0 & I \\ * & -\gamma I & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0, \quad (36) \]

then, the filtering error system is asymptotically stable with \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_\infty < \gamma \) and \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_2 < \beta \).

**Proof:** The proof is omitted due to the space limits.

We are now in a position to propose the gain-scheduling observer design method. Suppose that the filter gain \( L(\sigma(t)) \) is linearly dependent on the scheduling variables and has the following structure:

\[ L(\sigma(t)) = \sum_{i=1}^{4} \alpha_i(t)L_i, \quad (37) \]

where \( L_i, \forall i = 1, \ldots, 4 \), is the matrix to be determined in the following theorem.

**Theorem 3:** Consider the filtering error system in (21). Given positive scalars \( \varrho, \varrho_1, \varrho_2, \varrho_3, \gamma, \) and \( \beta \), if there exist matrices \( N = N^T > 0, \varepsilon > 0, M, \mathcal{L}_i \) and \( W_i = W_i^T > 0 \) such that, \( \forall i, j = 1, \ldots, 4 \) and \( i \leq j \), the following conditions hold

\[ \text{tr}(W_i) < \beta, \quad (38) \]

\[ \mathcal{L}_{ij} + \mathcal{L}_{ji} < 0, \quad (40) \]

then, the filtering error system is asymptotically stable with \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_\infty < \gamma \) and \( \max_{\alpha(t) \in \Lambda}\|T_{zw}(s, \alpha(t))\|_2 < \beta \). In addition, the gains in (37) can be determined as

\[ L_i = M^{-1}\mathcal{L}_i. \quad (41) \]

**Proof:** The proof is omitted due to the space limits.

For a fixed \( \beta \), a smaller \( \mathcal{H}_\infty \) performance index \( \gamma \) means that the effect from the variation of the cross-sensitivity factor is also smaller. Therefore, it is meaningful to derive the minimum \( \gamma^* \).

**Corollary 1:** The minimum \( \mathcal{H}_\infty \) performance index \( \gamma \) for a given \( \mathcal{H}_2 \) performance index \( \beta \) in Theorem 3 can be found by solving the following convex optimization problem:

\[ \gamma^* = \arg \min \gamma \quad \text{subject to} \quad (38) \quad (39) \quad (40) \]

\[ \forall i = 1, \ldots, 4, j = i, \ldots, 4, \text{for a given } \varrho. \]

**IV. SIMULATION RESULTS AND COMPARISONS**

In this section, the simulations and comparisons of the proposed robust mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) observer design for removal of NO\textsubscript{x} sensor ammonia cross-sensitivity are carried out using an experimentally-validated full vehicle simulator eX-Emission developed by the Center for Automotive Research at the Ohio State University [19]. Based on the FTP75 test cycle, the exhaust flow rate, the exhaust temperature, and the concentration of the exhaust NO\textsubscript{x} can be seen in Fig. 3.

By applying the proposed method in Corollary 1, with the same parameters in [4], the derived estimator gains are

\[
\begin{bmatrix}
885.2 \\
978.8
\end{bmatrix}, \quad \begin{bmatrix}
876.0 \\
963.7
\end{bmatrix},
\]

\[
\begin{bmatrix}
-235.7 \\
1308.5
\end{bmatrix}, \quad \begin{bmatrix}
-92.2 \\
1462.4
\end{bmatrix}.
\]

It is necessary to mention the computation in our method includes the off-line calculation and the online calculation.
With the off-line calculation results in (42), the online calculation contains (37). The on-line computational time for each step in [4] is $4.7395e^{-4}$s. However, the on-line computational time of the proposed method in this work is $2.0611e^{-6}$s which is much smaller. Note that the computational time is an average value among 10000 random simulations in MATLAB 2007 with the same computer. Since the online computational load of the gain-scheduling observer in this paper is much smaller than the one in [4], the proposed method has a significant advantage in the real-time application.

Suppose that the cross-sensitivity is time-varying. With the designed gain-scheduling estimator, it follows from Fig. 4 that the observer can estimate the time-varying cross-sensitivity factor well. In addition, for a constant cross-sensitivity, the transient response of the designed observer is much faster than the extended Kalman filter. The comparison is omitted due to the space limit.

V. CONCLUSIONS

In this work, we have devoted to the NOx sensor ammonia cross-sensitivity problem in SCR systems. By assuming that the variation of the cross-sensitivity is bounded, a design approach of the robust mixed $H_2/H_{\infty}$ gain-scheduling observer was proposed. Simulations and comparisons have been done to show the efficacy and the advantages of the proposed approach over the existing extended Kalman filter in [4].

REFERENCES