Open-Loop Nash Equilibrium in Polynomial Differential Games via State-Dependent Riccati Equation

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Abstract—This paper studies finite- as well as infinite-time horizon nonzero-sum polynomial differential games. In both cases, we explore the so-called state-dependent Riccati equations to find a set of strategies that guarantee an open loop-Nash equilibrium for this particular class of nonlinear games. We demonstrate that this solution leads the game to an $\epsilon$- or quasi-equilibrium and provide an upper bound for this $\epsilon$ quantity. The proposed solution is given as a set of $N$ coupled polynomial Riccati-like state-dependent differential equations, where each equation includes a $p$-linear form tensor representation for its polynomial part. We provide an algorithm for finding the solution of the state-dependent algebraic equation in the infinite-time case based on a Hamiltonian approach. A numerical procedure is detailed to find the solution for this set of strategies. Numerical examples are presented to illustrate the effectiveness of the approach.

Keywords: Differential games, Nash equilibrium, Polynomial systems.

I. INTRODUCTION

Among many different fields such as engineering, ecology, management and economics, we find situations that involve several Decision-Makers (or Players) with different goals or objectives interlinked by the same decision process. Particularly, when the evolution of the underlying decision process evolves in time, this type of problems are often optimized using the Theory of Dynamic Games. This theory was initiated in the works of Isaacs [1]: he focused mainly on zero-sum games. Later on, the nonzero-sum differential games were introduced in [2] and [3]. In such games, each player looks for minimization of his own individual criterion. The paper [2] derived sufficient conditions of existence of a linear feedback equilibrium for a finite planning horizon, but only in the case of linear quadratic games governed by linear dynamics and quadratic criterion. (see [4] for a detailed survey). Usually, to resolve a conflict situation, the Nash-equilibrium (or, in general case, $\epsilon$-Nash equilibrium) is applied ([5], [6], [7]). It is recognized that Nash equilibrium is a natural solution in a noncooperative context. However, if we deal with complex nonlinear dynamics, it seems more appropriate to apply the concept of $\epsilon$-Nash equilibrium, since it allows one more flexibility in the selection of equilibrium strategies (see [6], [8]).

In many applications, the originally linear modeling cannot fit all the situations in practice, which are mostly nonlinear by nature. Therefore, we need to extend the equilibrium concepts to a certain class of nonlinear systems, namely, polynomial systems. Polynomial dynamics represents an important class of nonlinear dynamical systems, since it can approximate a large variety of intrinsically nonlinear functions, keeping the complexity on a manageable and pre-specified level. Compared to the linear-quadratic case, there are not so many works on nonlinear differential games and, particularly, to the best of authors knowledge, no results have been obtained for polynomial differential games. A few recent papers related to nonlinear differential games can be mentioned. The paper [9] presents a solution in a particular case of a nonlinear game representing a pollution and resource management problem. The paper [10] identifies the potentially chaotic behavior in a Markovian Nash equilibrium in a duopoly discrete-time model of advertising competition. The recent paper [11] proposes an iterative adaptive dynamic programming method to solve a particular type of games called two players zero-sum games. All these publications express the interest in finding equilibrium strategies in complex non-linear systems.

In this paper, we develop the State-Dependent Riccati Equations (SDRE) approach (see [12]–[15]) for a nonlinear polynomial game and derive a set of controls that leads to an open loop $\epsilon$-Nash equilibrium. For one player optimization problem (optimal control), the SDRE method has been proven to work well in many particular situations, providing a simple procedure for designing feedback controls ([12]–[15]); however, the general case solution is quasi-optimal, that is, the SDRE approach leads only to an approximate result. This is why the SDRE method provides an $\epsilon$-equilibrium for a game problem. Nevertheless, fast convergence of the obtained solution to the optimal one, a feedback form for the equilibrium controls, and numerical feasibility, make the SDRE approach a valuable method. Both finite- and infinite-time cases are studied and solved for nonlinear polynomial games. We also provide efficient numerical procedures to obtain the equilibrium strategies.

The rest of the paper is organized as follows. Section 2 presents the polynomial game description, problem statement, and basic assumptions. Section 3 defines the open-loop Nash equilibrium for polynomial games and presents the main result for the finite-time case. In Section 4, the infinite-time solution is obtained outlining a procedure of solving a SDRE in this case. Several numerical examples are presented in Section 5. Section 6 concludes this study.
II. PROBLEM STATEMENT

Consider the following polynomial differential game, where the players’ dynamics is governed by the differential equation:

\[ \dot{x}(t) = f(t,x) + \sum_{j=1}^{N} B^j(t) u^j(t) + d(t), x(t_0) = x_0, \] (1)

and a quadratic cost functional as the individual performance index for each player:

\[ L^i_T(u^i, \hat{u}^i) = \frac{1}{2} x^T(t) Q^i_x x(t) + \frac{1}{2} \int_{t_0}^{T} x^T(t) Q^i(t) x(t) + \sum_{j=1}^{N} u^{jT}(t) R^{ij}(t) u^j(t) \, dt, \] (2)

where \( x(t) \in \mathbb{R}^n \) is the state vector of the game, \( u^j \) is the control (action) of each \( j \)-player, which varies within a given region \( U^j \subset \mathbb{R}^{m^j}, \) \( j \) denotes the number of players \( (j = 1, N), \) \( B^j(t) \in \mathbb{R}^{n \times m^j} \) are the control matrices and \( d(t) \in \mathbb{R}^n \) is a continuous known exciting signal. The performance index \( L^i_T(u^i, \hat{u}^i) \) is given in the Bolza form, where \( u^i \) is the control of the \( i \)-player and \( \hat{u}^i \) are the controls for the rest of the players (\( i \) is the counter-coalition collection of players counteracting to the player with index \( i \)). For each player, the purpose of the game is to achieve the minimization of his own performance index by selecting appropriate inputs. We also assume that:

\[ Q^i(t) = Q^{iT}(t) \geq 0, \quad R^{ij}(t) = R^{ijT}(t) \geq 0, \quad R^{ii}(t) = R^{iiT}(t) > 0, \quad i \neq j. \] (3)

We consider the nonlinear function \( f(t,x) \) as a polynomial of \( n \) variables, components of the state vectors \( x(t) \in \mathbb{R}^n; \) this requires a special definition of the polynomial for degrees \( n > 1. \) Following the previous work (see [15]), a \( p \)-degree polynomial of a vector \( x(t) \in \mathbb{R}^n \) is regarded as a \( p \)-linear form of \( n \) components of \( x(t), \) that is to say:

\[ f(t,x) = a_0(t) + a_1(t)x_1 + a_2(t)x_2^2 + \cdots + a_x(t)x_1 \times \cdots \times \text{times} \times x. \] (4)

Here, the involved parameters are: \( a_0 \) is a vector of dimension \( n, \) \( a_1 \) is a matrix of dimension \( n \times n, \) \( a_2 \) is a 3D tensor of dimension \( n \times n \times n, \) and \( a_x \) is an \( (s+1) \)D tensor of dimension \( n \times \cdots \times (s+1) \) \( \text{times} \times n, \) and \( x \) is a \( p \)D tensor of dimension \( n \times \cdots \times n \), obtained by \( p \) times spatial multiplication of the vector \( x \) by itself. It is also possible to represent such a polynomial in a summation form:

\[ f_k(t,x) = a_0 k(t) + \sum_{i} a_1 k_i(t) x_i + \sum_{ij} a_2 k_{ij}(t) x_i x_j + \cdots + a_x k_{i_1 \cdots i_s}(t) x_{i_1} \cdots x_{i_s}, \] (5)

For given available information sets \( \eta_i(t) \) and a given set of strategies \( \gamma^i \in \Gamma^i (i \in \mathbb{N}), \) the control actions are completely determined by the relations \( u^i = \gamma^i (\eta_i). \) Substituting the set \( u^i \) into the cost functional (2) for a fixed final time \( T, \) leads to the number \( L^i_T(u^i, \hat{u}^i), \ (i \in \mathbb{N}), \) that is, the cost incurred by player \( i \) defined in the control action space [8]. For fixed initial state \( x_0, \) we get the mapping defined by

\[ J^i_T : \Gamma^1 \times \Gamma^2 \times \cdots \times \Gamma^N \rightarrow \mathbb{R}, \quad (\gamma^1, \gamma^2, \ldots, \gamma^N) \mapsto L^i_T(u^i, \hat{u}^i), \]

which is known as the cost functional of the player \( i \) for the game in strategic form.

A. Basic Assumptions and Definitions

A1 The control matrices \( B^j(t) \ (j = 1 \cdots N), \) the polynomial function \( f(t,x), \) and the exciting input \( d(t) \) are assumed to be known and integrable on \([0, T]\) for every participant, i.e., \( d(\cdot) \in L^1(0, T; \mathbb{R}^n). \)

A2 The class \( U^i \) of admissible control actions \( u^i(t) \) \((i = 1 \cdots N)\) contains all non-stationary controls satisfying the uniform Lipschitz condition in \( t, \) that is, for any \( t \geq 0, \)

\[ ||u^i(t) - u^i(t')|| \leq C ||t - t'||. \] (6)

Additionally, all admissible controls are assumed to be quadratically integrable in \( t \) (as well as the corresponding dynamics \( x(t) \)) over the time interval \([0, T]\) for any \( x, \) that is, for any \( x \in \mathbb{R}^n \)

\[ u^i(\cdot) \in L^2(0, T; \mathbb{R}^m) \] and \( x(t) \in L^2(0, T; \mathbb{R}^n). \] (7)

A3 All the players have access only to a open-loop information pattern, that is:

\[ \eta_i(t) = \{x_0\}, t \in [0, T]. \] (8)

In the control actions space, we introduce the next definition:

Definition 1 (\( \varepsilon \)-Nash equilibrium): The players’ control actions \( u^{i*}(\cdot) \ (i = 1 \cdots N) \) are said to be in \( \varepsilon \)-Nash equilibrium if for any other admissible control actions \( u^i(\cdot) \ (i = 1, \ldots, N) \) the following inequalities hold:

\[ L^{i*}_T := L^i_T(u^{i*}, \hat{u}^i) \leq \inf_{u^i(\cdot) \in U^i} L^i_T(u^i, \hat{u}^i) + \varepsilon^i_T, \ i = 1, \ldots, N, \] (9)

where \( \varepsilon^i_T \geq 0 \) is a numerical level characterizing how far the control \( u^{i*}(\cdot) \ (i = 1, \ldots, N) \) deviates from the pure Nash or zero equilibrium \( (\varepsilon^i_T = 0). \)

The problem of finding equilibrium controls in a closed form as in the linear case, which admits an exact equilibrium solution \( (\varepsilon^i_T = 0) \) for the game (1)-(2), may not be an easy task. Nonetheless, we propose a certain feedback form, which provides a non-zero equilibrium. Evidently, upon applying such a set controls, the cost functional \( L^i_T(u', \hat{u}^i) \), possessing the same structure as the functional (2) and coinciding with \( L^i_T(u^i, \hat{u}^i) \) if exact controls are applied, turns out to be dependent on this set of controls. Hence, both cost functionals in the left- and right-hand sides of the inequality:

\[ \hat{L}^i_T := \hat{L}^i_T(u^i, \hat{u}^i) \leq \inf_{u^i(\cdot) \in U^i} \hat{L}^i_T(u^i, \hat{u}^i) + \varepsilon^i_T \] (10)

also depend on “non-zero” controls as well as \( \varepsilon^i_T. \) The relation between \( L^i_T \) and \( \hat{L}^i_T \) is discussed in a remark given in the next section.
III. Open-Loop Nash Equilibrium in Polynomial Games

A. Finite Time Case

For the problem of finding a set of Nash equilibrium strategies in the game (1)-(2), and given the open-loop information structure (8), the following theorem holds in the finite-time case.

Theorem 2: Let the controls \( u^i, (i = 1, N) \) be such that there exists the solution consisting of the \( N \) co-state functions \( \psi^i \) to the following set of differential equations:

\[
\psi^i(t) = -\left( a_1 + \ldots + a_n x^i(t) \right) \psi^i(t) \quad (11)
\]

with terminal conditions \( \psi^i(T) = Q^i x(T) \), where \( H^i(x, \psi^i, u^i, u^f, t) \) is the individual Hamiltonian defined as:

\[
H^i(x, \psi^i, u^i, u^f, t) = \frac{1}{2} x^T(t) Q^i x(t) + \sum_{j=1}^{N} u^j(t) R^j u^j(t) + \psi^T i \left( a_1 x(t) + \ldots + a_n x^i(t) \right) + \sum_{j=1}^{N} B^j u^j(t) + d(t);
\]

the vector \( x^i \) satisfies:

\[
\dot{x}^i = f(t, x^i) + \sum_{j=1}^{N} B^i u^j(t) + d(t), \quad x^i(0) = x_0
\]

and \( i \)-th control satisfies the minimality condition:

\[
 u^i(t) = \arg \min_{u^i(t) \in U^i} H^i(x^i, \psi^i, u^i, u^f, t).
\]

Then, the set of controls \( u^i \) \( (i = 1, N) \) in the form (15) provides an open-loop Nash equilibrium solution.

\[
 u^i(t) = - \left( P^i \right)^{-1} B^i \psi^i(t). \quad (15)
\]

The proof of this theorem employs the same techniques as that of the theorem 6.11 in [8], and is omitted to avoid repetitions.

Remark 3: The solution given in Theorem 2 corresponds to zero equilibrium solution \( (\varepsilon = 0) \), that is, once the control actions (15) are applied to the game, the existence of the solution to the \( N \) point boundary problem yields the open-loop Nash equilibrium solution.

In the linear-quadratic case, the open-loop Nash equilibrium solution, can be represented (see [41]) in the state feedback form. In what follows, we show that it is actually possible to provide the feedback representation for the solution given by (11)-(15) to a polynomial differential game as well, but the resulting control actions would lead to a near- or \( \varepsilon \)-Nash solution. In order to estimate how far this set of \( \varepsilon \)-controls deviates from the zero equilibrium define the following processes:

\( x(t) \) is the players’ dynamics when all of the players use the exact Nash equilibrium solution (15). This trajectory is further used to evaluate the cost \( L^i(t, u^i, u^f) \).

\( \tilde{x}(t) \) is the players’ dynamics when the players apply the feedback controls; in its turn, this trajectory contributes to \( \tilde{L}^i(t, u^i, u^f) \).

\( \check{x}(t) \) is the players’ dynamics when each player \( j \) uses the control (15), except player \( i \), \( j \neq i \), and \( u^i \) is an admissible control satisfying (6). The corresponding cost is \( \check{L}^i(t, u^i, u^f) \).

Since the closed-loop system is regulated with Nash controls (15) or linear-affine feedback controls, the trajectories \( x(t) \) and \( \check{x}(t) \) should be quadratically bounded, that is, there exist \( \beta_1, \beta_2 \) such that

\[
\int_0^T \|x\|^2 dt \leq \beta_2 < \infty; \quad \int_0^T \|\check{x}\|^2 dt \leq \beta_1 < \infty.
\]

The final states of this trajectories are also quadratically bounded, that is:

\[
\|x(T)\|^2 \leq \gamma_2; \quad \|\check{x}(T)\|^2 \leq \gamma_1.
\]

Due to the smoothness of the admissible controls (6)-(7), the trajectories \( \check{x}(t) \) and \( \tilde{x}(t) \) are also quadratically bounded, that is, there exist \( \beta_3, \beta_4 \) such that

\[
\int_0^T \|\check{x}\|^2 dt \leq \beta_3 < \infty; \quad \int_0^T \|\tilde{x}\|^2 dt \leq \beta_4 < \infty;
\]

as well as their final states:

\[
\|\check{x}(T)\|^2 \leq \gamma_3; \quad \|\tilde{x}(T)\|^2 \leq \gamma_4.
\]

B. Open-Loop \( \varepsilon \)-Nash Equilibrium via State Dependent Riccati Equations

The next theorem uses the SDRE technique to represent the open-loop solution in a feedback form.

Theorem 4: For the \( N \)-person finite-time polynomial differential game (1)-(2) satisfying the restriction (3), if there exist the solution to the coupled state-dependent Riccati equation

\[
\dot{P}^i(t) + Q^i(t) \sum_{j=1}^{N} B^i(R^i)^{-1} B^j P^j(t) = 0;
\]

\[
P^i(T) = Q^i,
\]
and the shifting vector
\[ \dot{p}^i(t) = \left( a_1 + 2a_2x(t) + \ldots + na_nx(t) \ast \ast \ast x(t) \right) p^i(t) \]
(21)

- \[ \sum_{j=1}^{N} B^j(R^{ij})^{-1} B^j T p^i(t) + P^i(t) d(t) = 0, \]
\[ p^i(T) = 0. \]

then the control actions:
\[ u^i(t) = -R_{ii}^{-1} B^i T P^i(t)x(t). \]
(28)

Remark 5: The condition to obtain the set of controls (22) is that the individual co-state vector satisfies the equation \[ \psi^i(t) = P^i(t)x + p^i(t). \] However, in the matrix case, as pointed out in [12] for one-player optimization problem, this approach provides only a near-optimal solution, because the condition (11) can be satisfied for an affine form of \( \psi^i(t) \) only asymptotically.

The theorem gives an upper bound for the \( \epsilon^T \), which is useful to determine how the obtained \( \epsilon \)-equilibrium is far from the zero equilibrium solution. The following remark is valid in the presented case ([7]). It is easy to notice that the definition (10) can be rewritten in the following equivalent form:
\[ L^*_T(u^i, u^{\ast}) \leq \inf_{u(\cdot) \in U^i} L^*_T(u^i, u^{\ast}) + \epsilon^T, \]
\[ L^*_T(u^i, u^{\ast}) + \left( \inf_{u(\cdot) \in U^i} L^*_T(u^i, u^{\ast}) - L^*_T(u^i, \tilde{u}^{\ast}) + \tilde{\epsilon}^T \right) \leq L^*_T(u^i, \tilde{u}^{\ast}) + \tilde{\epsilon}^T, \]
where
\[ \tilde{\epsilon}^T := \epsilon^T + \inf_{u(\cdot) \in U^i} L^*_T(u^i, \tilde{u}^{\ast}) - L^*_T(u^i, \tilde{u}^{\ast}), \]
which gives the comparisons of the cost functionals corresponding to the \( \epsilon \)-Nash equilibrium strategies to the same cost functionals corresponding to the exact Nash controls. The modified \( \tilde{\epsilon}^T \) can be estimated by calculations similar to those for \( \epsilon^T \). Consider the dynamics of the game (1) with initial condition \( x(t_0) = x_0 \), the set of SDRE (20) with terminal conditions \( P^i(T) = Q^i \), and the linear equations (21) with boundary conditions \( p^i(T) = 0 \). A shooting method [16] can be applied to solve this multipoint boundary problem.

IV. INFINITE TIME CASE

In this section, we consider the infinite-time-horizon polynomial game (1)-(2) that is, the criterion that each \( i \)-player is pursuing to minimize is given by:
\[ L^*_\infty(u_1, u_2) = \frac{1}{2} \int_0^\infty \left( x^T Q x + \sum_{j=1}^{N} u^T_j R_j u_j \right) dt, \]
subject to the polynomial dynamics (1), with \( d(t) \equiv 0 \). Thus, a quadratic cost function with \( Q^i = 0 \) is studied. In this case, the system of algebraic state-dependent Riccati equations take the form
\[ P^i \left( a_1 + a_2x(t) + \ldots + a_nx(t) \ast \ast \ast x(t) \right) \]
\[ + (a_1 + 2a_2x(t) + \ldots + na_nx(t) \ast \ast \ast x(t)) \]
\[ + Q^i - P^i \sum_{j=1}^{N} B^j(R^{ij})^{-1} B^j T P^i = 0, \quad i = 1, \ldots, N. \]
(27)

Corollary 6: For the infinite-time-horizon polynomial nonlinear game with the assumptions (3), non-exogenous input, and quadratic criterion (26), if the algebraic state-dependent Riccati equations (27) have positive definite solutions \( P^i(x) \) \((i = 1, 2)\), then the open-loop \( \epsilon \)-Nash equilibrium controls are given by:
\[ u^i(t) = -(R^i)^{-1} B^i T P^i(t)x(t). \]
Remark 7: The main difference of the standard Nash Riccati equation from (27) is that the solutions of (27) depend on the vector \( x(t) \). In the one-player optimization case, it is conventionally considered that we can find a solution at each "time instant", when the state-dependent coefficients of (27) are considered constants, and then apply standard methods for solving an algebraic Riccati equation.

A. Solving the Coupled State-Dependent Algebraic Riccati Equation by Hamiltonian Approach

The solutions to the system of equations (27) are closely related (see [17]) to the Hamiltonian matrix given by (for simplicity, we consider \( N = 2 \)):

\[
H = \begin{pmatrix}
    a_1 + \ldots + a_n x^{n-1} & -S_1 \\
    -S_1 & 0 \\
    -(a_1 + \ldots + a_n x^{n-1})^T & -S_2 \\
    0 & -(a_1 + \ldots + a_n x^{n-1})^T
\end{pmatrix}
\]

where \( S_i = B_i^T (R_i)^{-1} B_i^T \), \( i = 1, 2 \). If the vector \( x \) is fixed, the matrix \( H \) coincides with that in the linear-quadratic case. Indeed, it is known ([4] p. 280) that for a fixed vector \( x \) if \( V \subset \mathbb{R}^{3n \times n} \) is an \( n \)-dimensional invariant subspace of \( H \), then \( W_i \in \mathbb{R}^{n \times n} \), \( i = 1, 2 \), are three real matrices such that:

\[
V = \text{Im} \begin{bmatrix}
    W_0 \\
    W_1 \\
    W_2
\end{bmatrix},
\]

and if \( W_0 \) is invertible, then \( P_i := W_i W_0^{-1} \), \( i = 1, 2 \), is a solution to the system of coupled Riccati equations (27) (for \( N = 2 \)). Moreover, the solutions \( (P_1, P_2) \), are independent of the specific choice of a basis \( V \), which gives a method to find the Nash controls. It is also known that in the standard case (no dependence on \( x \)), the algebraic coupled Riccati equations have stabilizing solutions \( (P_1, P_2) \) if and only if the matrix \( H \) has an \( n \)-dimensional stable graph subspace and \( H \) has \( 2n \) eigenvalues. We extend this approach for the polynomial case as follows. As noted, in view of the polynomial structure of the game, it is possible to find the \( \varepsilon \)-Nash controls depending on the state \( x \) or, equivalently, calculate eigenvalues and eigenvectors of \( H \), which depends on \( x \). For this purpose, we propose the following algorithm:

Algorithm 8: 1.- Assume that \( x \) is fixed as a constant value.
2.- Calculate the eigenvalues of \( H \), which turn out to be \( x \)-dependent.
3.- For the particular choice of eigenvalues, calculate the eigenvectors (also \( x \)-dependent).
4.- Calculate the eigenspace of dimension \( (3 \times n) \times n \), generated by the linearly independent eigenvectors, which is given by:

\[
W = [ W_0 \ \ W_1 \ \ W_2 ]^T.
\]

5.- The solutions of the algebraic state-dependent Riccati equations (27) are given by:

\[
P_i = W_i W_0^{-1}.
\]

V. NUMERICAL EXAMPLES

A. Example 1

Consider the scalar two players nonlinear polynomial game:

\[
\dot{x} = x^3 + u_1 + u_2,
\]

with the following finite-time cost functions for each player:

\[
L_{10}^1(u_1, u_2) = \frac{1}{2} \int_0^{10} (x^2 + u_1^2 + u_2^2) dt + \frac{1}{2} x^2(10),
\]

\[
L_{10}^2(u_1, u_2) = \frac{1}{2} \int_0^{10} (0.1 x^2 + u_1^2 + u_2^2) dt.
\]

The coupled Riccati equations take the form:

\[
P^1 = -(3x P^1 - \sum_{j=1}^2 P^1 P^j + 1),
\]

\[
P^2 = -(3x P^2 - \sum_{j=1}^2 P^1 P^j + 0.1),
\]

therefore, the \( \varepsilon \)-Nash controls for both players are given by:

\[
u^i = -P^i x; \quad i = 1, 2.
\]

B. Example 2

For the following two players multi-state nonlinear game:

\[
\begin{align*}
    \dot{x}_1 &= x_2; \\
    \dot{x}_2 &= x_1^3 + u_1 + u_2,
\end{align*}
\]

Fig. 1. State of the Game Example 1.

Fig. 2. \( \varepsilon \)-Nash Control. Player 1.

Fig. 3. \( \varepsilon \)-Nash Control. Player 2.
each player has the same cost function:

$$J_i(u_1, u_2) = \frac{1}{2} \int_0^\infty (x_1^2 + x_2^2 + u_1^2 + u_2^2) \, dt.$$ 

The Hamiltonian matrix for this game is:

$$H = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-x_1 & 0 & 0 & -1 & 0 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -2x_1 \\
0 & -1 & 0 & 0 & -1 & 0
\end{pmatrix}$$

the solutions for the state-dependent Riccati equations are:

$$P_{12} = \frac{(2 + x_1 + \sqrt{x_1^2 + 12x_1 - 4})^2 + 3x_1 - \sqrt{x_1^2 + 12x_1 - 4}}{4(2 + 3x_1 - \sqrt{x_1^2 + 12x_1 - 4} - \sqrt{2 + 3x_1 + \sqrt{x_1^2 + 12x_1 - 4}})}$$

$$P_{21} = \frac{(2 + x_1 - \sqrt{x_1^2 + 12x_1 - 4})^2 + 3x_1 + \sqrt{x_1^2 + 12x_1 - 4}}{4(2 + 3x_1 + \sqrt{x_1^2 + 12x_1 - 4} - \sqrt{2 + 3x_1 - \sqrt{x_1^2 + 12x_1 - 4}})}$$

and the $\varepsilon$-Nash controls are

$$u_i = -P_{12}x_1 - P_{21}x_2.$$ 

VI. CONCLUSIONS

This paper studied finite -as well as infinite- time horizon nonzero-sum polynomial differential games. In both cases, we demonstrated that the so-called state-dependent Riccati equations provide a valuable technique to find a set of strategies that guarantee an open loop-Nash $\varepsilon$ -equilibrium for this particular class of nonlinear games. We designed an algorithm for finding the solution of the state-dependent algebraic equation in the infinite-time case, based on a Hamiltonian approach. Numerical examples were presented to illustrate the effectiveness of the approach.

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