Output regulation for attitude control: a global approach

Gerd S. Schmidt, Christian Ebenbauer and Frank Allgöwer

Abstract—This paper considers a certain class of output regulation problems for the rigid body equations in natural representation and gives an explicit solution with strong convergence properties.

I. INTRODUCTION

The output regulation problem is one of the classical problems in control theory [1]. The solution of the output regulation problem for linear systems is well studied, see [2]. For nonlinear systems, different output regulation problems were considered and solved, see [3]–[5]. In general, the output regulation problem deals with asymptotic tracking of a reference input or asymptotic rejection of a disturbance under the assumption that a model for the exogenous inputs is given by the solution set of a known differential equation [5]. If the full state information and the exogenous input are not available from measurement, an observer is used to reconstruct the unmeasured quantities. For nonlinear output regulation problems this directly leads to the question whether a separate design of an observer and a state feedback leads to a stable closed loop. For general nonlinear problems such questions are difficult to answer, especially if one asks for strong and possibly global convergence results.

The attitude control of rigid body systems is a nonlinear control problem with a long history, see e.g. [6]. The configuration manifold of the rigid body attitude is SO(3) = {Θ ∈ ℝ3×3 | ΘTΘ = I, det(Θ) = 1}. For practical reasons, other representations of the attitude of a rigid body are used frequently, e.g. quaternions or Euler angles [7]. For results on attitude control in the quaternion representation see e.g. [8], [9], for output regulation formulations of the attitude control problem in the quaternion representation see [10]. However, alternative representations of the attitude are not homeomorphic to SO(3), hence attitude control systems based on alternative representations face several problems, for details see e.g. [7], [11]. Questions of global convergence for attitude control in the natural representation are nontrivial due to the geometry of SO(3). The interest in the problem of global convergence with separate feedback and observer design for the attitude control problem has been there for many years. More specifically, as stated in [12]: “[I]t is hoped that a mechanical energy function approach for rigid body control […] can be combined with the observer presented in this note to lead to a globally stable, nonlinear, observer-based, rigid body controller in which the observer and controller errors can be separated, in much the same way as one can separate controller and observer poles in the output feedback controllers of linear system theory”.

In this paper, we consider a class of attitude control problems in the natural SO(3) representation from an output regulation perspective. We present a novel solution to the output regulation problem which also addresses the separation problem in the context of attitude control. The proposed internal model based controller consists of an observer and a state feedback such that the closed loop vector field is smooth. We give a detailed description of the global convergence behavior and show that the controller achieves convergence for almost all initial conditions. To the best of our knowledge no methodology with separate observer/controller design for the natural representation of the attitude dynamics with similar strong convergence properties was presented so far. The presented results differ from semiglobal approaches, which guarantee the existence of design parameters such that the desired equilibrium is semiglobally asymptotically stable. Furthermore, our approach does not require exponential convergence rates for the observer error, as required e.g. in [13]. The additional contributions with respect to [14] are as follows: we show that the method proposed in [14] can be successfully extended to the non-compact state space which is the case for the rigid body equations due to the dynamics equations. We explain in detail the additional necessary steps in the feedback design and show that the observer design proposed in [14] can be extended to the rigid body case.

The remainder is organized as follows: a detailed problem statement is given in Section II. In Section III we present our main results. We conclude the work in Section IV. Relevant literature is discussed at appropriate places.

II. PROBLEM STATEMENT

In the general formulation of the nonlinear regulation problem as discussed in [4], [5], a control system given by

\[
\dot{x} = F(w, x, u) \\
e = H(w, x)
\]

(1)

is considered, where \(x \in \mathbb{R}^n\) is the state, \(e \in \mathbb{R}^q\) is the regulated output, \(u \in \mathbb{R}^p\) is the controlled input and \(w \in \mathbb{R}^r\) is the exogenous input of the system. The exogenous input includes disturbances, unknown references and unknown parameters. An assumption of the nonlinear output regulation is that the exogenous inputs \(w\) are solutions of a (neutraly stable) differential equation

\[
\dot{w} = s(w),
\]

(2)

where the initial conditions are from a given set \(\mathcal{W}\). The knowledge that \(w: \mathbb{R} \to \mathbb{R}^r\) is the solution of a system (2) is not uncommon for design problems and is a special feature of output regulation [5]. This assumption is a trade-off between the advantageous but unrealistic situation that \(w\) is available as real-time measurement and the situation that nothing is known about \(w\). The problem of nonlinear output regulation is to design a controller of the form

\[
\dot{x}_c = F_e(x_c, e) \\
\dot{u} = H_e(x_c, e),
\]

(3)

where \(x_c \in \mathbb{R}^v\), such that all solutions of the closed loop system (1), (2) and (3) are bounded and such that \(e(t) \to 0\) as \(t \to 0\).

In this paper, we consider the rigid-body attitude control problem in an output regulation framework. The equations of motion for the rotational motion of the rigid body in a space fixed coordinate frame are given by

\[
\frac{d}{dt}(J\omega) = m \quad \text{and} \quad \frac{d}{dt}\Theta = Q(\omega)\Theta,
\]

(4)
where \( \omega \in \mathbb{R}^3 \) denotes the angular velocity, \( \Theta \in SO(3) \) denotes the attitude of the rigid body, \( m \) denotes external forces, see e.g. [15], [16]. The inertia matrix \( J \) is given by the inertia matrix in a body fixed coordinate frame \( J_0 = \text{diag}(J_1, J_2, J_3) \) by \( J(t) = \Theta(t) J_0 \Theta(t)^T \). The tangent space \( T_{\Theta}(SO(3)) \) at \( \Theta \) of \( SO(3) \) is given by \( \{ \Omega \Theta | \Omega = -\Omega^T \} \). \( Q : \mathbb{R}^3 \to T_{\Theta}(SO(3)) \) defined by
\[
Q(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0 \end{bmatrix}
\] is an isomorphism and we denote the inverse of \( Q \) by \( Q^{-1} \). For \( x, y \in \mathbb{R}^3 \), \( Q \) fulfills the identity \( x^T y = -\frac{1}{2} \text{tr}(Q(x)Q(y)) \).

Here, we consider a problem setup given by
\[
\dot{w} = Sw \\
\dot{z}_1 = m + Gw \\
\dot{\Theta}_1 = Q(J_1^{-1} z_1) \Theta_1 \\
\dot{z}_2 = u + p(Gw) \\
\dot{\Theta}_2 = (Q(J_2^{-1} z_2) + Q(q(z_2, \Theta_2))) \Theta_2 \\
E = \Theta_2 \Theta_1^{-1}
\] where \( w \in \mathbb{R}^n \), \( z_k \in \mathbb{R}^3 \), \( \Theta_k \in SO(3) \), \( G \in \mathbb{R}^{3 \times 1} \), \( S \in \mathbb{R}^{3 \times 3} \), \( p : \mathbb{R}^3 \to \mathbb{R}^1 \), \( q : \mathbb{R}^3 \times SO(3) \to \mathbb{R}^3 \) and \( \dot{z}_k = J_k \dot{\theta}_k \) for \( k \in \{ 1, 2 \} \). In (6), the state of the control system is given by \( (z_1, \Theta_1) \), the state of the exosystem is given by \( (w, z_1, \Theta_1) \). We assume that the reference system is affected by an unknown disturbance modeled through the linear exosystem \( \dot{w} = Sw \). Furthermore, we assume that the attitudes of the rigid bodies \( \Theta_1, \Theta_2 \) as well as \( z_2 \) are available from measurements. \( E \) is the regulated output. Therefore, \( E \) represents the error between the desired attitude \( \Theta_1 \) and the controlled attitude \( \Theta_2 \).

The goal is to find a controller of the form
\[
\dot{x}_c = F_c(x_c, \Theta_1, z_1, \Theta_2, E) \\
u = h_c(x_c, \Theta_1, z_1, \Theta_2, E)
\] with \( F_c, h_c \) smooth, that achieves
\[
E(t) = \Theta_2(t) \Theta_1^{-1}(t) \to I
\] for \( t \to \infty \) such that the states of the closed loop system stay bounded. An application scenario for a system class which includes (6) as special case will be presented in [17].

The resulting control problem includes a stabilization of an equilibrium point on \( SO(3) \). Due to geometrical properties of \( SO(3) \) and the smoothness requirement on the vector field, the closed loop system (6) and (7) is going to have several equilibrium points, for more details see Section III-B. Consequently, the best possible solution to the underlying stabilization problem is an almost global solution in the sense that all initial conditions converge to the desired equilibrium except the ones lying in stable manifolds of additional (undesired) unstable equilibria. Hence, we ask for a quite strong convergence result in presence of several equilibria, i.e. convergence of the error for almost all initial conditions. More specifically, this means convergence to the desired equilibrium point in the sense that (i) the \( \omega \)-limit set of every solution of the closed loop is contained in the set of equilibria and (ii) the desired equilibrium point \( E = I \) is asymptotically stable and (iii) all other equilibrium points are unstable. From a “practical” point of view, i.e. due to small perturbations, this means that we converge “always” to the desired equilibrium point. Summarizing, the considered regulation problem then is the following:

\begin{itemize}
\item[(RP)] Find a controller of the form (7) such that \( E(t) \to I \) as \( t \to \infty \) for almost all initial conditions.
\item[(a)] The set \( \Delta = \{ (\Theta_1, \Theta_2) \in SO(n) \times SO(n) | \Theta_2 \Theta_1^{-1} = I \} \) is positively invariant for the closed loop system (6), (7).
\item[(b)] All solutions of the closed loop system are bounded.
\end{itemize}

For convenience we denote the solution of a differential equation \( \dot{x} = f(x) \) by \( x(t) \) without mentioning the corresponding initial condition \( x(0) \). Furthermore, we use the notation \( \ast \) shorthand for \( \frac{d}{dt}(\cdot) \).

### III. Main Results

In the following, we solve the regulation problem (RP) successively. I.e we show that the feedback needs to fulfill a certain invariance condition (“internal model”) in Section III-A. Second, we solve the control problem under the assumption that the full state measurements are available in Section III-B. Third, we design an observer which reconstructs the full state information from the attitude measurements in Section III-C. Finally, in Section III-D we prove that the combination of the separate solutions leads to convergence for almost all initial conditions which solves (RP).

#### A. Invariance of \( \Delta \)

**Lemma 1**: Assume a controller of the form (36) solves the regulation problem as stated in (RP). Then the feedback \( u \) has the form
\[
u = -p(Gw) + (J_2^{-1} z_2 + \frac{\partial q}{\partial z_2} \dot{z}_2) \Theta_2 \Theta_1^{-1} \Theta_1 \Theta_2, m)
\]
\[
(\phi(1) \dot{z}_2, \Theta_1, \Theta_2, m) = (J_1^{-1} z_1 - J_1^{-1} z_2 - \frac{\partial q}{\partial z_2} \dot{z}_2) \Theta_1 \Theta_2, m)
\]
\[
E(t) \equiv 0 \text{ and } E(t) \equiv 0. \text{ With}
\]
\[
E = \Theta_2 \Theta_1^{-1} - \Theta_2 \Theta_1^{-1} \Theta_2 \Theta_1^{-1}
\]
\[
E = (Q(J_2^{-1} z_2) + Q(q(z_2, \Theta_2)))E - EQ(J_1^{-1} z_1)
\]
\[
E = Q(J_2^{-1} z_2 + Q(q(z_2, \Theta_2)) \dot{z}_2 - J_1^{-1} z_1)
\]
\[
E = Q(J_2^{-1} z_2 + Q(q(z_2, \Theta_2)) \dot{z}_2 - J_1^{-1} z_1)
\]
\[
E = Q(J_2^{-1} z_2 + Q(q(z_2, \Theta_2)) (u + p(Gw))
\]
\[
E = (Q(J_2^{-1} z_2) + Q(q(z_2, \Theta_2))) - EQ(J_1^{-1} z_1)E^T E.
\]

#### B. Solution of (RP) with full state measurements

Next we assume that we have full state measurements, i.e. that \( Gw, z_1 \) are available for the controller design. Furthermore we assume that \( m \) is available from measurements. To solve (RP), we need to find a feedback \( u \) depending on \( (Gw, z_1, \Theta_1, \Theta_2, m) \) which almost globally stabilizes \( I \) for \( E = \Theta_2 \Theta_1^{-1} \). The dynamics for \( E = I \) are given by
\[
\dot{E} = (Q(J_2^{-1} z_2) + Q(q(z_2, \Theta_2))) - EQ(J_1^{-1} z_1)E^T E.
\]

As mentioned before, we are looking for smooth feedback laws which imply several equilibria for the closed loop system. More
specifically, the Poincaré-Hopf Theorem states that a smooth vector field $v$ on a compact, oriented manifold $M$ with only finitely many equilibrium points has the property that the sum of the indices of $v$ equals the Euler characteristic of $M$ [18]. A compact, odd-dimensional, oriented manifold $M$ (e.g., $SO(3)$) has Euler characteristic zero [18, p.116]. An asymptotically stable equilibrium has an index of one, hence a globally asymptotically stable equilibrium is impossible.

We use a backstepping-like technique (see e.g. [19]) to stabilize $I$ for the $E$-dynamics utilizing $u$ in $z_2 = u + p(Gw)$. To explain the utilized controller structure, we use a coordinate transformation for the vector field (6). More specifically, let $T$ be a map of $\mathbb{R}^3 \times SO(3) \times \mathbb{R} \times \mathbb{R} \times SO(3)$ on itself defined by $T(w, z_1, \Theta_1, z_2, \Theta_2) = (w, z_1, \Theta_1, \zeta, \Theta_2)$, where $\zeta$ is defined by

$$
Q(\zeta) = Q\left(J^1z_2 + Q(q(z_2, \Theta_2)) - E Q\left(J^-1z_1\right) E^T + E - E^T\right).
$$

(14)

We assume for the following that $q$ is such that $T$ is a diffeomorphism. Then, the derivative of $\zeta$ is then given by

$$
\dot{Q}(\zeta) = U(w, z_1, \Theta_1, z_2, \Theta_2, m) :=
= Q\left(J^-1u + Q\left(J^-1p(Gw)\right) + Q\left(\frac{\partial q}{\partial z_2} u \right) + Q\left(\frac{\partial q}{\partial \Theta_2} \theta \right) \right)
+ Q\left(\frac{\partial \theta}{\partial \Theta_1} \theta \right) - EQ\left(J^-1z_1\right) E^T + E - E^T.
$$

(15)

The system (6) in $(w, z_1, \Theta_1, \zeta, E)$-coordinates is thus given by the first three equations of (6) and (13) and (15). From the definition of $\zeta$ from (14) we get: if $Q(\zeta) = 0$ then $E = (E^T - E^T)E$, which can be shown to be almost globally asymptotically stable, see e.g. [14]. Thus we have to choose $u$ such that $Q(\zeta)$ converges to zero. More specifically, we choose $u$ as

$$
u(z_1, z_2, Gw, \Theta_1, \Theta_2, m) = -p(Gw) + \left(J^-1u + \frac{\partial q}{\partial z_2} u \right) - Q\left(J^-1z_2 - q(z_2, \Theta_2)\right)
+ Q\left(\frac{\partial \theta}{\partial \Theta_1} \theta \right) - EQ\left(J^-1z_1\right) E^T + E - E^T - g(z_1, z_2, \Theta_1, \Theta_2, m)\right] .
$$

(16)

Then we get

$$
Q\left(\zeta\right) = E^T - E - Q(\zeta)
E = Q(\zeta) + E^T - E)E.
$$

(17)

The convergence behavior of (17) is established in:

**Lemma 2:** The solutions $(\zeta(t), E(t))$ of (17) converge asymptotically for almost all initial conditions to $(0, I)$.

**Proof.** The closed loop dynamics are given by (17). Consider $V_1 : \mathbb{R}^3 \times SO(3) \to \mathbb{R}$ defined by

$$
V_1(\zeta, E) := \frac{1}{2} \zeta^T \zeta + 3 - \text{tr}(E).
$$

(18)

$V_1$ is non-negative and $0$ at $(0, I)$ and radially unbounded in $\zeta$. The derivative $V_1$ of $V_1$ along the vector field (17) is

$$
V_1 = -\text{tr}(E) + \zeta^T u = -\text{tr}\left(\left(Q(\zeta) + E^T - E\right)E\right) + \zeta^T v
= -\frac{1}{2} \text{tr}\left(\left(E^T - E\right)\left(E - E^T\right)\right) + \frac{1}{2} \text{tr}(Q(\zeta) Q(\zeta))
= \frac{1}{2} \text{tr}\left(\left(E - E^T\right)^2\right) - \zeta^T \zeta.
$$

(19)

I.e. $V_1$ is negative unless $E$ is symmetric and $\zeta = 0$, which implies the convergence of $E$ to the set $\{\Theta \in SO(3) | E = E^T\} \times \{0\}$. According to Corollary 11 (Appendix), the Cartesian product of $[0]$ and the set of critical points of $E \mapsto -\text{tr}(E)$ coincides with the set where $V_1$ vanishes. Since (a) $V_1$ is negative semidefinite, (b) $(E, \zeta) = (0, I)$ is the unique isolated minimum of $E \mapsto -\text{tr}(E) + \zeta^T \zeta$ and (c) all critical points are saddle points or maxima of the function, we know that all equilibrium points of (17) except $(0, I)$ are unstable. Since $(0, I)$ is asymptotically stable, the solutions of (17) asymptotically converge to $(0, I)$ for almost all initial conditions.

Consequently, we have a solution to (RP) if full state measurements are available as summarized in the following:

**Lemma 3:** Assume that $m, Gw, z_1$ are known and that $q$ is such that the map $T$ defined by $T(w, z_1, \Theta_1, z_2, \Theta_2) = (w, z_1, \Theta_1, \zeta, \Theta_2)$, where $\zeta$ is defined by (14), is a diffeomorphism of the set $\mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times SO(3)$ on itself. Furthermore assume that $J^1z_1 + \frac{\partial q}{\partial z_2}$ is everywhere non-singular, then the state feedback law $u$ defined by (16) solves the output regulation problem (RP).

**Proof.** The proof summarizes the previous discussion. If $q$ fulfills the given conditions and if we choose $u$ according to (16), then the closed loop system in $(w, z_1, \Theta_1, \zeta, E)$-coordinates is given by

$$
\dot{w} = Sw
\dot{z}_1 = m + Gw
\dot{\Theta}_1 = Q\left(J^-1z_1\right) \Theta_1
\dot{\zeta} = -Q^-1(\dot{E} - E^T) - \zeta
\dot{E} = Q(\zeta) + E^T - E)E.
$$

(20)

Lemma 2 shows that almost all initial conditions of the $(E, \zeta)$ dynamics converge to $(I, 0)$ for $t \to \infty$.

**Remark 4:** If $q$ is only a function of $\Theta_2$ and not of $z_2$, i.e. $q = q(\Theta_2)$, the map $T$ as defined in Lemma 3 is a diffeomorphism, as one can verify by a direct calculation.

**C. Observer design**

In this section we discuss the first step to drop the assumption that full state measurement of the states of the exosystem in the control system (6) is available. More specifically, we use an observer which reconstructs $(Gw, z_1)$ from $\Theta_1$-measurements. Therefore, we consider here the system given by

$$
\dot{w} = Sw
\dot{z}_1 = -Gz + Gw + m
\dot{\Theta}_1 = Q\left(J^-1z_1\right) \Theta_1
$$

(21)

where $w \in \mathbb{R}^4$, $z \in \mathbb{R}^3$, $\Theta \in SO(3)$, $J = \Theta_1 \Theta_1^T$ and $Q : \mathbb{R}^3 \to T_I(\mathbb{R}^3)$ is defined by (5). This system corresponds to the exosystem. Since we assume that the external forces $m$ are available as real-time measurement, we drop $m$ in the following considerations from the vector fields. To allow a dissipative term for the $z$-vector field we have included $-Gz$ with $0 \leq \Gamma \in \mathbb{R}^{3 \times 3}$. The goal is to design an observer that allows to recover $(Gw, z)$ of the system (21) from the output $\Theta$. We use a Luemberger-type observer structure, i.e.

$$
\dot{w} = Sw + l_1(E_\Theta)
\dot{z}_1 = -Gz + Gw + l_2(E_\Theta)
\dot{\Theta}_1 = Q\left(J^-1z_1\right) + L(E_\Theta) \dot{\Theta}_1
$$

(22)

where $l_1, l_2 : SO(3) \to \mathbb{R}^3$ and $L : SO(3) \to T_I SO(3)$ are observer gains, $e_w := w - w$, $e_z := z - z$ and $E_\Theta := \Theta \Theta^{-1}$ are the errors. The
error dynamics are given by
\begin{align}
\dot{e}_w &= S e_w + l_1(E\Theta) \\
\dot{e}_z &= -\Gamma e_z + G e_w + l_2(E\Theta) \\
\dot{E}\Theta &= \hat{\Theta}\Theta^{-1} + \hat{\Theta}\Theta^{-1} - \hat{\Theta}\Theta^{-1} \Theta \Theta^{-1} = (Q(\bar{J}^{-1}z) + L(E\Theta))E\Theta - E\Theta Q(\bar{J}^{-1}z) \\
\dot{\hat{\Theta}}^{-1} &= \dot{\hat{\Theta}}\Theta^{-1} - \hat{\Theta}\Theta^{-1}
\end{align}
(23)
where \([A,B] = AB - BA\) for matrices \([A,B]\).

In addition to the observer structure, we impose an observability assumption concerning the influence of \((G e_w, e_z)\) on the \(E\Theta\) dynamics. Informally speaking, we assume that as long as the \((G e_w, e_z)\) dynamics do not converge to zero asymptotically, \(E\Theta\) cannot converge to the identity. Since \(G e_w\) and \(e_z\) can be considered as the inputs of the \(E\Theta\) system, this assumption is a special case of output-input stability as presented in [20]. Another interpretation is that we assume an asymptotic zero-input detectability as defined in [19]. Formally the assumption is:

**Definition 5:** Let \(x = f(x,w)\) be a system with output \(y = h(x)\). We call the output \(y\) of such a control system asymptotically w-detectable if \(y(t) \to 0\) for \(t \to \infty\) implies \(w(t) \to 0\) for \(t \to \infty\). □

**Remark 6:** The error dynamics (23) are asymptotically \((G e_w, e_z)\)-detectable for this assumption that \(E\Theta(t) \to I\) for \(t \to \infty\). Since \(E\Theta(t) = E\Theta(0) + \int_0^t E\Theta(t)dt\), we know that \(\lim_{t \to \infty} \int_0^t E\Theta(t)dt = I - E\Theta(0)\). Since \(E\Theta(t)\) is bounded \((SO(3)\) is compact) and since \(e_w\) and \(e_z\) are bounded (see (35)), one can show by differentiation of (23) that \(E\Theta(t)\) is bounded if \(m\) and \(z_1\) are assumed to be bounded (i.e. \(m(t)\) and \(m(t)\) bounded for all \(t \in \mathbb{R} \cup \{\infty\}\) and \(S\) has no zero eigenvalue). \(E\Theta\) is uniformly continuous by Barbacott’s Lemma [21]. \(E\Theta(t) \to I\) for \(t \to \infty\) implies \(E\Theta(t) \to 0\) for \(t \to \infty\). Moreover \(E\Theta(t) \to 0\) implies \(e_z(t) \to 0\) and this again implies \(G e_w(t) \to 0\) for \(t \to \infty\).

**Lemma 7:** Consider the system (21) together with the observer (22). Moreover, assume that the output \(y = E\Theta - I\) is asymptotically \((G e_w, e_z)\)-detectable for the error dynamics (23) and that there are matrices \(P_0 = P_1^T, P_1 = P_2^T, P_2\) such that
\[
\begin{bmatrix}
P_0 \\
\frac{1}{2} P_2^T S \\
\frac{1}{2} P_2 \\
\end{bmatrix} > 0,
\]
(24)
\[
X = \begin{bmatrix}
P_0 S + \frac{1}{2} P_2 G \\
\frac{1}{2} P_2^T S + \frac{1}{2} P_2 G \\
\end{bmatrix} 0,
\]
(25)
\[
l_1(E\Theta) = -\frac{1}{2} P_0^{-1} P_2 J_2(E\Theta),
\]
(26)
\[
l_2(E\Theta) = - (P_1 - \frac{1}{2} P_2^T P_0^{-1} J_2^{-1}(J_1^{-1})^T)\begin{bmatrix}
E_{\Theta 21} - E_{\Theta 32} \\
E_{\Theta 31} - E_{\Theta 13} \\
E_{\Theta 12} - E_{\Theta 21} \\
\end{bmatrix},
\]
(27)
\[
L(E\Theta) = E\Theta - E\Theta^T
\]
(28)
Then \((G e_w, e_z, E\Theta) = (G \hat{w} - G w, \hat{z} - z, \hat{\Theta} \Theta^{-1})\) converges for almost all initial conditions towards \((0,0,1) \in \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3)\). □

**Proof:** We consider the function \(V_2 : \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \to \mathbb{R}\) defined by
\[
V_2 := \frac{1}{2} tr((E\Theta - I)^T (E\Theta - I)) + [e_w^T e_z]^T \begin{bmatrix}
P_0 \\
\frac{1}{2} P_2^T S \\
\frac{1}{2} P_2 \\
\end{bmatrix} [e_w^T e_z]
\]
(29)
\[
= 3 - tr(E\Theta) + \frac{1}{2} e_w^T P_0 e_w + \frac{1}{2} e_z^T P_1 e_z + \frac{1}{2} e_w^T P_2 e_z.
\]
(30)
Since \(Q(\bar{J}^{-1}z)\) and \(L(E\Theta)\) are skew-symmetric, the third differential equation in (22) defines a flow on \(SO(3)\). Thus, if the initial condition for the third equation in (22) is in \(SO(3)\), then \(E\Theta = \Theta \Theta^{-1} \in SO(3)\). Since \(V_2(E\Theta, e_w, e_z) > 0\) for all \(E\Theta \in SO(3) \setminus \{I\}, e_w \in \mathbb{R}^3 \setminus \{0\}\) and \(e_z \in \mathbb{R}^3 \setminus \{0\}\), the function \(V_2\) is a Lyapunov function candidate to investigate the stability of \((0,0,1)\). The derivative \(\dot{V}_2\) of \(V_2\) along the vector field (23) is given by
\[
\dot{V}_2 = tr(L(E\Theta) E\Theta)
\]
(31)
\[
+ tr(Q(\bar{J}^{-1}e_z) E\Theta + e_z^T (P_1 l_2(E\Theta) + \frac{1}{2} P_1^T l_1(E\Theta)))
\]
\[
+ e_w^T (P_0 S + \frac{1}{2} P_2 G) e_w + e_w^T \frac{1}{2} P_2^T P_0 P_1 G e_z - e_z^T P_1 e_z \geq 0.
\]
(32)
In the following we discuss the expressions of each line in (31) separately starting with the last and ending with the first. Because of (27) the last term in (31) is zero. Because of the assumption (25), we obtain
\[
\dot{V}_2 = tr(Q(\bar{J}^{-1}e_z) E\Theta + e_z^T (P_1 l_2(E\Theta) + \frac{1}{2} P_1^T l_1(E\Theta)))
\]
(33)
\[
= tr(Q(\bar{J}^{-1}e_z) E\Theta + e_z^T (P_1 - \frac{1}{2} P_2^T P_0^{-1} P_2 l_2(E\Theta)) = 0.
\]
(34)
Finally using the observer gain \(L(E\Theta) = E\Theta - E\Theta^T\), we obtain
\[
\dot{V}_2 = tr((E\Theta - E\Theta^T) E\Theta) + \frac{1}{2} [e_w^T e_z]^T (X + X^T) [e_w^T e_z] \leq 0.
\]
(35)
\(\dot{V}_2\) is similar to \(\dot{V}_1\) in the proof of Lemma 2, which implies the convergence \(E\Theta \to I\) for almost all initial conditions and the boundedness of \(e_w\) and \(e_z\). Together with the \((G e_w, e_z)\)-detectability assumption this implies the asymptotic convergence of \((G e_w, e_z, E\Theta)\) to \((0,0,1)\) for almost all initial conditions. □

**D. Solution of (RP)**

In this section we use the feedback from Section III-B and the observer from Section III-C to solve the problem (RP) under the assumption that \(G w\) and \(z_1\) are not available from measurement.

Therefore, we consider the system as given in (6), let \(\hat{E}\Theta_0 = \hat{\Theta}_0 \Theta_0^{-1}\) and choose the observer plus a certainty equivalence feedback as follows:
\[
\dot{\hat{w}} = S \hat{w} + l_1(E\Theta_0)
\]
(36)
\[
\dot{\hat{z}} = \hat{G} \hat{w} + l_2(E\Theta_0)
\]
\[
\hat{\Theta}_0 = \begin{bmatrix}
Q(\bar{J}^{-1}z) + L(E\Theta_0) \\
u(z_1, z_2, \hat{G} \Theta_0, \Theta_0, m)
\end{bmatrix}
\]
where \(\hat{w}, \hat{z}, \hat{\Theta}_0\) are the estimates of \(w, z_1, \Theta_0\) respectively, where \(u\) is the function defined in (16), \(l_1(E\Theta_0)\) is given by (27), \(l_2(E\Theta_0)\) is given by (28) and \(L(E\Theta_0)\) is given by (29). To obtain the controller,
we have replaced $Gw$ by $G\tilde{w}$ and $z_1$ by $\tilde{z}_1$ in (16) where $\tilde{w}$ and $\tilde{z}_1$ are the states of the observer. Before giving the result, we discuss the key difficulty to establish convergence for almost all initial conditions for solutions of the closed loop.

In the closed loop system (6) and (7) we aim for $(Ge_w(t), e_1(t), E_{\Theta}(t), \zeta(t), E(t)) \to (0, 0, 0, 0, 0)$ for $t \to \infty$ for almost all initial conditions. Notice that we cannot use $V_1 + V_2$ with $V_1$ from the proof of Lemma 2 and $V_2$ from the proof of Lemma 7 because $V_1 + V_2$ is not a Lyapunov function for the closed loop system since the derivative $V_1 + V_2$ of $V_1 + V_2$ along the closed loop vector field (6) and (36) contains indefinite terms due to the nonlinear expressions $p$ and $q$. Consequently, we choose another approach to show that (36) solves (RP). Moreover, one could think that closed loop stability is trivially given due to the detectability assumption in Lemma 7. However, due to the multiple equilibria, this is by far not the case, as outlined below in more detail.

To show that (36) solves (RP), we utilize a generic result on the asymptotic behavior of solutions of differential equations which approach an invariant manifold $\mathcal{N}$, i.e. [22]. This result concerns bounded solutions of a system $\dot{x} = f(x)$, $x \in \mathcal{M}$ defined on a Riemannian manifold $\mathcal{M}$, where the $\omega$-limit set $\omega(x_0)$ of the solution $x(t)$ through $x_0$ lies in the closed embedded submanifold $\mathcal{N}$. Assume the existence of a function $V : \mathcal{O} \to \mathbb{R}$ on a tubular neighborhood $\mathcal{O}$ of $\mathcal{N}$ with $V|_{\mathcal{N}} \leq 0$ and $V|_{\mathcal{O}\setminus\mathcal{N}} < 0$ where $\mathcal{O} = \{y \in \mathcal{M} | V(y) = 0\}$, is called height function. We say the connected components $\{\mathcal{E}_k\}$ are contained in $V$, if each $\mathcal{E}_k$ lies in a level set of $V$ and $\{V(\mathcal{E}_k)\}_k$ has at most a finite number of accumulation points. Then:

**Theorem 8:** [22, Theorem 6] If the components $\{\mathcal{E}_k\}_k$ are contained in $V$, then $\omega(x(0)) \in \mathcal{E}_k$ for a unique $k$.

The closed loop vector field (in error coordinates) is given by the first three equations in (6) together with

$$
\begin{align*}
\dot{e}_w &= Se_w + l_1(E_{\Theta}) \\
\dot{e}_{z_1} &= Ge_w + l_2(E_{\Theta}) \\
E_{\Theta} &= (Q(J^{-1}_1e_1) + L(E_{\Theta}))E_{\Theta} + [Q(J^{-1}_1z_1), E_{\Theta}] \\
\zeta &= Q^{-1}(U(z_1, z_2, \Theta_1, \Theta_2, \tilde{w}, \tilde{m})) \\
E &= (Q(J^{-1}_2z_2) + Q(q(z_2, \Theta_2)) - EQ(J^{-1}_1z_1)^TE)
\end{align*}
$$

where $U$ is given by (15) and $u$ in $U$ is chosen as in (36) and $z_2 = \mathcal{T}_2(\zeta)$ (see (15) and Lemma 3). Denote in the following the rank of $G$ by $r$. The closed loop vector field plays the role of $f$ in the system $\dot{x} = f(x)$ for the height function Theorem (Theorem 8) and the state space manifold $\mathcal{M}$ is given by

$$\mathcal{M} := \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \quad (38)$$

where $\mathcal{X}_1 := \mathbb{R}^r \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^{3r}$, $\mathcal{X}_2 := \mathbb{R}^r \times \mathbb{R}^3 \times SO(3)$ and $\mathcal{X}_3 := \mathbb{R}^3 \times SO(3)$. I.e. the state is $x = (x_1, x_2, x_3)$ with $x_1 = (w(z_1, \Theta_1, Ge_w), z_2, Ge_w, e_1)$ and $x_3 = (\zeta, E)$. Because of Lemma 7, $e_2(t) = z_1(t) - z_1(t) \to 0$, $Ge_w(t) = G\tilde{w}(t) - Gw(t) \to 0$ and $E_{\Theta}(t) = \Theta_1(t)\Theta^{-1}(t) \to I$ for $t \to \infty$. Thus, the closed loop system (37) converges for almost all initial conditions to $\mathcal{N}$ defined by

$$\mathcal{N} := \mathcal{X}_1 \times \{0, 0, I\} \times \mathcal{X}_3 \quad (39)$$

$\mathcal{N}$ is a closed embedded submanifold of $\mathcal{M}$. The vector field of the system on $\mathcal{N}$ is given by the first three equations in (6) and (17) (i.e. the last two equations in (37)). The asymptotic behavior of the dynamics on $\mathcal{N}$ is the result of Lemma 2. More specifically, for almost all initial conditions we have $(\zeta(t), E(t)) \to (0, I)$ for $t \to \infty$. We know also that almost all solutions of the closed loop converge towards $\mathcal{N}$ asymptotically. However, this does not imply in the general situation that the solution also converges to the desired point on $\mathcal{N}$, i.e. the asymptotic behavior of the dynamics on $\mathcal{N}$ does not necessarily describe the asymptotic behavior of solutions which are not initialized on $\mathcal{N}$. The situation is illustrated in Figure 1. In the following we show that the existence of a suitable height function gives the desired result.

**Lemma 9:** $V_3 : \mathcal{M} \to \mathbb{R}$ defined by

$$V_3 := \frac{1}{2} \zeta^T \zeta - tr(E) \quad (40)$$

is a height function for the pair $(\mathcal{N}, f)$ where $f$ is the closed loop vector field on $\mathcal{M}$.

**Proof.** $V_3$ is defined on $\mathcal{M}$ and by Lemma 2, we have $V_3|_{\mathcal{N}} \leq 0$. As shown in Corollary 11, the set $\mathcal{E} := \{x \in \mathcal{M} | V_3(x) = 0\}$ is given by

$$\mathcal{E} := \mathcal{X}_1 \times \{0, 0, I\} \times \mathcal{F} \quad (41)$$

where $\mathcal{F} := \{(x, \Theta) \in \mathbb{R}^3 \times SO(3) | x = 0, \Theta^T = \Theta\}$. Corollary 11 also shows that $V_3|_{\mathcal{N}\setminus\mathcal{E}} < 0$.

**Theorem 10:** Assume that $m$ is a function of $(w, z_1, \Theta_1)$ such that $z_1$ is bounded, assume furthermore that $q(z_2, \Theta_2)$ is linear in $z_2$ and that (14) defines a diffeomorphism. In addition assume that the output $y = E_{\Theta} - I$ is asymptotically $(Ge_w, e_1)$-detectable for the closed loop. Then the certainty equivalence controller from (36) $L(E_{\Theta_0})$ is (29), $l_1(E_{\Theta_0})$ is (27) and $l_2(E_{\Theta_0})$ is (28) solves (RP).

**Proof.** First we show that $\zeta(z_2)$ is bounded. After substituting $u = u(z_1 + e_1, z_2, Gw + Ge_w, \Theta_1, \Theta_2, m)$ into (15) (this corresponds to the $\zeta$-equation in (37)) and after a tedious calculation utilizing that $q$ is linear in $z_2$, one obtains a differential equation of the form

$$\dot{\zeta} = -B(e_2(t), \zeta) + a(e_2, Ge_w, t), \quad (42)$$

where $B(0, t) = 0$ and $a(0, 0, t) = 0$. The solutions of this linear differential equation exist for all times and since $e_2(t), Ge_w(t) \to 0$ for $t \to \infty$ for almost all initial conditions, we see that $\zeta(t)$ is bounded. Since $z_1$ is bounded and $SO(3)$ is compact and because of Lemma 3 and 7, all solutions of the closed loop system are bounded. Moreover, since $m$ depends on $(w, z_1, \Theta_1)$ the closed loop is an autonomous system and therefore Theorem 8 can be used. Lemma 9 shows that $V_3$ defined by (40) is a height function for $(\mathcal{N}, f)$. $\mathcal{E}$ is given by (41) and since $V_3$ corresponds to $V$ in Corollary 11 the
connected components $\mathcal{F}_k$ of $\mathcal{F}$ are given by
\[
\{(x, \Theta) \in \mathbb{R}^3 \times SO(3) | \Theta = \Theta^T, \operatorname{tr}(\Theta) = 3 - 4k, x = 0 \} \quad (43)
\]
for $k \in \{0, 1\}$. The connected components $\delta_k = \mathcal{F}_k \times \mathcal{F}_k$ of $\delta$ are thus contained in level set of $V_1$ and, see Corollary 11. The number of connected components of $\delta$ is finite, hence the set $\{V_1(\delta_k)\}_{k=0}^{1}$ is finite. Thus, according to Theorem 8 we know that the $\omega$-limit set of a solution solution of (37) converges to $\mathcal{N}$ in a unique $\delta_k$.

In the final step of the proof we show the asymptotic stability of $(0, 0, I) \times (0, 0, I) \in \mathcal{F}_2 \times \mathcal{F}_3$ relative to the projection of the dynamics (37) on $\mathcal{M} = \mathcal{F}_2 \times \mathcal{F}_3$. This means, we consider the system given by (37) as a time-varying system where $z_1, w, m$ are functions of time, i.e. (37) defines a non-autonomous system with a state-space given by $\mathbb{R}^{n-r} \times \mathcal{M}$ and with states $(\bar{x}, \bar{\Theta}, \bar{e}_r, \bar{\zeta}, \bar{E})$.

Theorem 12: [23, Theorem 5] If on a neighborhood $\Omega_1$ of the origin there exists a $C^1$-function $V : \Omega_1 \to \mathbb{R}$ such that
\[
V(\bar{x}) \geq 0 \quad \text{for all } \bar{x} \in \Omega_1 \quad \text{and} \quad V(0) = 0.
\]
\[
V(\bar{x}) \leq 0 \quad \text{for all } \bar{x} \in \Omega_1 \quad \text{and} \quad t \geq 0.
\]
\[
\text{The restriction of } f \text{ is uniformly asymptotically stable on the positively invariant set } \{ x \in \Omega_1 | V(x) = 0 \}.
\]

Then the origin is an uniformly stable equilibrium point for $\dot{x} = f(x, t)$.

REFERENCES


