Abstract—In this paper, we study the semi-global stabilization of general linear discrete-time critically unstable systems subject to input saturation and multiple unknown input delays. Based on a simple frequency-domain stability criterion, we find upper bounds for delays that are inversely proportional to the argument of open-loop eigenvalues on the unit circle. For delays satisfying these upper bounds, linear low-gain state and finite dimensional dynamic measurement feedbacks are constructed to solve the semi-global stabilization problems.

I. INTRODUCTION

The ubiquitous presence of time delay in a variety of engineering applications has invoked intense research enthusiasm in the study of time-delay systems. Voluminous results have been reported. It is not a goal of this paper to provide a complete review of the enormous literature in this context, nor is it possible. A brief coverage of recent research progress on time-delay systems can be found in [13], [5], [12], [8], [4], [2] and references therein. Input saturation is also an important issue that is inevitable in virtually any controller design for physical systems. Neglect of saturation effect can result in grave consequence of performance deterioration even instability. As such, it has attracted and sustained attention from researchers for decades. Some important previous work is summarized in [1], [14], [17], [15], [6], [7].

In this paper, we study the stabilization of discrete-time linear system subject to both input saturation and delay. A discrete-time system with known input delays can be converted to a delay-free system by state augmentation and hence is easier to deal with. We are interested in the case where the delay is unknown. This problem has been previously studied for both continuous- and discrete-time systems. In [10], a nested-saturation type controller is developed for stabilization of an integrator-chain system. This type of controller is also used for discrete-time systems in [20]. A linear delay-independent low-gain feedback was first constructed in [11] to achieve semi-global stabilization for a chain of integrators. In [23], [22], a different low-gain design is utilized to solve the semi-global stabilization problem for a broader class of critically unstable linear systems that has all the eigenvalues at the origin in the continuous-time case and at 1 in the discrete-time case. In a recent paper [18], the authors consider general continuous-time critically unstable systems subject to input saturation and multiple unknown input delays and propose upper bounds for delays that are inversely proportional to the modulus of open-loop eigenvalues on the imaginary axis. For tolerable delays, linear low-gain feedbacks can be constructed which solve the semi-global stabilization problem.

The aim of this paper is to extend the results in [18] to discrete-time case. The analysis and design is based on a simple frequency-domain stability criterion for discrete linear time-delay systems. It turns out that the results of discrete-time systems are in a strict parallel with those of continuous-time systems. The upper bound of tolerable delays found here is also inversely proportional to the argument of eigenvalues on the unit circle. If all the delays satisfy the proposed upper bounds, linear state and finite dimensional dynamic measurement feedback can be constructed using the $H_2$ low-gain design technique to achieve the semi-global stabilization.

The paper is organized as follows. Two stabilization problems are formulated in Section II. Some preliminary results, including a stability criterion for linear discrete time-delay systems and some key properties of $H_2$ low-gain feedback, are presented in Section III. The main results of this paper are developed in Section IV. In this part, we first stabilize the linearized system without saturation using the low-gain feedback and then show that by proper selection of a tuning parameter, the same controller will solve the semi-global stabilization problems in the presence of saturation. Proofs of several technical lemmas are not included due to space limitation.

A. Notations

In this paper, standard notations are used. For any open set $\mathcal{G} \subset \mathbb{C}$, $\partial \mathcal{G}$ and $\overline{\mathcal{G}}$ denote its boundary and closure. For $z_0 \in \mathbb{C}$ and $r > 0$, $\mathcal{D}(z_0, r)$ denotes an open disc centered at $z_0$ with radius $r$. Among all, the unit open disc centered at the origin is of particular importance and will be used very often, as such we denote specially $\mathcal{D}_0 := \mathcal{D}(0, 1)$ and $\mathbb{C}_0 := \partial \mathcal{D}(0, 1)$. For any $K_1, K_2 \in \mathbb{N}$ and $K_1 \leq K_2$, $[K_1, K_2] := \{k \in \mathbb{N} \mid K_1 \leq k \leq K_2\}$. Let $\ell^\infty(K)$ denote the Banach space of finite sequences $\{y_1, \ldots, y_K\} \subset \mathbb{C}^n$ with norm $\|\cdot\|_\infty := \max_{k \in K} \|y_k\|$. For column vectors $x_1, \ldots, x_m$, we simply use $x = [x_1; \ldots; x_m]$ to denote the stack vector.
II. PROBLEM FORMULATION

Consider a discrete-time linear system subject to input saturation and delay
\[
\begin{cases}
x(k + 1) = Ax(k) + \sum_{i=1}^{m} B_i \sigma(u_i(k - \kappa_i)), \\
y(k) = Cx(k), \\
x(\theta) = \phi_{0+k}, \theta \in [-K, 0]
\end{cases}
\]
where \( x \in \mathbb{R}^n, u_i \in \mathbb{R}, \kappa_i \in [0, K_i], K_i \in \mathbb{N} \) and \( K = \max\{K_i\} \). The initial condition \( \phi \in \ell^n_w(K) \). Let
\[
u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}.
\]

We can formulate two semi-global stabilization problems as follows:

**Problem 1:** The semi-global asymptotic stabilization via state feedback problem for system (1) is to find, for any set of positive integers \( K_i > 0 \) and a priori given bounded set of initial conditions \( W \subseteq \ell^n_w(K) \) with \( K = \max\{K_i\} \), a delay-independent linear state feedback controller \( u = Fx \) such that the zero solution of the closed-loop system is locally asymptotically stable for any \( \kappa_i \in [0, K_i] \) with \( W \) contained in its domain of attraction, i.e. the following properties hold for all \( \kappa_i \in [0, K_i] \), \( i = 1, \ldots, m \):
1. \( \forall \epsilon > 0, \exists \delta \) such that if \( \|\phi\|_{\infty} \leq \delta, \|x(k)\| \leq \epsilon \) for all \( k \geq 0 \);
2. \( \forall \phi \in W, x(k) \to 0 \) as \( k \to \infty \).

**Problem 2:** The semi-global asymptotic stabilization via measurement feedback problem for system (1) is to find an integer \( q \), and find, for any set of positive integers \( K_i > 0 \) and any a priori given bounded set \( W \subseteq \ell^{n+q}_w(K) \) with \( K = \max\{K_i\} \), a delay-independent linear finite dimensional measurement feedback controller
\[
\begin{cases}
\chi(k + 1) = A_k \chi(k) + B_k y(k), \\
u(k) = C_k \chi(k) + D_k y(k),
\end{cases}
\]
such that the zero solution of the closed-loop system is locally asymptotically stable for all \( \kappa_i \in [0, K_i] \) with \( W \) contained in its domain of attraction, i.e. the following properties hold for all \( \kappa_i \in [0, K_i] \):
1. \( \forall \epsilon > 0, \exists \delta \) such that if \( \|\phi; \psi\|_{\infty} \leq \delta, \|x(k)\| \leq \epsilon \) for all \( k \geq 0 \);
2. \( \forall (\phi; \psi) \in W, (x(k), \chi(k)) \to 0 \) as \( k \to \infty \).

Since the input of (1) is bounded, it is well known that the following assumption is necessary for semi-global stabilization.

**Assumption 1:** \((A, B)\) is stabilizable, \((A, C)\) is detectable and \( A \) has all its eigenvalues in the closed unit disc \( \mathbb{C}^\circ \).

III. PRELIMINARIES

In this section, we shall present stability criteria for discrete time-delay system which are the basic of this paper and recall the standard low-gain feedback design and some of its properties.

A. Stability of discrete linear time-delay systems

Consider system
\[
x(k + 1) = Ax(k) + \sum_{i=1}^{m} A_i x(k - \kappa_i), \quad (3)
\]
where \( x(k) \in \mathbb{R}^n \) and \( \kappa_i \in \mathbb{N} \). Suppose \( A + \sum_{i=1}^{m} A_i \) is Schur stable. The next lemma is a standard result.

**Lemma 1:** System (3) is asymptotically stable if and only if
\[
\det \left[ zI - A - \sum_{i=1}^{m} z^{-\kappa_i} A_i \right] \neq 0, \quad \forall z \notin D_0, \quad \forall \kappa_i \in [0, K_i]. \quad (4)
\]

Define for \( \alpha \in [0, 1] \)
\[
F_\alpha(z) = \det \left[ zI - A - (1 - \alpha) \sum_{i=1}^{m} A_i - \alpha \sum_{i=1}^{m} z^{-\kappa_i} A_i \right].
\]

The results of this paper are all based on the following lemma.

**Lemma 2:** The system (3) is asymptotically stable if
\[
\det (F_\alpha(z)) \neq 0, \quad \forall z \in \mathbb{C}^\circ, \forall \alpha \in [0, 1]. \quad (5)
\]

**Proof:** Suppose (5) holds but (4) does not hold, that is, \( F_1(z) \) has zeros in \( \mathbb{C} \setminus D_0 \). However, since \( A + \sum_{i=1}^{m} A_i \) is Schur stable, all the zeros of \( F_0(z) \) must be in the open unit disc \( D_0 \). Note that the zeros of \( F_\alpha(z) \) move continuously as \( \alpha \) varies. Hence, there exists \( \alpha_0 \in (0, 1] \) such that \( \det (F_{\alpha_0}(z_0)) = 0 \) for some \( z_0 \in \mathbb{C}^\circ \), which contradicts (6).

Consider a special case of (3) where \( A_i = B_i F_i \), that is,
\[
x(k + 1) = Ax(k) + \sum_{i=1}^{m} B_i F_i x(k - \kappa_i). \quad (7)
\]

Assume that \( \bar{A} = A + \sum_{i=1}^{m} B_i F_i \) is Schur stable. The following variation of Lemma 2 is more convenient to use.

**Lemma 3:** The system (7) is asymptotically stable if
\[
\det [I + \alpha G(z)(I - D(z))] \neq 0, \quad \forall z \in \mathbb{C}^\circ, \forall \alpha \in [0, 1], \quad (8)
\]
where \( G(z) = F(zI - A - BF)^{-1} B, \quad D(z) = \text{diag}(z^{-\kappa_i})_{i=1}^{m} \) and
\[
B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}.
\]

**Proof:** The proof is straightforward. Note that
\[
\det \left[ zI - A - (1 - \alpha) \sum_{i=1}^{m} B_i F_i - \alpha \sum_{i=1}^{m} z^{-\kappa_i} B_i F_i \right] = \det [zI - A - BF + \alpha B(I - D(z))F] = \det [zI - A - BF] \times \det [I + \alpha(zI - A - BF)^{-1} B(I - D(z))] = \det [zI - A - BF] \times \det [I + \alpha F(zI - A - BF)^{-1} B(I - D(z))].
\]
Since $A + BF$ is Schur stable, (6) holds if and only if (8) holds.

Remark 1: Lemma 2 and 3 are discrete-time counterparts of Lemma 2 and 3 in [21] (see also the work in [3], [9]). However, the conditions (6) and (8) are only sufficient for discrete-time system.

B. $H_2$ low-gain state feedback and compensator

Consider a discrete-time linear system

$$\begin{align*}
    x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0 \\
    y(k) &= Cx(k), \\
    z(k) &= u(k).
\end{align*}$$

Let Assumption 1 hold. Recall the following definition from [19]. An $H_2$ low-gain sequence is a family of parameterized matrices $F_\epsilon$ with $\epsilon \in (0, 1]$ such that the following properties hold

1) $A + BF_\epsilon$ is Schur stable for any $\epsilon \in (0, 1]$

2) the closed-loop system of (9) and $u = F_\epsilon x$ satisfies

$$\lim_{\epsilon \to 0} \|z\|_2 = 0, \quad \forall x_0 \in \mathbb{R}^n.$$  

The $H_2$ low-gain sequence can be constructed as

$$F_\epsilon = -(B'P_\epsilon B + I)^{-1}B'P_\epsilon A$$  

(11)

where for $\epsilon \in (0, 1]$, $P_\epsilon$ is the positive definite solution of $H_2$ Algebraic Riccati Equation

$$P_\epsilon = A'P_\epsilon A + \epsilon I - A'P_\epsilon B(BP_\epsilon B + I)^{-1}B'P_\epsilon A.$$  

(12)

It is known that under Assumption 1, $P_\epsilon \to 0$, and thus $F_\epsilon \to 0$, as $\epsilon \to 0$. Moreover, we also have the following lemma

Lemma 4: Define transfer function $G_\epsilon(z) = F_\epsilon(zI - A - BF_\epsilon)^{-1}B$. We have

$$\|I + G_\epsilon\|_\infty \leq \sqrt{1 + \lambda_{\text{max}}(B'P_\epsilon B)}.$$  

(13)

Proof: Define $R(z) = I - F_\epsilon(zI - A)^{-1}B$. $R(z)$ satisfies the following return difference equality ([16]):

$$R(z^{-1})'(I + B'P_\epsilon B)R(z) = I + \epsilon B'(z^{-1}I - A)^{-1}(zI - A)^{-1}B.$$  

This implies for $z \in \mathbb{C}^\circ$

$$(1 + \lambda_{\text{max}}(B'P_\epsilon B)) R(z)^* R(z) \geq I,$$

and hence

$$\sigma(I - F_\epsilon(zI - A)^{-1}B) \geq \frac{1}{\sqrt{1 + \lambda_{\text{max}}(B'P_\epsilon B)}}, \quad z \in \mathbb{C}^\circ.$$  

(14)

By matrix inversion lemma,

$$\tilde{\sigma}(I + F_\epsilon(zI - A - BF_\epsilon)^{-1}B) \leq \frac{1}{\sqrt{1 + \lambda_{\text{max}}(B'P_\epsilon B)}}, \quad z \in \mathbb{C}^\circ,$$

which yields (13).

Remark 2: An immediate consequence of Lemma 4 is the following relations which will be useful in our analysis.

$$\|I + G_\epsilon\|_\infty \leq \eta := \sqrt{1 + \lambda_{\text{max}}(BP_\epsilon B)},$$

$$\|G_\epsilon\|_\infty \leq 1 + \eta, \quad \forall \epsilon \in (0, 1]$$  

(15)

The low-gain state feedback $u = F_\epsilon x$ can be realized with an observer, which we refer to as low-gain compensator

$$\begin{align*}
    x(k+1) &= Ax(k) + BF_\epsilon x(k) - K(y(k) - Cx(k)), \\
    u(k) &= F_\epsilon x(k).
\end{align*}$$

(16)

with $\gamma(0) = \chi_0$, where $K$ is such that $A + KC$ is Schur stable. It can be shown that (16) is a generalized $H_2$ low-gain “sequence” as it satisfies the aforementioned two properties of an $H_2$ low-gain. First, it is easy to see $A + BF_\epsilon$ is Schur stable, where

$$A = \begin{bmatrix}
A & 0 \\
-KC & A + KC + BF_\epsilon
\end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ B \end{bmatrix}$$  

and $F_\epsilon = \begin{bmatrix} 0 & F_\epsilon \end{bmatrix}$.

The next lemma proves property (10) for the closed-loop of (9) and (16).

Lemma 5: The closed-loop of (9) and (16) satisfies

$$\lim_{\epsilon \to 0} \|z\|_2 = 0, \quad \forall x_0, \chi_0 \in \mathbb{R}^n.$$  

IV. MAIN RESULT

Now we are in good position to solve the two stabilization problems formulated in Section II. We will develop the results for a linearized system ignoring saturation utilizing the low-gain state feedback and compensator. Then it can be shown that the input of the resulting closed-loop systems can be made sufficient small to avoid saturation for a compact set of initial conditions, which will lead to the solution of Problem 1 and 2.

A. Global stabilization of linear discrete-time system with input delay

We first consider the stabilization problem for system (1) in the absence of saturation

$$\begin{align*}
    x(k+1) &= Ax(k) + \sum_{i=1}^{m} B_i u(k - \kappa_i), \\
    y(k) &= Cx(k).
\end{align*}$$

(17)

Since the system (17) is linear, it is possible to achieve the global stabilization via linear feedback. We shall show that this in fact can be achieved by a low-gain feedback $u = F_\epsilon x$ with $F_\epsilon$ given by (11).

Define

$$\omega^i_{\text{max}} = \max\{\omega \in [0, \pi] \mid \exists v \in \mathbb{C}^n, A'v = e^{i\omega}v, v'B_i \neq 0\}.$$

Clearly, $\omega^i_{\text{max}}$ is the largest argument of eigenvalues that are, at least partially, controllable via input $u_i$. It will be made clear in the following theorem and its proof that this $\omega^i_{\text{max}}$ dictates the delay tolerance in the channel $u_i$.

Theorem 1: Consider system (17). Let Assumption 1 hold and $F = F_\epsilon$ be given by (11) and (12) with $\epsilon \in (0, 1]$. For any $K_j < \frac{1}{3\omega^i_{\text{max}}}$, there exists $\epsilon^* \in (0, 1]$ such that the system(17) where $u = F_\epsilon x$ with $F_\epsilon$ given by (11) is asymptotically stable for $\epsilon \in (0, \epsilon^*)$ and $\kappa_i \in [0, K_i]$.

In a special case where $A$ has all the eigenvalues equal to 1, Theorem 1 immediately implies that any bounded delay can
be tolerated with using low-gain feedback \( u = F \epsilon x \). This is stated in the following corollary.

**Corollary 1:** Consider system (17). Let Assumption 1 hold and \( F = F \epsilon \) be given by (11) and (12) with \( \epsilon \in (0, 1] \). Suppose \( A \) has all the eigenvalues equal to 1. For any given positive integers \( K_i \), there exists \( \epsilon^* \in (0, 1] \) such that the system (17) where \( u = F \epsilon x \) with \( F \epsilon \) given by (11) is asymptotically stable for \( \epsilon \in (0, \epsilon^* \) and \( \kappa_i \in \left[0, K_i \right] \).

**Proof of Theorem 1:** Consider the closed-loop system

\[
x(k + 1) = Ax(k) + \sum_{i=1}^{m} B_i F_{\epsilon,i} x(k - \kappa_i),
\]

where \( F_{\epsilon,i} \) is the \( i \)-th row of \( F \epsilon \). Let \( G_{\epsilon}(z) = F_{\epsilon}(z I - A - B F_{\epsilon})^{-1} B \). It follows from Lemma 3 that the system (18) is asymptotically stable if

\[
det\left[I + \alpha G_{\epsilon}(e^{j\omega}) (I - D(e^{j\omega}))\right] \neq 0,
\]

\[
\forall \omega \in [-\pi, \pi), \forall \alpha \in [0, 1].
\]

where \( D(z) = \text{diag}(z^{-\kappa_i}) \). Due to symmetry, we only need to consider the \( \omega \in [0, \pi] \). Assume \( A \) has \( r \) eigenvalues on the unit circle which are denoted by \( e^{j\omega_q}, q = 1, \ldots, r \) with \( \omega_q \in [0, \pi] \). Given \( K_i < \frac{2\pi}{\max_q \omega_q} \) for \( i = 1, \ldots, m \), there exists a \( \delta > 0 \) such that

1. The neighborhoods \( \mathcal{E}_q := [\omega_q - \delta, \omega_q + \delta] \cap [0, \pi], q = 1, \ldots, r \), around these eigenfrequencies are mutually disjoint;
2. If \( e^{j\omega_q} \) is at least partially controllable through input \( i \),

\[
\omega K_i \leq \frac{\pi}{2} - \frac{1}{2} \left( \frac{\pi}{2} - K_i \omega_q^i \right), \quad \forall \omega \in \mathcal{E}_q.
\]

**Lemma 6:** The following properties hold:
1. If \( e^{j\omega_q} \) is not controllable via input \( u_i \), for some \( i \), then

\[
\lim_{\epsilon \to 0^+} F_{\epsilon}(e^{j\omega} I - A - B F_{\epsilon})^{-1} B_i = 0,
\]

uniformly in \( \omega \) for \( \omega \in \mathcal{E}_q \).
2. There exists \( \epsilon^* \) such that for \( \epsilon \in (0, \epsilon^* \),

\[
\|F_{\epsilon}(e^{j\omega} I - A - B F_{\epsilon})^{-1} B\| \leq \frac{1}{\beta}, \forall \omega \in \Omega,
\]

where \( \Omega := [0, \pi] \setminus \bigcup_{q=1}^{r} \mathcal{E}_q \). Owing to Lemma 6, we find that there exists an \( \epsilon^* \) such that (19) is satisfied if for all \( q = 1, \ldots, r \),

\[
det[I + \alpha G_{\epsilon}(e^{j\omega}) (I - D(e^{j\omega}))] \neq 0,
\]

\[
\forall \omega \in \mathcal{E}_q, \forall \kappa_i \in \left[0, K_i \right], \alpha \in [0, 1].
\]

provided \( \epsilon \leq \epsilon^* \) where \( D(e^{j\omega}) \) equals \( D(e^{j\omega_q}) \) with \( \kappa_i = 0 \) for all \( i \)'s such that the eigenvalue \( e^{j\omega_q} \) is not controllable via input channel \( i \). Clearly, \( D(e^{j\omega_q}) \) is still unitary. Moreover, by (20), we find that

\[
\text{Re}(\hat{D}_q(e^{j\omega_q})) > \frac{1}{2} I, \quad \forall \omega \in \mathcal{E}_q.
\]

Let’s consider (21). We can write

\[
I + \alpha G_{\epsilon}(e^{j\omega}) (I - D(e^{j\omega})) = (1 - \alpha) I + \alpha \hat{D}_q(e^{j\omega}) + \alpha (I + G_{\epsilon}(e^{j\omega})) (I - \hat{D}_q(e^{j\omega})).
\]

Note that,

\[
(1 - \alpha) I + \alpha \hat{D}_q(e^{j\omega}) = (1 - \alpha)^2 + \alpha^2 I + 2\alpha(1 - \alpha) \text{Re}(\hat{D}_q(e^{j\omega})) \geq \alpha I.
\]

Therefore

\[
\hat{\alpha} (1 - \alpha) I + \alpha \hat{D}_q(e^{j\omega}) \geq \sqrt{\alpha} \geq \alpha.
\]

This together with (22) imply that (21) holds if

\[
\hat{\sigma}(I + G_{\epsilon}(e^{j\omega})) (I - \hat{D}_q(e^{j\omega})) < 1
\]

By (20), for any \( q = 1, \ldots, r \), there exists \( \xi_q \in (0, 1) \) solely depending on \( K_i \) such that we get \( \hat{\sigma}(I + G_{\epsilon}(e^{j\omega})) < 1 - \xi_q \) for \( \omega \in \mathcal{E}_q \). According to (13), there exists a \( \epsilon_2 \leq \epsilon_1 \) such that for \( \epsilon \in (0, \epsilon_2] \), \( \hat{\sigma}(I + G_{\epsilon}(e^{j\omega})) < 1/(1 - \xi_q) \) for any \( q = 1, \ldots, r \). Therefore, condition (19) is satisfied. \( \square \)

The next theorem is concerned with measurement feedback.

**Theorem 2:** Consider system (17). Let Assumption 1 hold. For any positive integers \( K_i < \frac{2\pi}{\max_q \omega_q} \), there exists an \( \epsilon^* \) such that for \( \epsilon \in (0, \epsilon^* \), the closed-loop system of (17) and low-gain compensator (16) is asymptotically stable for \( K_i \in [0, K_i] \).

**Corollary 2:** Consider system (17). Let Assumption 1 hold and \( A \) has all its eigenvalues equal to 1. For any positive integers \( K_i \), there exists an \( \epsilon^* \) such that for \( \epsilon \in (0, \epsilon^* \), the closed-loop system of (17) and low-gain compensator (16) is asymptotically stable for \( K_i \in [0, K_i] \).

**Proof of Theorem 2:** The closed-loop system is given by

\[
\begin{cases}
    x(k + 1) = Ax(k) + \sum_{i=1}^{m} B_i F_{\epsilon,i} x(k - \kappa_i) \\
    y(k + 1) = (A + B F_{\epsilon} + K C)x(k),
\end{cases}
\]

It is well known that system (24) without delay is asymptotically stable. Define

\[
G_{\epsilon}^m(z) = -F_{\epsilon}(z I - A - B F_{\epsilon})^{-1} K C(z I - A - K C)^{-1} B.
\]

Obviously, \( G_{\epsilon}^m(z) \) is stable. It follows from Lemma 3 that (24) is global asymptotically stable if

\[
det[I + \alpha G_{\epsilon}^m(e^{j\omega}) (I - D(e^{j\omega}))] \neq 0,
\]

\[
\forall \omega \in [-\pi, \pi], \forall K_i \in [0, K_i] \forall \alpha \in [0, 1],
\]

where \( D(z) = \text{diag}(z^{-\kappa_i}) \). We have the following lemma

**Lemma 7:** Let \( G_{\epsilon}(z) = F_{\epsilon}(z I - A - B F_{\epsilon})^{-1} B \). Then

\[
\lim_{\epsilon \to 0} (G_{\epsilon}^m(e^{j\omega}) - G_{\epsilon}(e^{j\omega})) = 0
\]

uniformly \( \omega \).

If, by Theorem 1, there exists an \( \epsilon^1 \) such that for all \( \epsilon \in (0, \epsilon^1) \) we have (19) satisfied with \( G_{\epsilon}(e^{j\omega}) \), then we can find an \( \epsilon_2 \leq \epsilon_1 \) such that (25) holds for all \( \epsilon \in (0, \epsilon_2] \). \( \square \)
B. Semi-global stabilization subject to input saturation

In this subsection, we shall show that the low-gain state feedback and compensator which stabilize the linearized systems (17) also solve the semi-global stabilization problem for the same linear system with input saturation (1) by a proper selection of the tuning parameter $\epsilon$ with respect to a set of initial conditions.

**Theorem 3:** Consider the system (1). Let Assumption 1 hold. The semi-global asymptotic stabilization via state feedback problem can be solved by the low-gain feedback (11). Specifically, for a set of non-negative integers $K_i < \frac{\pi}{2m_{\text{max}}}$, $i = 1, ..., m$ and any a priori given compact set of initial conditions $W \subset \ell^2_{m_{\text{max}}}(K)$ where $K = \max\{K_i\}$, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, the low-gain feedback (11) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $W$ for any $\kappa_i \in [0, K_i]$, $i = 1, ..., m$. In the special case where all the eigenvalues of $A$ are 1, the low-gain feedback allows any bounded but arbitrarily large input delays. This recovers the partial results in [22].

**Corollary 3:** Consider the system (1). Let Assumption 1 hold and $A$ has all its eigenvalues equal to 1 for any given set of non-negative integers $K_i$, $i = 1, ..., m$ and any a priori given compact set of initial conditions $W \subset \ell^2_{m_{\text{max}}}(K)$ where $K = \max\{K_i\}$, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, the low-gain feedback (11) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $W$ for any $\kappa_i \in [0, K_i]$, $i = 1, ..., m$. Consider the system (1). Let Assumption 1 hold and $A$ has all its eigenvalues equal to 1. For any given set of non-negative integers $K_i$, $i = 1, ..., m$ and any a priori given compact set of initial conditions $W \subset \ell^2_{m_{\text{max}}}(K)$ where $K = \max\{K_i\}$, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, the low-gain feedback (11) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $W$ for any $\kappa_i \in [0, K_i]$, $i = 1, ..., m$.

**Proof of Theorem 3:** The closed-loop system can be written as

$$x(k) = Ax(k) + \sum_{i=1}^{m} B_i \sigma(F_i x(k - \kappa_i))$$  (26)

Since $K_i < \frac{\pi}{2m_{\text{max}}}$, the local Lyapunov stability of the origin for sufficiently small $\epsilon$ follows from Theorem 1, that is, there exists $\epsilon_1 \in (0, 1]$ such that for $\epsilon \in (0, \epsilon_1]$, the origin of (26) is locally stable.

It remains to show the attractivity. It suffices to prove that for system (18) with initial condition in $W$, there exists $\epsilon_2 \leq \epsilon_1$ such that for $\epsilon \in (0, \epsilon_2]$, we shall have that

$$\|F_k x(k - K)\| \leq 1, \forall k \geq 0.$$  

This will imply that for system (26) no saturation will be active for all $k \geq 0$, and hence, the system is linear and stable for $\epsilon \leq \epsilon_1$. This will complete the proof.

Define two linear time invariant operators $g_\epsilon$ and $\delta$ with the following transfer matrices:

$$G_\epsilon(z) = F_\epsilon(zI - A - BF_\epsilon)^{-1} B$$

$$\Delta(z) = I - D(z) = \text{diag}(1 - z^{-\kappa_i})_{i=1}^{m}.$$  

Note that the operators $g_\epsilon$ and $\delta$ have zero initial conditions. From the proof of Theorem 1, we know that (19) is satisfied which guarantees that there exists a $\mu > 0$ such that

$$\sigma(I + G_\epsilon(z)\Delta(z)) > \mu, \quad \forall z \in \mathbb{C}^0, \forall K_i \in [0, K_i]$$

for all $\epsilon \leq \epsilon_1$ and this $\mu$ only depends on $K_i$ provided that $\epsilon \leq \epsilon_1$. This implies that

$$\|(I + G_\epsilon(z)\Delta(z))^{-1}\| \leq \frac{1}{\mu}.$$  

Note that for $k \geq 0$

$$x(k + 1) = (A + BF_\epsilon)x(k) - \delta(F_\epsilon x)(k) + Bu_\epsilon(k),$$

where

$$u_\epsilon(k) = \begin{bmatrix} v_1(k) \\ \vdots \\ v_m(k) \end{bmatrix}, \quad v_i(k) = \begin{cases} F_\epsilon \phi(k - \tau_i), & k < \kappa_i, \\ 0, & k \geq \kappa_i. \end{cases}$$

Since $u_\epsilon(k)$ vanishes for $k \geq K$, $\phi \in W$ is bounded and $F_\epsilon \to 0$, we have for any $\phi \in W$, $\|v_\epsilon\| \to 0$ and $\|v_\epsilon\| \to 0$ as $\epsilon \to 0$.

We have

$$F_\epsilon x(k) = F_\epsilon (A + BF_\epsilon)^k x(0) - (g_\epsilon \circ \delta)(F_\epsilon x)(k) + g_\epsilon (u_\epsilon)(k),$$

and hence

$$F_\epsilon x(k) = (1 + g_\epsilon \circ \delta)^{-1} \left[ F_\epsilon (A + BF_\epsilon)^k x(0) + g_\epsilon (u_\epsilon)(k) \right].$$  

(27)

Let $w_\epsilon(k) = g_\epsilon (u_\epsilon)(k)$. By the definition of $g_\epsilon$, we have

$$\|w_\epsilon\| \leq \|G_\epsilon(z)\| \|v_\epsilon\| \leq (1 + \eta) \|v_\epsilon\|$$

where $\eta$ is given by (15). Hence for any given initial condition $\phi$, $\|w_\epsilon\| \to 0$ as $\epsilon \to 0$. Then from (27), we get

$$\|F_\epsilon x(k)\| \leq \|F_\epsilon (A + BF_\epsilon)^k x(0)\| + \|(1 + G_\epsilon(z)\Delta(z))^{-1}\| \|w_\epsilon\|$$

$$\leq \frac{1}{\mu} \|F_\epsilon (A + BF_\epsilon)^k x(0)\| + \frac{1}{\mu} \|w_\epsilon\|.$$  

Since, by (10), $\|F_\epsilon (A + BF_\epsilon)^k x(0)\| \to 0$ and $u_\epsilon \to 0$ as $\epsilon \to 0$ and $\mu$ is independent of $\epsilon$ (provided $\epsilon$ is smaller than $\epsilon_1$), there exists an $\epsilon_2$ such that $\|F_\epsilon x\| \leq 1$ for $\epsilon \in (0, \epsilon_1]$ and $\phi \in W$. This implies that $\|F_\epsilon x(k)\| \leq \|F_\epsilon x\| \leq 1$ for $k \geq 0$. At last, since $\phi \in W$, there exists $\epsilon^* \leq \epsilon_2$ such that $\|F_\epsilon x(k)\| \leq 1$ for $k \geq -K$.

The next theorem solves Problem 2.

**Theorem 4:** Consider the system (1). Let Assumption 1 hold. The semi-global asymptotic stabilization via measurement feedback problem can be solved by the low-gain compensator (16). Specifically, for any a priori given compact set of initial conditions $W \subset \ell^2_{m_{\text{max}}}(K)$ and a set of positive integers $K_i < \frac{\pi}{2m_{\text{max}}}$, $i = 1, ..., m$, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, the origin of the closed-loop system of (1) and (16) is local asymptotic stable for any $K_i \in [0, K_i]$, $i = 1, ..., m$ with the domain of attraction containing $W$.

**Corollary 4:** Consider the system (1). Let Assumption 1 hold and $A$ has all its eigenvalues equal to 1. For any a priori given compact set of initial conditions $W \subset \ell^2_{m_{\text{max}}}(K)$ and any given set of positive integers $K_i$, $i = 1, ..., m$, there exists an $\epsilon^*$ such that for any $\epsilon \in (0, \epsilon^*)$, the origin of the closed-loop system of (1) and (16) is local asymptotic stable for any $K_i \in [0, K_i]$, $i = 1, ..., m$ with the domain of attraction containing $W$. 

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Proof of Theorem 4: The closed-loop system can be written as

\[
\begin{align*}
    x(k+1) &= Ax(k) + \sum_{i=1}^{m} B_i \sigma(F_i x(k - k_i)) \\
    \chi(k+1) &= (A + BF_0 + K \chi(k) - KCx(k) \\
    x(\theta) &= \phi(\theta), \forall \theta \in [-K_1, 0] \\
    \chi(\theta) &= \psi(\theta), \forall \theta \in [-K_1, 0].
\end{align*}
\]  

(28)

Suppose $K_1$’s satisfy the bound $K_1 < \frac{\pi}{\lambda_{\max}}$. By Theorem 2, there exists an $\varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_1]$, the closed-loop system without saturation is asymptotically stable. Then the local stability of (28) for $\varepsilon \leq \varepsilon_1$ follows.

Define two linear time invariant operators $G_\varepsilon^m$ and $\delta$ with $z$ transform

\[
G_\varepsilon^m(z) = -F_\varepsilon(zI - A - BF_\varepsilon)^{-1} KC(zI - A - KC)^{-1} B
\]

\[
\Delta(z) = I - D(z) = \text{diag}(1 - z^{-k_i})_{i=1}^m.
\]

From the proof of Theorem 2, we know that (25) holds for $\varepsilon \leq \varepsilon_1$. There exists a $\mu > 0$ such that

\[
\sigma(I + G_\varepsilon^m(z) \Delta(z)) > \mu, \forall z \in \mathbb{C}, \forall K_1 \in [0, K_1].
\]  

(29)

where $\mu$ is independent of $\varepsilon$ provided that $\varepsilon \leq \varepsilon_1$. It follows from Lemma 7 that $G_\varepsilon^m(z) \to G_\varepsilon(z)$ uniformly on $\mathbb{C}$ such that $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1} B$. Since $\|G_\varepsilon\|_{\infty} \leq 1 + \eta$ for any $\varepsilon \in (0, 1]$ with $\eta$ given by (15), there exists an $\varepsilon_2$ such that

\[
\|G_\varepsilon\|_{\infty} \leq 2(1 + \eta).
\]  

(30)

Given (29), (30) and Lemma 5 hold, we can use exactly the same argument as in the proof of Theorem 3 to prove that there exists an $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*)$,

\[
\|F_\varepsilon x(k - K)\| \leq 1, \forall k \geq 0, (\phi, \psi) \in W.
\]

V. Conclusion

In this paper, the semi-global stabilization problems for general uncritically unstable systems subject to input saturation and multiple unknown delays are solved. We propose upper bounds on delays based on a frequency-domain stability criterion for linear discrete time-delay system and constructed a low-gain state feedback and compensator to achieve the semi-global stabilization with feasible delays.

REFERENCES