Abstract—This paper proposes an optimal control algorithm for a polynomial system with a quadratic criterion over infinite horizon. The designed regulator gives a closed form solution to the infinite horizon optimal control problem for a polynomial system with a quadratic criterion. The obtained solution consists in a feedback control law obtained by solving a Riccati algebraic equation dependent on the state. Numerical simulations in the example show advantages of the developed algorithm.

I. INTRODUCTION

The optimal control was initiated in the 50’s and 60’s, and since then it has been applied to solving different types of problems. The optimal regulator was obtained (see, for example, [11]–[6]) for linear systems with finite and infinite horizons, as well as stochastic systems ([7]–[11]). Nevertheless, the optimal regulator problem for nonlinear systems is still open, since only computational approaches have been proposed for such problems ([12]–[22]). To the best of the authors’ knowledge, the infinite-time optimal regulator has not been developed in a closed form for a polynomial system with a quadratic criterion.

This paper proposes an optimal control algorithm for a polynomial system with a quadratic criterion over infinite horizon. The designed regulator gives a closed form solution to the infinite horizon optimal control problem for a polynomial system and a quadratic criterion. The obtained solution consists in a feedback control law obtained by solving a Riccati algebraic equation dependent on the state. The appendix contains the proofs.

The paper then presents an illustrative example showing an advantage of the designed optimal regulator, compared to the control obtained by a linear approximation of a polynomial system.

The organization of the paper is given by: The infinite-time optimal control problem is stated for a polynomial system with a quadratic criterion in Section 2. In Section 3, the optimal regulator is obtained using a co-state equation of the maximum principle, and, in Section 4, the optimal regulator is obtained by means of the Hamilton-Jacobi-Bellman equation.

II. PROBLEM STATEMENT

The dynamic system state \(z(s)\) is given by a nonlinear differential equation

\[
\frac{dz(s)}{ds} = h(z, s) + G(s)v(s), \quad z(s_0) = z_0;
\]

(1)

Here, \(h(z, s)\) is a nonlinear polynomial function, \(v(s) \in R^n\) is the control input, \(z(s) \in R^n\) is the state vector. The coefficient \(G(s)\) is a matrix function of dimension \(n \times m\).

The vector function \(h(z, s) \in R^n\) is a polynomial of \(n\) components of the state vector \(z\), with time-dependent coefficients (see [23] for definition). Following [23], a \(r\)-degree polynomial of a vector is defined as a \(r\)-linear form of its components

\[
h(z, s) = \beta_0(s) + \beta_1(s)z + \beta_2(s)zz^T + \ldots + \beta_r(s)z \times \ldots \times z,
\]

where \(\beta_i, i = 1, \ldots, r,\) is a tensor of dimension \(n \times \ldots \times r \times n,\) and \(z \times \ldots \times z\) is a tensor of dimension \(n \times \ldots \times n\). Each component of the polynomial can be represented as a summation

\[
h_i(z, s) = \beta_0(s) + \sum_j \beta_{1j}(s)z_j(s) + \ldots + \sum_{j,k} \beta_{2jk}(s)z_j(s)z_k(s) + \ldots + \sum_{i_1, \ldots, i_r} \beta_{r_{i_1 \ldots i_r}}(s)z_{i_1}(s) \ldots z_{i_r}(s),
\]

\(l, j, k, i_1 \ldots i_r = 1, \ldots, n.\)

The criterion \(J\) is given by the following quadratic functional of \(z\) and \(v\)

\[
J(u) = \frac{1}{2} \int_{s_0}^{s} v^T(s)R(s)v(s)ds + \frac{1}{2} \int_{s_0}^{s} z^T(s)L(s)z(s)ds,
\]

(2)

where \(R\) and \(L\) are symmetric matrices, with \(R > 0\) and \(L \geq 0,\) and \(a^T\) denotes transpose to a vector (matrix) \(a.\)

The optimal control problem is to minimize the criterion \(J\) along trajectory \(z^*(s), s \in [s_0, \infty]\) through the selection of control \(v^*(s), s \in [s_0, \infty],\) which is generated upon substituting \(v^*(s)\) into the state equation (1).

III. CONTROL DESIGN. I. MAXIMUM PRINCIPLE

A. Optimal control problem solution 1.

The following theorem gives the solution to the optimal control problem stated above.

Theorem 1. The control law

\[
v^*(s) = -R^{-1}(s)G^T(s)m(z(s))
\]

(3)
presents the optimal regulator for the polynomial system (1) with respect to the quadratic criterion (2), where the matrix function \( m(z) \) is the solution of the equation

\[
\frac{dm(z)}{dz}(\beta_1(s)z + \beta_2(s)zz^T + \ldots + \beta_r(s)z \cdots r \times \text{times} z - G(s)R^{-1}(s)G^T(s)m(z)) = 0
\]

where \( G(s)R^{-1}(s)G^T(s) \) is the solution of the differential equation

\[
Lz - (\beta_1(s) + 2\beta_2(s)z + \ldots + r\beta_r(s)z \cdots r \times \text{times} z)^Tm(z)
\]

with the initial condition \( m(0) = 0 \).

Substituting the optimal control (3) in to the polynomial system (1) yields the optimally controlled state governed by the equation

\[
\frac{dz(s)}{ds} = h(z,s) - G(s)R^{-1}(s)G^T(s)m(z), \quad z(s_0) = z_0.
\]

**B. Example I.**

Consider a quadratic scalar state equation

\[
\dot{z}(s) = z^2(s) + v(s), \quad z(0) = z_0.
\]

The control law is to minimize

\[
J = \frac{1}{2} \int_0^\infty v^2(s)ds \quad \text{through the selection of control } v(s), t \in [0, \infty), \text{ i.e., using the minimal overall energy of control } v \text{ so that it minimizes the overall energy of the state } z.
\]

The control law used in this example is calculated as (3) where the matrix \( m(z) \) is the solution of the differential equation (4), that is \( v(s) = -R^{-1}(s)G^T(s)m(z) \) (see Subsection 2.1). From equations (6) and (7), we see that \( G(s) = 1 \) and \( R(s) = 1 \), therefore, the control takes the form

\[
v(s) = -m(z(s));
\]

where \( m(z) \) satisfies the equation

\[
\frac{dm(z)}{dz}(z^2 - m(z)) = -z - 2zm(z)
\]

with the initial condition \( m(0) = 0 \). The solution of the differential equation \( m(z) \) is given by

\[
m(z) = z^2 + z\sqrt{z^2 + 1},
\]

hence, the form of control is

\[
v(s) = -z^2 - z\sqrt{z^2 + 1},
\]

and the state equation (6) takes the form

\[
\dot{z}(s) = z^2(s) - z^2(s) - z(s)\sqrt{z^2(s) + 1}, \quad z(0) = z_0.
\]

Consider the initial condition \( z(0) = 1 \) for the system (6), controlled by (10)–(11), to perform numerical simulations.

Fig. 1 shows the results of applying the regulator (10)–(11), which displays the state graph \( z(s) \), controlled by (10), the control function (10) \( v(s) \), and the criterion graph \( J(s) \) in the interval \([0, 20]\). At the final moment \( T = 20 \), the criterion (7) takes the value \( J(20) = 0.9428 \).

The designed regulator (10)–(11) is compared to the linear regulator for the linearized system

\[
\dot{z}(s) = 2z(s) + v(s), \quad z(0) = z_0.
\]

The control law is given by

\[
v(s) = -M(s)z(s),
\]

where \( M(s) \) satisfies the Riccati algebraic equation for the linearized system

\[
0 = 1 + 4M(s) - M(s)^2.
\]

The solution of the Riccati algebraic equation is given by

\[
M = 2 + \sqrt{5},
\]

hence, the form of control is

\[
v(s) = -(2 + \sqrt{5})z,
\]

and the state equation (6) takes the form

\[
\dot{z}(s) = z^2(s) - (2 + \sqrt{5})z(s), \quad z(0) = z_0.
\]

Consider again the initial condition \( z(0) = 1 \) for the system (6), controlled by (14)–(17), to perform numerical simulations.

Fig. 1 shows the results of applying the regulator (14)–(17), which displays the state graph \( z(s) \), controlled by (17), the control function (17) \( v(s) \), and the criterion graph \( J(s) \) in the interval \([0, 20]\). At the final moment \( T = 20 \), the criterion (7) takes the value \( J(20) = 1.3266 \).

**IV. CONTROL DESIGN. II.**

HAMILTON-JACOBI-BELLMAN EQUATION

A. Optimal control problem statement II.

The following theorem gives another solution to the optimal control problem stated in Section 1.

**Theorem 2.** The control law

\[
v^*(s) = R^{-1}(s)G^T(s)M(s)z(s),
\]

presents the optimal regulator for the polynomial system (1) with respect to the quadratic criterion (2), where the matrix function \( M(s) \) is the solution of the Riccati algebraic equation

\[
0 = L(s) - [\beta_1(s) + \beta_2(s)z(s) + \beta_3(s)z(s)z^T(s) + \ldots + \beta_r(s)z(s) \cdots r \times \text{times} z(s)]M(s) - M(s)[\beta_1(s) + \beta_2(s)z(s) + \beta_3(s)z(s)z^T(s) + \ldots + \beta_r(s)z(s) \cdots r \times \text{times} z(s)] - M(s)G(s)R^{-1}(s)G^T(s)M(s).
\]

Substituting the optimal control (3) into the polynomial system (1) yields the optimally controlled state governed by the equation

\[
\frac{dz(s)}{ds} = h(z,s) - G(s)R^{-1}(s)G^T(s)m(z), \quad z(s_0) = z_0.
\]

**Remark 1.** Note that the "curl" condition, discussed in [24], holds, since the nonlinear function \( h(z,s) \) in (1) is a specially defined polynomial.
B. Example II.

Consider a quadratic scalar state equation

\[ \dot{z}(s) = z^2(s) + v(s), \quad z(0) = z_0. \]  \hfill (22)

The control problem is to minimize

\[ J = \frac{1}{2} \int_0^T v^2(s)ds + \frac{1}{2} \int_0^T z^2(s)ds. \]  \hfill (23)

through the selection of control \( v(s), \ t \in [0, \infty) \).

The control law used in this example is calculated as \( (19) \), where the matrix \( M(s) \) is the solution of the Riccati algebraic equation \( (20) \), that is, \( v^*(s) = -R^{-1}(s)G^T(s)M(s)z(s) \) (see Subsection 3.1). From equations \( (22) \) and \( (23) \), we see that \( G(s) = 1 \) and \( R(s) = 1 \), therefore, the control takes the form

\[ v(s) = -M(s)z(s), \]  \hfill (24)

where \( M(s) \) satisfies the Riccati algebraic equation

\[ 0 = 1 - z^T(s)M(s) - M(s)z(s) - M(s)M(s). \]  \hfill (25)

The solution of the Riccati algebraic equation is given by

\[ M(s) = z(s) + \sqrt{z^2(s) + 1}, \]  \hfill (26)

hence, the form of control is

\[ v(s) = -(z(s) + \sqrt{z^2(s) + 1})z, \]  \hfill (27)

and the state equation \( (22) \) takes the form

\[ \dot{z}(s) = z^2(s) - z^2(s) - z(s)\sqrt{z^2(s) + 1}, \quad z(0) = z_0. \]  \hfill (28)

Consider the initial condition \( z(0) = 1 \) for the system \( (22) \), controlled by \( (26) - (27) \), to perform numerical simulations.

Fig. 2 shows the results of applying the regulator \( (26) - (27) \), which displays the state graph \( (22) z(s) \), controlled by \( (27) \), the control function \( (27) v(s) \), and the criterion graph \( (23) J(s) \) in the interval \( [0, 20] \). At the final moment \( T = 20 \), the criterion \( (23) \) takes the value \( J(20) = 0.9428 \).

As in Example 1, this example is compared with the linear feedback control for the linearized system given by

\[ v(s) = -(2 + \sqrt{5})z(s), \]  \hfill (29)

and the state equation \( (16) \) takes the form

\[ \dot{z}(s) = z^2(s) - (2 + \sqrt{5})z(s), \quad z(0) = z_0. \]  \hfill (30)

Remark 2. As observed, the control obtained by the algebraic Riccati equation coincides with the control obtained from the co-state equation.
Consider two-dimensional quadratic state equations

\[ \dot{z}_1(s) = z_2(s), \quad z_1(0) = z_{10}, \]  
\[ \dot{z}_2(s) = z_1^2(s) + v(s), \quad z_2(0) = z_{20}. \]  

The control problem is to minimize

\[ J = \frac{1}{2} \int_0^\infty v^2(s)ds + \frac{1}{2} \int_0^\infty (z_1^2(s) + z_2^2(s))ds. \]

through the selection of control \( v(s), t \in [0, \infty] \).

The control law used in this example is calculated as (19), where the matrix \( M(s) \) is the solution of the Riccati algebraic equation (20), that is, \( v^*(s) = -R^{-1}(s)G^T(s)M(s)z(s) \) (see Subsection 3.1). From equations (31) and (32), we see that \( G(s) = [0 \quad 1]^T \) and \( R(s) = 1 \), therefore, the control takes the form

\[ v(s) = -M_{21}(s)z_1(s) - M_{22}(s)z_2(s), \]  

where \( M(s) \) satisfies the Riccati algebraic equations

\[ 0 = 1 - z_1(s)^T M_{12}(s) - M_{21}(s)z_1(s) - M_{12}(s)M_{21}(s), \]  
\[ 0 = -z_1(s)^T M_{22}(s) - M_{11}(s) - M_{12}(s)M_{22}(s), \]  
\[ 0 = -M_{11}(s) - M_{22}(s)z_1(s) - M_{22}(s)M_{21}(s), \]  
\[ 0 = 1 - M_{21}(s) - M_{12}(s) - M_{22}(s)M_{22}(s). \]

The solution to the Riccati algebraic equations is given by

\[ M_{11}(s) = \sqrt{2z_1(s) + 1 + 2\sqrt{z_1^2(s) + 1}}, \]  
\[ M_{12}(s) = z_1(s) + \sqrt{z_1^2(s) + 1}, \]  
\[ M_{21}(s) = z_1(s) + \sqrt{z_1^2(s) + 1}, \]  
\[ M_{22}(s) = \sqrt{2z_1(s) + 1 + 2\sqrt{z_1^2(s) + 1}}, \]

the form of control is

\[ v(s) = -(z_1(s) + \sqrt{z_1^2(s) + 1})z_1(s) \]  
\[ -(\sqrt{2z_1(s) + 1 + 2\sqrt{z_1^2(s) + 1}}z_2(s), \]  

and the state equations (31) and (32) take the form

\[ \dot{z}_1(s) = z_2(s), \quad z_1(0) = z_{10}, \]  
\[ \dot{z}_2(s) = z_1^2(s) - (z_1(s) + \sqrt{z_1^2(s) + 1})z_1(s) \]  
\[ -(\sqrt{2z_1(s) + 1 + 2\sqrt{z_1^2(s) + 1}}z_2(s), \quad z_2(0) = z_{20}. \]

Consider the initial conditions \( z_1(0) = 1 \) and \( z_2(0) = 1 \) for the system (31)–(32), controlled by (26)–(27), to perform numerical simulations.

Fig. 3 shows the results of applying the regulator (39)–(43) to the system (31),(32), which displays the state graphs (31) \( z_1(s) \) and (32) \( z_2(s) \), controlled by (43), the control function.
(43) \( v(s) \), and the criterion graph (33) \( J(s) \) in the interval \([0, 20]\). At the final moment \( T = 20 \), the criterion (33) takes the value \( J(20) = 5.5459 \).

The designed regulator (39)–(43) is compared to the linear regulator for the linearized system
\[
\begin{align*}
\dot{z}_1(s) & = z_2(s), \quad z_1(0) = z_{10}, \quad (46) \\
\dot{z}_2(s) & = 2z_1(s) + v(s), \quad z_2(0) = z_{20}. \quad (47)
\end{align*}
\]

The linear control law is given by
\[
v(s) = -M(s)z(s), \quad (48)
\]
where \( M(s) \) satisfies the Riccati algebraic equation for the linearized system
\[
\begin{align*}
0 & = 1 - 2M_{12}(s) - 2M_{21}(s) - M_{12}(s)M_{21}(s), \quad (49) \\
0 & = -2M_{22}(s) - M_{11}(s) - M_{12}(s)M_{22}(s), \quad (50) \\
0 & = -M_{11}(s) - 2M_{22}(s) - M_{22}(s)M_{21}(s), \quad (51) \\
0 & = 1 - M_{21}(s) - M_{12}(s) - M_{22}(s)M_{22}(s). \quad (52)
\end{align*}
\]

The solution of the Riccati algebraic equation is given by
\[
\begin{align*}
M_{11}(s) & = \sqrt{5 + \sqrt{5} \sqrt{5}}, \quad (53) \\
M_{12}(s) & = 2 + \sqrt{5}, \quad (54) \\
M_{21}(s) & = 2 + \sqrt{5}, \quad (55) \\
M_{22}(s) & = \sqrt{5 + \sqrt{5}}, \quad (56)
\end{align*}
\]
the form of control is
\[
v(s) = -(2 + \sqrt{5})z_1(s) - (\sqrt{5 + \sqrt{5}})z_2(s), \quad (57)
\]
and the state equation (31) and (32) takes the form
\[
\begin{align*}
\dot{z}_1(s) & = z_2(s), \quad z_1(0) = z_{10}, \quad (58) \\
\dot{z}_2(s) & = z_1^2(s) - (z_1(s) + \sqrt{z_1^2(s) + 1})z_1(s) \\
& - (\sqrt{2z_1(s) + 1 + 2\sqrt{z_1(s) + 1}})z_2(s), \quad z_2(0) = z_{20}. \quad (59)
\end{align*}
\]

Consider the initial conditions \( z_1(0) = 1 \) and \( z_2(0) = 1 \) for the system (58) and (59), controlled by (53)–(57), to perform numerical simulations.

Fig. 3 shows the results of applying the regulator (48)–(57) to the system (58),(59), which displays the state graphs (58) \( z_1(s) \) and (59) \( z_2(s) \), controlled by (57), the control function (57) \( v(s) \), and the criterion graph (33) \( J(s) \) in the interval \([0, 20]\). At the final moment \( T = 20 \), the criterion (33) takes the value \( J(20) = 6.9481 \).
V. CONCLUSIONS

The optimal regulator has been designed for polynomial systems and a quadratic criterion over infinite horizon, using the Hamilton-Jacobi-Bellman differential equation. The obtained solution consists in a feedback control law obtained by solving the Riccati algebraic equation dependent on the state. The obtained theoretical results have been numerically verified in an illustrative example. The simulation results show an advantage for the designed feedback control compared to the control obtained by linearization.

REFERENCES