Disturbance Propagation Analysis in Vehicle Formations: an Information–Theoretic Approach

Yingbo Zhao, Paolo Minero, and Vijay Gupta

Abstract—We consider the problem of disturbance propagation in a control system where a group of vehicles aims to move in a formation with a tree topology. The control law at every vehicle depends on its position error with respect to its parent vehicle in the tree as well as on coded information transmitted by other vehicles across side communication channels. A lower bound on the integral of the log sensitivity function of the position errors for any vehicle with respect to a stochastic disturbance acting on the lead vehicle is presented. The effect of the side information channels is illustrated through some examples. It is also shown that in some cases the lower bound is achievable through appropriate design of the controllers and encoder/decoder pairs.

I. INTRODUCTION

Formation control has received significant interest over the past few decades [1], [8], [11], [17], [18]. The task of formation control is to arrange multiple autonomous agents (e.g., vehicles) in a specified geometric structure. Applications of formation control include formations of unmanned aerial vehicles [4], [6], [22], teams of multiple mobile robots [9], [16] and automated highway systems [13], [14].

In this paper, we consider a leader-follower formation where there is one independent leader that all the other agents aim at following, either directly or indirectly. It is assumed that there is a specific predecessor assigned to each follower. In other words, the formation structure graph is a directed tree (see the solid lines in Fig. 1). We concentrate on the class of problems where the movement of the formation is one dimensional and the agents in the formation are dynamically decoupled, meaning that the action of one agent does not affect the dynamics of other agents directly.

Even in the simple case where the formation is a string, a small disturbance affecting a certain agent in the formation can propagate and amplify to affect the other agents [2], [23]. A typical result (see e.g. [21]) considers a lower bound on the peak of the frequency response magnitude of the transfer function from a deterministic disturbance at the leader vehicle to the states of other vehicles in the string, where individual controllers have access to information from a specified set of vehicles. Such a lower bound reflects the worst case disturbance amplification under specified information constraint.

The setup that we consider in this paper assumes that there is a stochastic disturbance affecting the leader and the performance metric of interest is the log sensitivity integral [12]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) \, d\omega, \quad (1)$$

where $S_{d,e_i}(\omega)$ represents the ratio of the power spectral densities of the tracking error at the $i$-th vehicle and the disturbance at the leader. Note that as opposed to most literature in string stability, we consider power spectral densities since we assume the disturbance to be stochastic. For a single vehicle (e.g., just the leader), the performance metric in (1) has been shown to be subject to a “Bode-like” fundamental limitation [19]:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e}(\omega) \, d\omega \geq \sum_{\lambda \in \mathcal{U}(A)} \log |\lambda|, \quad (2)$$

where $\mathcal{U}(A)$ denotes the set of unstable poles of the open-loop system transfer function. The inequality in (2) holds regardless of the control law used by the system, which implies that the effect of the disturbance on the error cannot be arbitrarily reduced, no matter what control law is used.

There does not seem to be much work on Bode integral formula for distributed control systems (as against multi-input multi-output (MIMO) systems, but still centralized control systems). One obvious bound can be obtained by considering the corresponding centralized control problem; however, that bound does not lead to any insight on the role of the communication graph.

In our earlier work [25], [26], we showed that for a string with the “predecessor-following” information flow topology (i.e., where there is no communication between non-adjacent agents), the log sensitivity integral for the $i$-th agent in the
platoon satisfies
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,1}(\omega) d\omega \geq \sum_{\lambda \in U(A_i)} \log |\lambda| + \sum_{l=0}^{i-1} \bar{\Lambda}_l, \]
where the term $\bar{\Lambda}_l$ is a function of the stationary open-loop gains of the predecessors of the $i$-th vehicle (for the exact definition, see (10)). Note that the lower bound for the disturbance propagation performance for the $i$-th vehicle is independent of the control law of the $i$-th vehicle itself, which makes it a “Bode-like” fundamental limitation.

However, the communication graph considered in [25], [26] is limited by the requirement that every agent in the formation has only its immediate predecessor as its in-neighbor, as depicted by the solid lines in Fig. 1. In [24] a more general communication graph was considered by allowing the leader to be an in-neighbor for any other agent.

In this paper, we further extend the result in two ways: (i) we allow arbitrary tree topologies as the desired formation, and (ii) we allow an arbitrary communication graph.

The main result of this paper is a lower bound to the integral of the log sensitivity function from the disturbance acting on the leader to the position error of any agent. This bound holds for a class of nonlinear controllers that includes, in particular, all LTI controllers. Similar to the classical Bode integral formula [5], the derived bound depends on the open-loop unstable poles but is independent of the controller chosen at the agent. Unlike in the centralized SISO case, however, here the bound also depends on the plants and controllers of other agents in the formation. Furthermore, the lower bound characterizes the effect of the communication graph. In particular, the lower bound depends on how much side information that the agent receives from its in-neighbors, and whether the side information is delayed.

The rest of the paper is organized as follows. The problem formulation is stated in Section III. Some preliminary results for deriving the main results are provided in Section IV. Section V presents the main result of the paper. A few special cases to illustrate the main result are listed in Section VI. Section VII concludes the paper.

Notation: Throughout the paper, we denote random variables using boldface letters. For any $k \leq j$ we use the notation $x^k_j = (x(k), x(k+1), \ldots, x(j))$ to denote finite segment of a sequence $x(1), x(2), \ldots$ and we omit the subscript $k$ when it is equal to 1.

II. REVIEW OF BASIC INFORMATION THEORY

Throughout this paper, we follow the notation in [10]. The differential entropy $h(x^k)$ of a continuous random vector $x^k$ with probability density function $f(x^k)$ (in short $x^k \sim f(x^k)$) is defined as
\[ h(x^k) \triangleq - \int f(x^k) \log f(x^k) dx^k = -\mathbb{E}[\log f(x^k)]. \]
Let $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be a continuously differentiable bijective function. Then,
\[ h(\psi(x^k)) = h(x^k) + \mathbb{E}[\log |\det(J)|], \]
where $J$ is the Jacobian matrix of $\psi$ with respect to $x^k$. The maximum entropy theorem states that if $x \sim f(x)$ has covariance matrix $K \succ 0$, then
\[ h(x) \leq \frac{1}{2} \log((2\pi e)^k |K|) \]
with equality if and only if $x$ is Gaussian distributed.

Let $x^k \sim f(x^k)$ and $y^k | \{x^k = x^k\} \sim f(y^k | x^k)$. Then the conditional differential entropy of $y^k$ given $x^k$ is defined as
\[ h(y^k | x^k) \triangleq -\mathbb{E}_x f(x^k | y^k) \log f(y^k | x^k)]. \]

The mutual information $I(x^k; y^k)$ between continuous random vectors $(x^k, y^k) \sim f(x^k, y^k)$ is defined as
\[ I(x^k; y^k) = \int f(x^k, y^k) \log \frac{f(x^k, y^k)}{f(x^k) f(y^k)} dx^k dy^k. \]
Let $z^k \sim f(z^k)$ and $(x^k, y^k) \{z^k = z^k\} \sim f(x^k, y^k | z^k)$. Denote the mutual information between $x^k$ and $y^k$ given $\{z^k = z^k\}$ by $I(x^k; y^k | z^k = z^k)$. Then, the conditional mutual information $I(x^k; y^k | z^k)$ between $x^k$ and $y^k$ given $z^k$ is defined as
\[ I(x^k; y^k | z^k) = \int I(x^k; y^k | z^k = z^k) f(z^k) dz^k. \]
The chain rule of differential entropy states that $h(x^k) = \sum_{i=1}^k h(x_i | x^{i-1})$ and implies the chain rule of mutual information $I(x^k; y^k) = \sum_{i=1}^k I(x_i; y^k | x^{i-1})$.

The above definitions can be extended to stationary stochastic processes. The differential entropy rate $h(x)$ of a stationary continuous-valued stochastic process $x = \{x_i\}_{i=1}^\infty$ is defined as
\[ h(x) \triangleq \lim_{n \rightarrow \infty} \frac{h(x^n)}{n} = \lim_{n \rightarrow \infty} h(x_n | x^{n-1}). \]
If $x$ with $x(k) \in \mathbb{R}^k$ has power spectral density $\Phi_x(\omega)$, then
\[ h(x) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \log ((2\pi e)^k |\Phi_x(\omega)|) d\omega \]
with equality if and only if $x$ is a stationary Gaussian process.

The mutual information rate $I(x; y)$ between two stationary continuous-valued stochastic processes $x$ and $Y$ is defined as
\[ I(x; y) \triangleq \lim_{n \rightarrow \infty} \frac{I(x^n; y^n)}{n}. \]
The readers are referred to [7, Chapter 2] for more details.

III. PROBLEM FORMULATION

Consider a formation of $n+1$ agents with a single independent leader. Every agent is modeled as a single-input single-output (SISO) system, whose dynamics are given by
\[ x_i(k+1) = A_i x_i(k) + B_i u_i(k), \]
\[ y_i(k) = C_i x_i(k), \quad i = 0, \ldots, n, \]
where $x_i(k) \in \mathbb{R}^{n_i}$ is the system state, $y_i(k) \in \mathbb{R}$ is the system output, $u_i(k) \in \mathbb{R}$ is the control input, and $(A_i, C_i)$ is an observable pair. Without loss of generality, denote the leader as agent 0. The leader of the formation aims at following a deterministic reference command signal.
while the objective of the followers is to keep a constant distance with respect to a designated target agent in the formation. For every agent \( i \), where \( 1 \leq i \leq n \), we assign another agent \( j \in \{0, \ldots, n\} \setminus \{i\} \) as its predecessor and a corresponding constant spacing \( \delta_i \). The \( i \)-th agent regulates its output \( y_i(k) \) to satisfy \( y_i(k) = r_i(k) + \delta_i \), where \( r_i(k) \) is the reference input for agent \( i \). The leader regulates its output \( y_0(k) \) to satisfy \( y_0(k) = r_0(k) \). The tracking errors are defined as

\[
e_0(k) = r_0(k) - y_0(k) + d(k),
\]

\[
e_i(k) = r_i(k) - y_i(k) + \delta_i, \quad i = 1, \ldots, n,
\]

where \( d \) is a stochastic disturbance process at the input of the leader’s controller.

A communication graph is also specified for the formation that assigns to every agent a set of other agents (its in-neighbors) that it can receive information from. Thus, every agent \( i \) receives the signal \( Z_i(k) \) at time \( k \) from agent \( j \) if \( j \in N_{in}(i) \), where \( N_{in}(i) = \{0, \ldots, n\} \setminus \{i\} \) denotes the set of in-neighbors of agent \( i \). Similarly, every agent \( i \) also transmits information to its out-neighbors, denoted by \( N_{out}(i) \). Fig. 2 depicts the model for an individual vehicle in the formation, where the vectors are defined as

\[
Z_i(k) \triangleq \{Z_{ji}(k)|j \in N_{in}(i)\} \in \mathbb{R}^{|N_{in}(i)|},
\]

\[
X_i(k) \triangleq \{X_{ij}(k)|j \in N_{out}(i)\} \in \mathbb{R}^{|N_{out}(i)|},
\]

and \( D_j, E_j \) represent the decoder and the encoder at agent \( j \), respectively. In particular, \( D_j \) is a mapping \( \mathbb{R}^{|N_{in}(i)|} \times (k+1) \rightarrow \mathbb{R} \) given by \( D_j(k, Z_i^k) = \hat{r}_i(k) \), and \( E_i \) is a mapping \( \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{|N_{out}(i)|} \) given by \( E_i(k, Y_i^k) = X_i(k) \).

Fig. 2. Model of an individual agent in the formation

To design its control input, the controller of every agent \( i \) has access to the error \( e_i \) and the side information \( \hat{r}_i \). The control law of the \( i \)-th agent is given by

\[
u_i(k) = u_{i,k}(\hat{r}_i^k, e_i^k),
\]

for some time-varying and possibly non-linear function \( u_{i,k} : \mathbb{R}^{2(k+1)} \rightarrow \mathbb{R} \).

For simplicity, we assume that the input-output delay of each agent is unity, which means that \( u_i(k) \) does not affect \( y_i(k) \) but affects \( y_i(k+1) \). This assumption implies that \( C_i B_i \neq 0 \). This assumption can be relaxed along the lines of our earlier work [24, 26].

Since the formation structure graph is a directed tree, there exists a unique path from the leader to each follower. Denote the path from the leader to the \( i \)-th follower as \( 0 = i(0) \rightarrow i(1) \rightarrow \cdots \rightarrow i(l_i) = i \), where \( i(j) \in \{1, \ldots, n\} \setminus \{i\} \) for \( j = 1, \ldots, l_i - 1 \) represents the agents in the path between the leader and the \( i \)-th follower and \( l_i \) represents the length of the path, i.e., the number of followers in the path.

If for all \( k \in \mathbb{Z}^+ \) and \( m \leq k \),

\[
\sup_{\{u_j, D_j, E_j\}_{j=0}^m} \left( I(\tilde{r}_i^m; r_i(k)|r_i^{k-1}) \right) = 0,
\]

then the side information \( Z_i \) is called delayed side information (DSI) because it provides only historical information about \( r_i \). Equation (6) indicates that if the side information \( Z_i \) is delayed with respect to \( r_i \), then there do not exist any controllers or encoder/decoder pairs such that the decoded message \( \tilde{r}_i \) can provide future knowledge about \( r_i \).

Similarly, if there exist \( k \in \mathbb{Z}^+ \) and \( m \leq k \), such that

\[
\sup_{\{u_j, D_j, E_j\}_{j=0}^m} \left( I(\tilde{r}_i^m; r_i(k)|r_i^{k-1}) \right) > 0,
\]

then the side information \( Z_i \) is called preview side information (PSI).

Suppose that the communication graph, the open-loop dynamics of the agents and the distributions of \( d \) and \( x_i(0) \)’s are all given. Then the mutual information rate \( \bar{I}(r_i; \tilde{r}_i) \) is a function of all the (distributed) control laws. Denote

\[
\sup_{\{u_j, D_j, E_j\}_{j=0}^n} \left( \bar{I}(r_i; \tilde{r}_i) \right) = C_i.
\]

The quantity \( C_i \) characterizes the rate at which information can be reliably transmitted through the side channels to agent \( i \). If \( Z_i \) is DSI, then it is obvious that \( C_i = 0 \). If \( Z_i \) is PSI, then \( C_i \) seems intractable to calculate for a general case. However, there are many important special cases where the value of \( C_i \) is known, some examples in Section VI.

To ensure that all the information-theoretic quantities are well defined, we assume that the control laws (5) are such that the random processes describing the closed-loop dynamics of the vehicles have well defined continuous joint probability density functions and are asymptotically stationary. In addition, we make the following assumptions:

**Assumption 1:** The disturbance \( d \) is a zero-mean stationary Gaussian process with power spectral density \( \Phi_d(\omega) \).

**Assumption 2:** The initial conditions \( x_0(0), \ldots, x_n(0) \) are mutually independent random variables with finite differential entropies and they are all independent of the disturbance process \( d \). Besides, the conditional differential entropy \( h(x_i(0)|r_i, \tilde{r}_i) \) is also finite for \( i = 0, \ldots, n \).

Assumption 2 intuitively indicates that there is still uncertainty left about the initial condition \( x_i(0) \) at the \( i \)-th controller even with the reference input \( r_i \) and side information \( \tilde{r}_i \) known.

Because of the coupling among the vehicles in the formation through their common task, the disturbance acting on the leader propagates downstream and affects other vehicles in the formation. The objective of this paper is to analyze the sensitivity from the disturbance \( d \) to the spacing errors (4), which is defined as follows.

**Definition 1:** The sensitivity function \( S_{x,y}(\omega) \) between two stationary stochastic processes \( x \) and \( y \) with power
spectral densities $\Phi_x(\omega)$ and $\Phi_y(\omega)$, respectively, is defined as

$$S_{x,y}(\omega) \triangleq \sqrt{\frac{\Phi_y(\omega)}{\Phi_x(\omega)}}.$$  \hfill (9)

We refer the reader to [20] for a discussion on the relationship between $S_{x,y}(\omega)$ and the classical sensitivity function.

Assumption 3: All the closed-loop subsystems are mean-square stable, i.e., $\sup_{\omega \in \mathbb{R}} \mathbb{E} \left[ x_T^T(k) x(k) \right] < \infty$.

Assumption 4: For every $(r_i^k, e_i^{k-1}) \in \mathbb{R}^{2k+1}$, the control law $u_{i,k}(r_i^k, e_i^{k-1}, x)$ is a continuously differentiable bijective function of $x \in \mathbb{R}$.

In the sequel, for every $(r_i^k, e_i^{k-1}) \in \mathbb{R}^{2k+1}$, we denote by $u'_{i,k}(r_i^k, e_i^k)$ the partial derivative of $u_{i,k}(r_i^k, e_i^{k-1}, x)$ with respect to its last coordinate as evaluated at $e_i(k)$, i.e.,

$$u'_{i,k}(r_i^k, e_i^k) \triangleq \frac{\partial}{\partial x} u_{i,k}(r_i^k, e_i^k, x) \big|_{x=e_i(k)}.$$  

Notice that $|u'_{i,k}(r_i^k, e_i^k)|$ represents the controller gain of the $i$-th agent at time $k$ and $|C_i B_i|$ represents the gain of the plant of the $i$-th agent. Thus $|C_i B_i u'_{i,k}(r_i^k, e_i^k)|$ characterizes the loop gain. As is well known, a large loop gain is desirable for good tracking performance. Therefore, we assume that $\mathbb{E} \left[ C_i B_i u'_{i,k}(\hat{r}_i, e_i) \right] > 1$ for $i = 0, \ldots, n$. The case where $\mathbb{E} \left[ C_i B_i u'_{i,k}(\hat{r}_i, e_i) \right] \leq 1$ is covered in the discussion following the main result for completeness. Define

$$\bar{\Lambda}_i \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \mathbb{E} \left[ \log |C_i B_i u'_{i,k}(\hat{r}_i, e_i)| \right]$$  \hfill (10)

as the stationary loop gain of agent $i$, which will be used in the main result.

IV. PRELIMINARY RESULTS

In order to prove Theorem 1, we first establish a number of preliminary results.

Lemma 1: Suppose that Assumptions 2 and 3 hold. Then,

$$I(\epsilon_i; x_i(0)|\hat{r}_i) \geq \sum_{\lambda \in U(A_i)} \log |\lambda|.$$  \hfill (11)

If there is no inter-vehicular communication, then (11) reduces to [20, Lemma 4.1] (i.e., $I(\epsilon_i; x_i(0)|\hat{r}_i) \geq \sum_{\lambda \in U(A_i)} \log |\lambda|$). Based on Lemma 1, we give another result, providing a lower bound for $\bar{h}(e_i)$ in the presence of side information $\hat{r}_i$.

Lemma 2: Suppose that Assumptions 2 and 3 hold. Then, for every $i = 0, \ldots, n$, if $Z_i$ is DSI

$$\bar{h}(e_i) \geq \bar{h}(r_i) + \max \left( \sum_{\lambda \in U(A_i)} \log |\lambda| - C_i, 0 \right),$$  \hfill (12)

while if $Z_i$ is PSI

$$\bar{h}(e_i) \geq \bar{h}(r_i) + \sum_{\lambda \in U(A_i)} \log |\lambda| - C_i,$$  \hfill (13)

where $C_i$ is defined in (8).

An important implication of (12) is that the contribution of DSI to the disturbance rejection performance is limited. The reason is that DSI can only help to stabilize the open-loop system but cannot reduce the external innovation $h(r_i(k)|r_i^{k-1})$ (see (6)). In other words, DSI can at most reduce the right hand side of (12) by $\sum_{\lambda \in U(A_i)} \log |\lambda|$. On the other hand, since PSI can reduce $h(r_i(k)|r_i^{k-1})$ and help with stabilization, its effect on the right hand side of (13) is not constrained by the saturation at $\sum_{\lambda \in U(A_i)} \log |\lambda|$.

Based on Lemma 2 we make the following remarks:

1) If the plant of the $i$-th vehicle is stable, then there exists no DSI that can improve upon the disturbance propagation performance given in (1).

2) In general, PSI is more useful than DSI in the sense that it does not suffer from the saturation at $\sum_{\lambda \in U(A_i)} \log |\lambda|$.

Note that the integral of the log sensitivity function (1) can be lower bounded by the difference of the entropy rates $\bar{h}(d)$ and $\bar{h}(e_i)$ of the disturbance and error processes as follows

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( \log \Phi_{e_i}(\omega) - \log \Phi_{d}(\omega) \right) d\omega$$  \hfill (14)

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \log \Phi_{e_i}(\omega) d\omega - \bar{h}(d)$$  \hfill (15)

$$\geq \bar{h}(e_i) - \bar{h}(d).$$  \hfill (16)

Here (14) follows from the definition of sensitivity function in (9), (15) and (16) follow from the maximum entropy theorem and Assumption 1, and equality in (16) holds iff $e_i$ is a Gaussian process with power spectral density $\Phi_e$.

Note that if we can relate the entropy rate $\bar{h}(r_i)$ to $\bar{h}(d)$, then by Lemma 2 and (16), a fundamental limit of performance is obtained. The relation between $\bar{h}(r_i)$ and $\bar{h}(d)$ is established by the following result.

Lemma 3: Suppose that Assumptions 2–4 hold. Then, for every $i = 0, \ldots, n$, if $Z_i$ is DSI

$$\bar{h}(y_i) \geq \bar{h}(r_i) + \bar{A}_i,$$

while if $Z_i$ is PSI

$$\bar{h}(y_i) \geq \bar{h}(r_i) + \max(\bar{A}_i - C_i, 0),$$

where $\bar{A}_i$ is defined in (10).

Lemma 3 indicates that the differential entropy rate of the output $y_i$ is at least as large as a “scaled” version of the entropy rate of the input $r_i$, where the additive scaling factors depend on the corresponding stationary loop gain and cannot be affected by any delayed side information.

Since there is a unique path from the leader to the $i$-th vehicle, by using Lemma 3 for every vehicle between the leader and the $i$-th follower we can lower bound $y_{i-1}$, i.e., $r_i$, in terms of $\bar{h}(d)$. Together with Lemma 2, a lower bound on the disturbance propagation performance is obtained in Theorem 1.
V. MAIN RESULT

Consider the tracking error $e_i$ of the $i$-th follower, $i = 0, \ldots, n$.

**Theorem 1:** Let Assumptions 1-4 hold. Then, for every $i = 0, \ldots, n$ it holds that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega)d\omega \geq \sum_{\lambda \in U(A_i)} \log |\lambda| - \bar{C}_i + \sum_{j=0}^{i-1} (\bar{\Lambda}_{i(j)} - \bar{C}_{i(j)}),$$

(17)

where $\bar{\Lambda}_{i(j)}$ is defined in (10),

$$\bar{C}_i = \begin{cases} \min(\sum_{\lambda \in U(A_i)} \log |\lambda|, C_i), & \text{for DSI}, \\ C_i, & \text{for PSI}; \end{cases}$$

and

$$\bar{C}_{i(j)} = \begin{cases} 0, & \text{for DSI}, \\ \min(\bar{\Lambda}_{i(j)}, C_{i(j)}), & \text{for PSI}. \end{cases}$$

A few remarks are in order.

1) Similar to the Bode integral formula for SISO systems, the right hand side of (17) depends on the open-loop poles of the $i$-th agent and is independent of the choice made for the $i$-th controller. Therefore, (17) characterizes a fundamental limitation for all control laws $u_i$ satisfying Assumption 4.

2) In the special case where $C_j = 0$ for all $0 \leq j \leq i-1$, (i.e., there is no side information flow in the vehicle formation), Theorem 1 recovers the result in [25] with the predecessor-following control strategy. Besides, if there is only limited leader information available to the followers, Theorem 1 recovers the result in [24].

3) In the special case where the control mappings in (5) are chosen as

$$u_i(k) = f(k, \bar{p}^i, e_i^{k-1}) + F_i e_i(k),$$

for some $f(k, \cdot) : \mathbb{R}^{2k+1} \rightarrow \mathbb{R}$ and some real scalar constant $F_i$, the loop gain becomes time-invariant and is given by $|C_i B_i F_i|$. Hence the right hand side of (17) reduces to

$$\sum_{\lambda \in U(A_i)} \log |\lambda| - \bar{C}_i + \sum_{j=0}^{i-1} \log |(CBF)_{i(j)}| - \bar{C}_{i(j)}).$$

4) With a large loop gain the system can achieve good tracking performance. However, (17) reveals that large open-loop gains amplify the disturbance as it propagates along the formation. As a result, a vehicle formation with large loop gains may enjoy good tracking performance when there is no disturbance, but the performance degrades fast when the disturbance is present (see [26] for a numerical example).

5) If $\bar{\Lambda}_j < 0$, meaning that the open-loop gain is smaller than 1 (which is undesirable for the tracking performance), then term $\max(\bar{\Lambda}_j - C_j, 0)$ in (17) is replaced by $\Lambda_j$.

VI. APPLICATION TO THREE SPECIFIC SCENARIOS

A. Vehicle platoon control using predecessor-following strategy

In our first example, the vehicle formation is one-dimensional (also known as a string formation) [3], [15]. In the “predecessor following” information flow structure [23], each vehicle’s controller has access to only the spacing error with respect to its predecessor. For this setup, Theorem 1 reduces to the following corollary, which is also presented in [25].

**Corollary 1:** Consider the above problem setup where there is no information flow between non-adjacent agents in the platoon. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega)d\omega \geq \sum_{\lambda \in U(A_i)} \log |\lambda| + \sum_{j=0}^{i-1} \bar{\Lambda}_j.$$  

(18)

For agents that can be modeled as a scalar system, under the further assumption that all the initial conditions $x_i(0)$’s are Gaussian distributed, it has been proved in [25] that the equality sign in (18) can be achieved by the class of controllers given by

$$u_i(k) = a_i^{k+1} \left( \mathbb{E}[x_i(0) | e_i^{k-1}] - \mathbb{E}[x_i(0) | e_i^k] \right),$$

for $0 \leq i \leq n$.

B. Vehicle platoon control with limited leader information

In this example, we consider the formation control problem of a string of vehicles under the assumption that vehicle $i$ has access to its position error relative to the preceding vehicle (numbered $i-1$) as well as to limited information about the platoon leader, which is communicated from the leader over a side communication channel with capacity $C_i$ – see Fig. 3.

In this case, Theorem 1 reduces to the following corollary, which is also discussed in [24].

**Corollary 2:** Consider the setup where the $i$-th follower receives PSI from the leader through a communication channel with capacity $C_i$. Then it follows that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega)d\omega \geq \sum_{\lambda \in U(A_i)} \log |\lambda| - C_i + \sum_{j=0}^{i-1} \max(\bar{\Lambda}_j - C_j, 0).$$

It has also been shown in [24] that the lower bound for the disturbance propagation performance in Corollary 2 can be achieved by appropriate design of control laws and encoder/decoder pairs.
C. Vehicle formation control with noiseless DSI

Consider a vehicle formation control setup where agent $i$ may have DSI transmitted from other agents through a noiseless channel with infinite capacity, see Fig. 4. In this case, Theorem 1 reduces to the following corollary.

**Corollary 3:** Consider the above setup where the $i$-th agent receives noiseless DSI from other agents, i.e., $\mathcal{C}_i \to \infty$. Then it holds that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_{d,e_i}(\omega) d\omega \geq \sum_{j=0}^{i-1} \lambda_{i(j)}. \quad (19)$$

From Corollary 3 it can be seen that the effect of unstable open-loop poles can be compensated by DSI and the disturbance propagation performance can be improved only for agents with unstable plants. In other words, the contribution of DSI to the disturbance rejection performance is limited in the sense that even all the agents in the formation have infinite DSI, the performance can only be improved by $\sum_{\lambda \in \mathcal{D}(A_i)} \log |\lambda|$, which is 0 if the agent is open-loop stable.

VII. Conclusion

In this paper, we studied the problem of disturbance propagation in leader-follower vehicle formations. By using information-theoretic techniques we derived a lower bound on the integral of the log sensitivity function of tracking errors for every vehicle with respect to a stochastic disturbance acting on the formation leader. The derived bound holds for a class of possibly nonlinear controllers. The main result in this paper includes earlier results as special cases.

References


