Realtime Generation of the Bell States by Linear-Nonlocal Measurements and Bang-bang Control

Thanh Long Vu1, Jaspreet Singh Dhupia1

Abstract—We present a realtime feedback control scheme to deterministically produce the maximally entangled Bell states of two partially separated atoms. The recently introduced concept of SWM-(simultaneous weak measurements)-induced quantum state reduction is generalized to the case of anti-commutative observables. This generalization enables the probabilistic preparation of the Bell states via the SWMs of one linear observable and one nonlocal observable. The deterministic generation of the desired Bell state is then carried out by combining SWM-induced quantum state reduction with feedback control. The realtime implementation of the proposed control scheme is guaranteed by using the time delay control approach in which the computation time of filter-based control is compensated for by the delay time in control input.

Index Terms—Quantum feedback control, entanglement, Bell states, weak measurement, stochastic nonlinear control

I. INTRODUCTION

The most intriguing nonclassical feature of quantum systems is entanglement, i.e., the correlation between partially separated subsystems. In this paper, we are interested in the deterministic generation of the Bell states, which are maximally entangled states of two qubits and play a central role in quantum information science [1]. We consider a quantum system consisting of a couple of two-level atoms placed in two distant cavities $C_1$ and $C_2$ as in Fig. 1. As each two-level atom can be seen as a qubit, this quantum system is a two-qubit system.

A fundamental physical principle states that Local Operations and Classical Communication (LOCC) cannot generate entanglement between initially unentangled states; see [2] for a proof of this principle. As such, in order to produce entanglement, we usually need some nonlocal effects. In [3], entanglement between two atoms was produced via the nonlocal effect of the Hamiltonian $H_0$. However, the entanglement generated is not maximal.

This paper will present a way to produce maximal entanglement by the nonlocal effect of measurement-based feedback control, instead of the nonlocal effect of the free Hamiltonian and control Hamiltonian. In the measurement-based quantum feedback control [4], [5], the quantum system is continuously measured by the homodyne detectors $D = \{D_1, ..., D_m\}$. The measurement records $y_t = [y_{1t}, ..., y_{mt}]$ are sent to a filter to extract information about the system and the filter state (i.e., the estimate state conditioned on the measurement records) $\rho_t$ is then fed back via the controller $u_t$ and magnetic fields $L_1, L_2$ to drive the system dynamics.

Fig. 1. Measurement-based feedback control of two atoms.

Finally, we show that if we use single measurement, then it is hard to prepare the Bell states even when the measured quantity (observable) and the free Hamiltonian are nonlocal. This difficulty motivated us to consider the simultaneous weak measurements of multiple observables in the generation of the Bell states. In [6], we introduced the concept of SWM-(simultaneous weak measurement)-induced quantum state reduction. We showed that under the SWMs of two commutative observables $A_1$ and $A_2$, i.e., $A_1A_2 = A_2A_1$, the filter state almost surely converges to the common set of the eigenspaces of $A_1$ and $A_2$, defined as SWM-induced space. This concept was then applied to probabilistically generate the Bell states by using the SWMs of nonlocal observables $\sigma_i^x \otimes \sigma_j^z$ and $\sigma_i^y \otimes \sigma_j^z$, where $\sigma_i^{x,y,z}$ are the Pauli operators and $\otimes$ denotes the tensor product of operators.

Note that implementing nonlocal operations is more difficult than local operations or linear combination of local observables. As such, it is desirable to use nonlocal effect as less as possible. For this purpose, we generalize the concept of simultaneous weak measurements-induced quantum state reduction to the case of non-commutative observables $A_1$ and $A_2$, i.e., $A_1A_2 = -A_2A_1$. This generalization allows us to probabilistically produce the Bell states by the SWMs of one linear observable, i.e., a linear combination of local observables, and one nonlocal observable, making this scheme different from the scheme of using only nonlocal observables as in [6]. In addition, the free Hamiltonian is local, instead being nonlocal as in [6].

For the deterministic generation of the desired Bell states, similar to [6], we harness a combination of simultaneous weak measurements-induced quantum state reduction with time delay bang-bang control. The time to compute the filter-based control input is fully compensated for by the delay time in the control input, through which the realtime implementation of the proposed control scheme is guaranteed. An advantage of the proposed control scheme is that we do...
not require any constraint on the time delay. Therefore, the proposed measurement-based feedback scheme can be implemented in the real time regardless the length of computation time.

The main contributions of the paper are as follows:

(i) The concept of SWM-induced quantum state reduction in [6] is generalized to the case of anti-commutative observables;

(ii) The Bell states are probabilistically produced by using the SWMs of one linear observable and one nonlocal observable, instead of utilizing only nonlocal observables as in [6]. We also show that by this way, the Bell states can be generated with many choices of the measured observables different from those in [6];

(iii) The time delay bang-bang control is combined with the generalized SWM-induced quantum state reduction to deterministically generate the desired Bell state in the real time.

Notations:

\( i = \sqrt{-1} \) (we use the Roman character \( i \) to distinguish the imaginary unit from the index \( i \));

\( A^1, A^\dagger \) are the conjugate transpose and complex conjugate of the matrix \( A \); \( \text{Tr}(A) \) is the trace of matrix \( A \); \( [A, B] := AB - BA \) is the commutator of the matrices \( A, B \); \( I_n \) is the identity matrix;

\( \otimes \) : the tensor product of operators or the Kronecker product of matrices;

\( \mathbb{R}, \mathbb{C} \) : the sets of real and complex numbers, respectively;

\( \mathcal{L} \) : the infinitesimal generator;

\( \langle 0 |, | 1 \rangle \), \( \langle + |, | - \rangle \) are Dirac notations of the two eigenstates of the qubit [7], or equivalently by density matrices:

\[
\phi^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} \ 1 & 0 & 0 & \pm 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pm 1 & 0 & 0 & 1 \end{bmatrix},
\psi^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 1 & \pm 1 & 0 \\
\pm 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}
\]

(2)

in the standard basis \( \{ |0 \rangle, |1 \rangle \} \).

**Theorem 2.1:** For the quantum system with nonlocal free Hamiltonian \( H_0 = J \sigma_1^z \otimes \sigma_2^z \), it is impossible to make sure that the filter state almost surely converges to one of the Bell states by the single weak measurement of neither \( \sigma_1^z \otimes \sigma_2^z \) nor \( \sigma_1^x \otimes \sigma_2^x \).

**Proof:** We consider the typical continuous weak measurement of the nonlocal observable \( A = \sigma_1^x \otimes \sigma_2^x \). The analysis is similar when we utilize single weak measurement of the nonlocal observable \( \sigma_1^z \otimes \sigma_2^z \).

We have the following stochastic master equation and measurement record:

\[
d\rho = -i[H_0, \rho] dt + \Gamma_A [D[A]\rho] dt + \sqrt{\eta_A \Gamma_A} H[A] \rho dw_t \tag{3}
\]

\[
dy = \text{Tr}(A\rho) dt + \frac{1}{2\sqrt{\eta_A \Gamma_A}} dw_t
\]

where \( D[A]\rho := A\rho A^\dagger - \frac{1}{2}(\rho A^\dagger A + A^\dagger A)\rho \) and \( \mathcal{H}[A] \rho := A\rho + \rho A^\dagger [\text{Tr}(A \rho A^\dagger)] \rho; \Gamma_A \) and \( \eta_A \) are measurement strength and efficiency and \( dw_t \) is the Wiener increment.

Consider the Lyapunov function candidate \( U(\rho) = \text{Tr}(A^2\rho) - \text{Tr}^2(A\rho) \), which is the variance of the filtering process along \( A \). Since \( A \) and \( H_0 \) are commutative, a straightforward computation gives the infinitesimal generator associated with (3) acting on \( U(\rho) \):

\[
\mathcal{L} U(\rho_i) = -4\eta_A \Gamma_A U(\rho_i)^2 \leq 0 \tag{4}
\]

Applying Theorem 2.1 in Ref. [8], we achieve

\[
\mathbb{P}\{ \lim_{t \to \infty} U(\rho_i) = 0 \} = 1. \tag{5}
\]

As such, the weak measurement of the observable \( A \) renders the variance \( U(\rho_i) \) of the filtering process along \( A \) to 0 almost surely. Let the density matrix \( \rho = [\rho_{ij}]_{4 \times 4} \in \mathcal{S} \). Then

\[
U(\rho) = \text{Tr}(A^2\rho) - \text{Tr}^2(A\rho) = \text{Tr}(\rho_{14} + \rho_{23} + \rho_{32} + \rho_{41})^2 \tag{6}
\]

Noticing the positivity and self-adjointness of \( \rho \), we have

\[
|\rho_{14} + \rho_{23} + \rho_{32} + \rho_{41}| \leq |\rho_{14} + \rho_{41}| + |\rho_{23} + \rho_{32}| = 2|\text{Re}(\rho_{14})| + 2|\text{Re}(\rho_{23})| \\
\leq 2|\rho_{14}| + 2|\rho_{23}| \\
\leq 2\sqrt{\rho_{14}^2 + 2\rho_{23}^2} \rho_{33} \\
\leq \rho_{11} + \rho_{44} + \rho_{22} + \rho_{33} = 1 \tag{7}
\]

As such, by (6), it holds that \( U(\rho) = 0 \) iff

\[
\rho_{14} = \rho_{41} = \pm \rho_{11} = \pm \rho_{44} \text{ and } \rho_{23} = \rho_{32} = \pm \rho_{22} = \pm \rho_{33} \tag{8}
\]
Therefore, the weak measurement of the observable $A$ renders $ρ_t$ to the set $Φ_A$ almost surely, where

$$Φ_A = \{ρ = [ρ_{ij}]_{4×4} ∈ S : ρ_{11} = ρ_{44} = ± ρ_{12} = ± ρ_{23} = ± ρ_{33} \}. \quad (9)$$

Consider the subset of $Φ_A$ :

$$ω_A = \{ρ ∈ S : ρ = \begin{bmatrix} a & 0 & 0 & ±a \\ 0 & b & ±b & 0 \\ 0 & ±b & b & 0 \\ ±a & 0 & 0 & a \end{bmatrix}, \quad a, b ∈ \mathbb{R}^+, a + b = 1/2 \} \quad (10)$$

As $H_0 = Jσ_1^z ⊗ σ_2^z$ and $A = σ_1^x ⊗ σ_2^z$, it can be checked that $H_0$ and $A$ are commutative with all points in the set $ω_A$. As such, any point in the set $ω_A$ is an equilibrium of the filter (3). Therefore, under the single weak measurement of the nonlocal observable $A$, the filter state can converge to any point in $ω_A$, i.e., the limit set contains the set $ω_A$. We note that though the set $ω_A$ contains all the Bell states, it is impossible to make sure that the filter state converges to one of the Bell states almost surely. Theorem 2.1 is proved.

From Theorem 2.1, it can be observed that if we use single weak measurement, then it is difficult to generate the Bell states even when the free Hamiltonian and the measured observable are nonlocal. This difficulty motivated us to consider the effect of simultaneous weak measurements of multiple observables in the next section.

III. GENERALIZED SWM-INDUCED QUANTUM STATE REDUCTION

In [6], we introduced an interesting property of quantum systems subject to simultaneous weak measurements, termed as SWM-induced quantum state reduction, which was stated that under SWMs of two Hermitian, commutative observables $A_1$ and $A_2$, the filter state almost surely converges to the common set of eigenspaces of $A_1$ and $A_2$. This concept was applied to produce the Bell states via the SWMs of nonlocal observables $σ_1^z ⊗ σ_2^z$ and $σ_1^x ⊗ σ_2^z$.

As implementing nonlocal operations is more difficult than local operations or linear combination of local observables, it is desirable to utilize as less nonlocal effect as possible. In this section, we further generalize the SWM-induced quantum state reduction concept to the case of anti-commutative observables, which shall enable us to probabilistically produce the Bell states by the SWMs of either $σ_1^x ⊗ σ_2^z$ or $σ_1^x ⊗ σ_2^z$ and one additional linear observable, while the free Hamiltonian is local.

**Theorem 3.1**: Consider the system with free Hamiltonian $H_0$ under simultaneous weak measurements of two Hermitian observables $A_1$ and $A_2$ that are commutative or anti-commutative with each other and with $H_0$. Then, the filter state almost surely converges to the SWM-induced space, which is the common set of eigenspaces of $A_1$ and $A_2$.

**Proof**: We have the SME and measurement records:

$$dρ = -i[H_0, ρ]dt + Γ_{A_1}D[A_1]ρdt + \sqrt{η_{A_1}Γ_{A_1}H[A_1]}ρdw_{A_1} + Γ_{A_2}D[A_2]ρdt + \sqrt{η_{A_2}Γ_{A_2}H[A_2]}ρdw_{A_2} \quad (11)$$

$$dy_1 = Tr(A_1ρ)dt + \frac{1}{2\sqrt{η_{A_1}Γ_{A_1}}}dw_{A_1}$$

$$dy_2 = Tr(A_2ρ)dt + \frac{1}{2\sqrt{η_{A_2}Γ_{A_2}}}dw_{A_2}$$

where $dw_{A_1}$ and $dw_{A_2}$ are independent Wiener increments. Consider the Lyapunov function candidate

$$V(ρ) = U_1(ρ) + U_2(ρ) \quad (12)$$

which is a combination of the variances $U_1(ρ)$ and $U_2(ρ)$ of the filtering process along $A_1$ and $A_2$:

$$U_1(ρ) = Tr(A_1^2ρ) - Tr^2(A_1ρ), \quad U_2(ρ) = Tr(A_2^2ρ) - Tr^2(A_2ρ). \quad (13)$$

A simple computation gives the infinitesimal generator associated with (11) acting on $Tr^2(A_iρ), i = 1, 2$,

$$\mathcal{L}Tr^2(A_iρ) = 4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i) + 4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i) \quad (14)$$

where $U_{i2}(ρ) := Tr(A_1A_2ρ) - Tr(A_1ρ)Tr(A_2ρ)$ if $A_1$ and $A_2$ are commutative, and $U_{i2}(ρ) := Tr(A_1ρ)Tr(A_2ρ)$ if $A_1$ and $A_2$ are anti-commutative. Due to the cyclic property of trace, the infinitesimal generator associated with (11) acting on $Tr(A_i^2ρ), i = 1, 2$, is

$$\mathcal{L}Tr(A_i^2ρ) = Tr(-iA_i^2[H_0, ρ]) + \sum_{j=1}^2 Γ_{A_j}A_i^2D[A_j]ρ_i$$

$$= Tr(-i[A_i^2, H_0]ρ_i)$$

$$+ Tr(\sum_{j=1}^2 Γ_{A_j}A_jA_iA_j - \frac{1}{2}A_i^2A_j^2 - \frac{1}{2}A_j^2A_i^2)ρ_i) \quad (15)$$

As $A_i$ is either commutative or anti-commutative with $H_0$, we have $A_i^2H_0 = ±A_iH_0A_i = ± H_0A_i^2 = A_i^2H_0$. Since $A_i$ and $A_j$ are commutative or anti-commutative, we have

$$A_iA_j = ± A_iA_jA_iA_j = ± A_iA_iA_jA_i = A_i^2A_j^2 = A_i^2A_j^2 \quad (16)$$

Therefore $\mathcal{L}Tr(A_i^2ρ) = 0, i = 1, 2$, and the infinitesimal generator associated with (11) acting on $V(ρ)$ is

$$\mathcal{L}V(ρ) = \sum_{i=1}^2 \mathcal{L}Tr(A_i^2ρ) - \mathcal{L}Tr^2(A_iρ)$$

$$= -4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i) - 4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i)$$

$$- 4(η_{A_1}Γ_{A_1} + η_{A_2}Γ_{A_2})U_{i2}^2(ρ_i) \quad (17)$$

where $U_{i2}(ρ) := Tr(A_1A_2ρ) - Tr(A_1ρ)Tr(A_2ρ)$ if $A_1$ and $A_2$ are commutative, and $U_{i2}(ρ) := Tr(A_1ρ)Tr(A_2ρ)$ if $A_1$ and $A_2$ are anti-commutative. Hence,

$$\mathcal{L}V(ρ) ≤ -4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i) - 4η_{A_i}Γ_{A_i}U_{i2}^2(ρ_i)$$

$$≤ -η_mV(ρ) ≤ 0, \quad (18)$$
where \( \eta_m = 2\min\{\eta_{A_1}, \eta_{A_2}, \eta_{A_3}\} > 0 \). Similar to Section II, we conclude that \( \lim_{t \to \infty} V(\rho_t) = 0 \) almost surely, leading to
\[
\mathbb{P}\{ \lim_{t \to \infty} U_1(\rho_t) = 0 \} = 1, \tag{19}
\]
and
\[
\mathbb{P}\{ \lim_{t \to \infty} U_2(\rho_t) = 0 \} = 1. \tag{20}
\]

Therefore, the SWMs of two commutative observables \( A_1 \) and \( A_2 \) render both variances \( U_1(\rho) \) and \( U_2(\rho) \) of the filtering process along \( A_1 \) and \( A_2 \) to 0 almost surely. Let \( \Phi_{A_i} = \{ \rho \in \mathcal{S} : U_i(\rho) = 0 \} \), \( i = 1, 2 \), which are eigenspaces of \( A_1 \) and \( A_2 \). Then, Eqs. (19) and (20) imply that the filter state \( \rho_t \) almost surely converges to the common set \( \Phi_{A_1} \cap \Phi_{A_2} \) of eigenspaces of \( A_1 \) and \( A_2 \). Theorem 3.1 is proved.

We define the common set \( \Phi_{SWM} := \Phi_{A_1} \cap \Phi_{A_2} \) of eigenspaces of \( A_1 \) and \( A_2 \) as the "SWM-induced space." An illustration of the SWM-induced space is given in Fig. 2.

IV. PROBABILISTIC GENERATION OF THE BELL STATES

In this section, by exploiting the generalized SWM-induced quantum state reduction, we prove that the Bell states can be probabilistically produced by the SWMs of either \( \sigma_1^z \otimes \sigma_2^z \) or \( \sigma_1^x \otimes \sigma_2^x \) and one additional linear observable. To highlight the efficiency of the proposed scheme, we consider the free Hamiltonian \( H_0 = 0 \) which is trivially local.

The two Bell states \( \psi^\pm \) in (2) can be probabilistically generated as follows.

**Theorem 4.1:** Consider the system with a free Hamiltonian \( H_0 = 0 \) subject to the SWMs of the anti-commutative linear observable \( A_1 = \sigma_1^z \otimes I_2 - I_2 \otimes \sigma_2^z \) and nonlocal observable \( A_2 = \sigma_1^x \otimes \sigma_2^x \). Then, from any initial state, the filter state \( \rho_t \) converges to one of the two states \( \{ \phi^+, \phi^- \} \) almost surely.

**Proof:** As \( A_1 \) and \( A_2 \) are anti-commutative with each other and trivially commutative with \( H_0 \), Theorem 3.1 is applicable. On the other hand, it can be checked that the eigenspace of \( A_1 \) and eigenspace of \( A_2 \) are tangent at \( \phi^+ \) and \( \phi^- \), i.e., the SWM-induced space \( \Phi_{SWM} \) associated with \( A_1 \) and \( A_2 \) reduce to \( \{ \phi^+, \phi^- \} \). Applying Theorem 3.1, the SWMs of \( A_1 \) and \( A_2 \) render the filter state \( \rho_t \) to one of the two states \( \{ \phi^+, \phi^- \} \) almost surely. Theorem 4.1 is proved.

Similarly, the two Bell states \( \psi^\pm \) in (2) can be probabilistically generated as follows.

**Theorem 4.2:** Consider the system with a free Hamiltonian \( H_0 = 0 \) subject to the SWMs of two anti-commutative linear observable \( A_1 = \sigma_1^z \otimes I_2 + I_2 \otimes \sigma_2^z \) and nonlocal observable \( A_2 = \sigma_1^x \otimes \sigma_2^x \). Then, from any initial state, the filter state \( \rho_t \) converges to one of the two states \( \{ \psi^+, \psi^- \} \) almost surely.

**Proof:** Similar to that of Theorem 4.1.

We note that the above choice of anti-commutative observables is not unique. Indeed, we can also produce the Bell states by the SWMs of other observables.

**Theorem 4.3:** Consider the system with a free Hamiltonian \( H_0 = 0 \) subject to the SWMs of the anti-commutative nonlocal observable \( A_1 = \sigma_1^z \otimes \sigma_2^z \) and linear observable \( A_2 = \sigma_1^x \otimes I_2 + I_2 \otimes \sigma_2^x \). Then, from any initial state, the filter state \( \rho_t \) converges to one of the two states \( \{ \phi^-, \psi^- \} \) almost surely.

**Proof:** Similar to that of Theorem 4.1.

**Theorem 4.4:** Consider the system with a free Hamiltonian \( H_0 = 0 \) subject to the SWMs of two anti-commutative nonlocal observable \( A_1 = \sigma_1^z \otimes \sigma_2^z \) and linear observable \( A_2 = \sigma_1^x \otimes I_2 - I_2 \otimes \sigma_2^x \). Then, from any initial state, the filter state \( \rho_t \) converges to one of the two states \( \{ \phi^+, \psi^+ \} \) almost surely.

**Proof:** Similar to that of Theorem 4.1.

**Remark 4.1:** Theorems 4.1-4.4 present a way to produce the maximally entangled Bell states by harnessing the nonlocal effect of one nonlocal measurement to produce maximal entanglement through the probabilistic generation of the Bell states.

V. DETERMINISTIC GENERATION OF THE DESIRED BELL STATE

In this section, similar to [6], we harness the SWM-induced state reduction and feedback control to deterministically generate any desired Bell state \( \rho_d \) from any initial state, i.e., to globally asymptotically stabilize the desired Bell state.

Consider the system with free Hamiltonian \( H_0 = 0 \) subject to the SWMs of two anti-commutative observables \( A_1 \) and \( A_2 \) associated with \( \rho_d \) as in Section IV, and the feedback control...
control given by the local control Hamiltonian $H_1 = (\sigma^+_1 + \sigma^-_1) \otimes I_2/2$. This control Hamiltonian $H_1$ can be physically implemented by applying the local magnetic fields along the $z-$axis and $x-$axis of the first qubit, in which the strength of these fields is adjusted by the control $u \in \mathbb{R}$ to be designed. We have the SME and measurement records:

$$dp = -i[H_1, p]dt + \Gamma_{A_i}D[A_i]p dt + \sqrt{\eta_{A_i}}\Gamma_{A_i}\mathcal{H}[A_i]p dw_{A_i} + \Gamma_{A_i}D[A_i]p dt + \sqrt{\eta_{A_i}}\Gamma_{A_i}\mathcal{H}[A_i]p dw_{A_i},$$

where $dw_{A_i}$ and $dw_{A_2}$ are independent Wiener increments.

For the deterministic generation of the desired Bell state, we employ the bang-bang control in hysteresis time delay form to render the system trajectory $\rho_t$ from any initial state to the desired state $\rho_d$ almost surely. This control has two modes of which the 1-mode pushes the system trajectory $\rho_t$ off the other Bell states $\rho_0 \neq \rho_d$ in a finite time and then the 0-mode drives $\rho_t$ to the desired Bell state $\rho_d$ almost surely.

To present the control, we denote:

$$D(\rho) := 1 - Tr(\rho \rho_d),$$

$$S_{>\alpha} := \{\rho \in S : \alpha \leq D(\rho) \leq 1\},$$

$$S_{\geq \alpha} := \{\rho \in S : \alpha \leq D(\rho) \leq 1\},$$

$$S_{<\alpha} := \{\rho \in S : 0 \leq D(\rho) < \alpha\},$$

$$S_{\leq \alpha} := \{\rho \in S : 0 \leq D(\rho) \leq \alpha\}.$$

**Theorem 5.1**: Consider the quantum filter (21). Then, from any initial state, the following bang-bang control with $\gamma < 1/4$ will render the filter state to the desired Bell state $\rho_d$ almost surely:

1. $u_t = 1$, if $\rho_{t-\tau} \in S_{>\gamma/2}$;
2. $u_t = 0$, if $\rho_{t-\tau} \in S_{\leq 1-\gamma}$;
3. If $\rho_{t-\tau} \in \Phi := S_{<1-\gamma/2} \cap S_{>1-\gamma}$, then $u_t = 0$ if $\rho_{t-\tau}$ last entered $\Phi$ through the boundary $S_{1-\gamma}$ and $u_t = 1$ otherwise.

**Proof**: The proof of Theorem 5.1 is similar to that of Theorem 3 in [6] and omitted here.

**Remark 5.1**: As the free Hamiltonian $H_0$ and control Hamiltonian $H_1$ are local, the proposed scheme only utilizes the nonlocal effects of one nonlocal measurement and feedback control to deterministically generate any desired maximally entangled Bell states. This is the main advantage of the proposed scheme, distinguishing it from other schemes in the literature such as [3].

**Remark 5.2**: Another advantage of the proposed scheme is that we do not require any constraint on the time delay $\tau$. Therefore, this scheme can implemented in the real time regardless the length of computation time.

**VI. NUMERICAL ILLUSTRATION**

In this section, we illustrate the effectiveness of the above SWM-induced quantum state reduction and time-delay bang-bang control schemes in the generation of the Bell states for two-qubit system.

Consider the desired Bell state

$$\rho_d = \phi^- = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

and the anti-commutative observables $A_1 = \sigma^+_1 \otimes I_2 - I_2 \otimes \sigma^+_2$ and $A_2 = \sigma^+_1 \otimes \sigma^+_2$ associated with $\rho_d = \phi^-$ and $\phi^+$ as in Theorem 4.1. Let the measurement strengths $\Gamma_{A_1} = \Gamma_{A_2} = 1$ and the measurement efficiencies $\eta_{A_1} = \eta_{A_2} = 0.8$. The numerical illustration is carried out with the initial condition $\rho_0 = \text{diag}(1, 0, 0, 0)$ which is an unentangled state.

The distance between a state $\rho$ and the desired Bell state $\rho_d = \phi^-$ is quantified by the function $D_{\phi^-}(|\rho|) = 1 - Tr(\rho \phi^-)$, which satisfies that $0 \leq D_{\phi^-}(|\rho|) \leq 1, \forall \rho \in S$.\n
$D_{\phi^-}(|\rho_d|) = 0,$ and $D_{\phi^-}(|\rho|) > 0$ for all $\rho \neq \rho_d$. Similarly, the distance between a state $\rho$ and the antipodal Bell state $\phi^+$ is quantified by the function $D_{\phi^+}(|\rho|) = 1 - Tr(\rho \phi^+)$.\n
**A. Probabilistic Generation of the Bell States**

This subsection illustrates the effectiveness of the SWM-induced quantum state reduction associated with two anti-
FIG. 5. SWM-induced quantum state reduction of four arbitrary sample paths under bang-bang control. (a) Average distance from $\rho_t$ to $\rho_d = \phi^-$; (b) Distance from $\rho_t$ to $\phi^+$. (c) Time delay control input $u_t = u(\rho_t - \tau)$ with $\tau = 0.25, \gamma = 0.2$.

commutative observables $A_1 = \sigma_I^x \otimes I_2 - I_3 \otimes \sigma_3^y$ and $A_2 = \sigma_I^y \otimes \sigma_2^x$ in the generation of the two Bell states $\phi^-$ and $\phi^+$. The simulation results with four arbitrary sample paths are showed in Fig. 3. It can be observed from Fig. 3 that, in all sample paths, the filter state is driven from the unentangled initial state $\rho_0$ to one of the two Bell states $\phi^-$ and $\phi^+$. Fig. 4 shows the average of distances $D_{\phi^-}(\rho_t)$ and $D_{\phi^+}(\rho_t)$ over 100 sample paths. It can be observed from Fig. 4 that the distance $D_{\phi^-}(\rho_t)$ converges in average to 0.46 ($D_{\phi^-}(\rho_\infty) = 0.46$), and the distance $D_{\phi^+}(\rho_t)$ converges in average to 0.54. Therefore, the simulation results in Figs. 3 and 4 show that the SWMs can be used to deterministically produce the maximal entanglement, though the generation of Bell states is probabilistic.

B. Deterministic Generation by Bang-Bang Control

This subsection move towards with the combination of SWM-induced quantum state reduction and time delay bang-bang control for the global stabilization of the desired Bell state: $\rho_d = \phi^-$. The simulation data is as in the previous subsection. The delay time $\tau = 0.25$, and the control parameter $\gamma = 0.2$. Fig. 6 shows the SWM-induced quantum state reduction of 4 arbitrary sample paths $\rho_t$ under the proposed control. It can be seen that at some time periods, the system trajectory may tend to the antipodal Bell states $\phi^+$, showed by the fact that $D_{\phi^+}(\rho_t)$ tends to 0, but then, the control drives it back to the desired Bell state $\rho_d = \phi^-$. This makes all the sample paths eventually converge to $\rho_d$. Therefore, the SWM-induced quantum state reduction and bang-bang control are effective in the deterministic generation of the desired Bell state.

VII. CONCLUSIONS

In this paper, we have presented a realtime weak measurement-based feedback control scheme to deterministically generate the Bell states of two separated atoms from any initial state. The concept of SWM-induced quantum state reduction has been generalized to the case of anti-commutative observables. This generalization allowed us to probabilistically produce the Bell states by the SWMs of one linear observable and one nonlocal observable. The realtime deterministic generation of the desired Bell state was then performed by combining this concept with time delay bang-bang control. The computation time of filter state and control input was fully compensated for by the time-delay control, enabling the proposed measurement-based feedback control to be implemented in the real time.

REFERENCES