Aspects and Comparison of Matrix Decompositions in Unscented Kalman Filter

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Abstract—The paper deals with state estimation of nonlinear Gaussian systems with a special focus on the unscented Kalman filter. Its algorithm is based on specification of a set of so-called sigma points which are generated according to the covariance matrix of the state decomposed into a product of a matrix and its transpose. The paper analyzes utilization of different matrix decompositions within the unscented transform, which is a core of the unscented Kalman filter. It is shown that different decompositions may lead to significant differences in quality of approximations provided by the transform. The influence of the decompositions on the filter is demonstrated in an example.

Keywords: state estimation, unscented transform, unscented Kalman filter, Cholesky decomposition, SVD

I. INTRODUCTION

In many fields such as target tracking, satellite navigation, signal processing, fault detection and adaptive and optimal control problems knowledge of a state plays a crucial role. This leads to significant development of state estimation methods for nonlinear discrete-time stochastic systems. Currently, the research mostly focuses on methods based on the Bayesian approach and on the optimization approach.

Application of the Bayesian approach to the recursive state estimation problem leads to the Bayesian recursive relations (BRRs). The BRRs provide probability density functions (PDFs) of the state conditioned by the measurements representing a full description of the state.

A closed-form solution to the BRRs is available only for a few special cases such as a linear Gaussian system. In other cases, an approximate method must be used. The approximate methods include particle filtering, point-mass method and the Gaussian sum method.

The optimization approach is largely based on the criterion of minimizing the mean square error and for linear Gaussian systems leads to the celebrated Kalman filter. As opposed to the Bayesian approach providing the conditional pdf of the state, the optimization approach provides a point estimate of the state and corresponding covariance matrix. An analytical solution of the state estimation problem based on the optimization approach can be found in a few special cases only, as is the case of the Bayesian approach. For other cases, the methods usually follow the Kalman filter framework and utilize some approximate techniques such as linearization of the nonlinear functions. For example, the extended and the second-order Kalman filters approximate nonlinear functions in the state or the measurement equation by the Taylor series up to the first or second order.

Within the last decade, novel approximate methods based on the polynomial interpolation [1]–[3] or on the unscented transform (UT) [2]–[7] have been published. The unscented Kalman filter (UKF) [4], the Gauss-Hermite filter [2] or the cubature Kalman Filter [7] represent these methods. These local filters are often referred to as the sigma point Kalman filters (SPKFs).

This paper focuses on the UKF although the results can also be applied to other SPKFs. Its design utilizing the UT requires computation of a matrix decomposition (MD) of covariance matrix of the state. The decomposition is used for specification of the position of sigma points. Note that the sigma points capture the same mean and covariance matrix irrespective of the MD choice [8]. Majority of the papers dealing with the SPKFs do not specify detail of the decomposition, some only mention the method for the MD computation. In principle, there are many ways to compute the MD, the Cholesky decomposition (ChD) and the singular value decomposition (SVD) belong among the most popular. In [9] and [10] it was suggested to use the SVD due to its higher numerical stability, especially when the covariance matrix is nearly singular. In [11] a difference between usage of the ChD and the SVD decomposition in the UT was studied in several numerical examples and in [12] the ChD was compared with several methods for computing the principal square root of the covariance matrix such as the diagonalization, the Schur decomposition and some iterative approaches. However, a thorough analysis is still missing.

Hence, the goal of the paper is to provide an analysis of usage of different MDs within the UT, more specifically to study influence of the MD on quality of the UT approximation and consequently of the UKF estimates. The analysis will be done for a function given by L2-norm squared and polar to Cartesian coordinates conversion. Based on the analysis a recommendation on the MD choice will be given.

The paper is organized as follows. Section II deals with the state estimation problem, a brief description of the UT and its utilization within the UKF. Then, an overview of different MDs is provided in Section III and in Section IV the analysis of impact of the MD on quality of the UT approximation is given. Section V presents a numerical illustration of the UKF with different MDs and concluding remarks are given in Section VI.

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II. STATE ESTIMATION BY UNSCENTED KALMAN FILTER

Consider the following nonlinear stochastic system

\[ x_{k+1} = f_k(x_k) + w_k, \quad k = 0, 1, 2, \ldots, \quad (1) \]

\[ z_k = h_k(x_k) + v_k, \quad k = 0, 1, 2, \ldots, \quad (2) \]

where the vectors \( x_k \in \mathbb{R}^{n_x} \) and \( z_k \in \mathbb{R}^{n_z} \) represent the underlying state of the system and measurement at time instant \( k \), respectively. \( f_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x} \) and \( h_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z} \) are known vector mappings, and \( w_k \in \mathbb{R}^{n_w} \), \( v_k \in \mathbb{R}^{n_v} \) are the state and measurement white noises and are mutually independent. The means \( E[w_k], E[v_k], \) and covariance matrices \( \text{var}[w_k] = Q_k, \text{var}[v_k] = R_k \), of the noises, \( w_k \) and \( v_k \), respectively are supposed to be known. The mean and covariance of the initial state \( x_0 \), \( E[x_0] \) and \( \text{var}[x_0] = P_0 \), are supposed to be known as well, and the initial state is independent of both noises.

The aim of optimization-based state estimation is to find the state estimate in the form of the first two conditional moments, i.e. the mean \( \hat{x}_k \) and covariance matrix \( P_k \) as \( \text{cov}[x_k] = \text{cov}(x_k|x^k) \) and the covariance matrix \( P_k | x_k \) is given by

\[ \hat{x}_k = \sum_{i=0}^{2n_x} W_i \hat{X}_i, \quad \text{var}[x_k] = \sum_{i=0}^{2n_x} W_i \hat{X}_i \hat{X}_i^T. \]

The sigma points \( \{ \hat{X}_i \} \) and corresponding weights \( \{ W_i \} \) are often given by

\[ \hat{X}_0 = \hat{x} + \sqrt{(n_x + \kappa)} \delta_0, \quad W_0 = \frac{\kappa}{n_x + \kappa}, \quad (3) \]

\[ \hat{X}_i = \hat{x} + \sqrt{(n_x + \kappa)} \delta_i, \quad W_i = \frac{1}{2(n_x + \kappa)}, \quad (4) \]

\[ \hat{X}_{n_x+i} = \hat{x} - \sqrt{(n_x + \kappa)} \delta_i, \quad W_{n_x+i} = W_i, \quad (5) \]

where \( i = 1, \ldots, n_x, \delta_0 = 0 \) and \( \delta_i = \sqrt{n_x} \), \( i = 1, 2, \ldots, 2n_x \). The notation \( \langle \dots \rangle \) represents the \( i \)-th column of the matrix in argument and it holds that \( \sqrt{K} \sum_i \langle \sqrt{K} \rangle = \hat{P}_x \).

The term \( \kappa \) is the scaling parameter influencing accuracy of the approximation. Note that there are also other UTs such as the higher order UT [13] or the scaled UT [14].

Then, each point is transformed through the function

\[ y_i = g(\hat{X}_i), \quad \forall i, \quad (6) \]

and the resulting characteristics of \( y \), i.e. its mean and covariance matrix, and the cross-covariance matrix \( P_{xy} \)

\[ \hat{y} = E[y] = E[g(x)], \quad (7) \]

\[ P_y = \text{cov}[y] = E[(y-\hat{y})(y-\hat{y})^T], \quad (8) \]

\[ P_{xy} = E[(x-\hat{x})(y-\hat{y})^T] \quad (9) \]

are given by

\[ \hat{y} = \sum_{i=0}^{2n_x} W_i y_i, \quad (10) \]

\[ P_y = \sum_{i=0}^{2n_x} W_i (y_i - \hat{y})(y_i - \hat{y})^T, \quad (11) \]

\[ P_{xy} = \sum_{i=0}^{2n_x} W_i (X_i - \hat{x})(y_i - \hat{y})^T. \quad (12) \]

Note that these results only approximate the true mean and covariance matrices which cannot generally be computed.

For the sake of brevity, the UKF algorithm will not be given here and can be found for example in [15].

III. MATRIX DECOMPOSITIONS - AN OVERVIEW

The specification of the sigma points in the UT requires a MD of the covariance matrix \( P_x \). For its decomposition denoted as \( \sqrt{P_x} \) the relation \( \sqrt{P_x} \sqrt{P_x}^T = P_x \) must hold.

A frequently used decomposition is the ChD defined by

\[ \sqrt{P_x} = LL^T, \quad (13) \]

where \( L \) is a lower triangular matrix. Note that the ChD is defined for a symmetric positive definite matrix \( P_x \) and is unique, however it can be extended for a positive semi-definite matrix [16] in which case the decomposition looses its uniqueness.

For ill-conditioned covariance matrices the SVD is recommended [9], [10]. The SVD is defined for general real rectangular matrices \( A \) in the following form

\[ A = UDV^T, \quad (14) \]

where \( D \) is a diagonal matrix of singular values and \( U \) and \( V \) are orthogonal matrices. For the symmetric positive definite matrix the SVD simplifies to the spectral decomposition

\[ P_x = UDU^T, \quad (15) \]

where \( D \) is a diagonal matrix of eigenvalues and \( U \) is an orthogonal matrix. Based on the SVD, the matrix \( \sqrt{P_x} \) is given as

\[ \sqrt{P_x} = \sqrt{D}U \sqrt{D}^T, \quad (16) \]

where \( \sqrt{D} = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) \) and \( \lambda_i \) is the \( i \)-th eigenvalue of \( P_x \). In fact, the matrix \( \sqrt{P_x} \) can be given by

\[ \sqrt{P_x} = \sqrt{D}O, \quad (17) \]

where the matrix \( O \) is an arbitrary \( n_x \) by \( n_x \) orthogonal matrix as

\[ P_x = (U \sqrt{DO})(U \sqrt{DO})^T = U \sqrt{DO}^T \sqrt{DO}^T = UDU^T. \]

The choice of \( O \) influences position of the sigma points, which are given by the columns of \( \sqrt{P_x} \). For any choice of \( O \) the points lie on a ellipsoid in \( \mathbb{R}^{n_x} \) whose principal axes are given by \( \lambda_i(U) \). Among different choices of \( O \) the following will be studied further:

1) For \( O = I \), where \( I \) denotes the identity matrix, the sigma points will lie on the principal axes of the ellipsoid, and hence will be orthogonal, i.e. \( \langle \sqrt{P_x} \rangle_1, \langle \sqrt{P_x} \rangle_2 = 0 \) for \( i \neq j \). For this choice the sigma points also have the largest span, i.e. maximum distance from the centre given by the mean, among all other choices of \( O \).

2) It must be noted that the ChD can be seen as a special case of the decomposition (17) for \( O = Q \) where \( Q \) is obtained from the QR decomposition of \( (U \sqrt{D})^T \).
To show this, consider \((U\sqrt{D})^T = QR\) and by left multiplication by \(Q^T\) it holds that
\[
Q^T(U\sqrt{D})^T = (U\sqrt{DQ})^T = QTQR ... 
\]
where \(A \otimes B\) and \(A \otimes^k\) denote Kronecker product and power.

The choice \(O = U^T\) leads to \(\sqrt{P_x}\) being a symmetric matrix and hence the decomposition corresponds to the matrix principal square root of \(P_x\). Note that different ways of computing the matrix square root (diagonalization, Schur decomposition and iterative approaches) and with the ChD were studied in [12] with respect to their influence on the UKF performance.

Note that for a diagonal matrix \(P_x\) the three above mentioned choices of \(O\) lead to the same position of sigma points, which is caused by the fact that for a diagonal \(P_x\) the SVD returns \(U = I\) (hence \(Q = I\)). Nonetheless, even for a diagonal covariance matrix it is possible to obtain different sigma points using a suitable orthogonal matrix \(O\).

**Computational costs**

To compute the above mentioned MD, it is possible to use the ChD, SVD or spectral decomposition. Basically, all these algorithms require \(\mathcal{O}(n_3)\) floating point operations (FLOPS). However, a closer look shows that the ChD is the cheapest among the methods as it uses \(\frac{1}{2}n^3\) FLOPS, while for example the SVD uses \(\frac{5}{6}n^3\) FLOPS [17]. Hence, if computational time or timeliness is critical, the ChD should be used.

**IV. MATRIX DECOMPOSITIONS IN THE UT - ANALYSIS**

The analysis consists of two parts where numerical comparisons of usage of MDs within the UT for computing the mean and covariance matrix of random variables will be illustrated. The first illustration will be followed by an analytical justification of the results.

**A. Numerical comparison - case I**

Consider a Gaussian random variable \(x \in \mathbb{R}^2\) with mean \(E[x] = \hat{x} = [m_1, m_2]^T\) and covariance matrix
\[
\text{var}[x] = P_x = \begin{bmatrix} 
\sigma_1^2 & \sigma_1\sigma_2\rho \\
\sigma_1\sigma_2\rho & \sigma_2^2 
\end{bmatrix}.
\]

Let the random variable \(y\) be given as \(y = g(x) = x^T\hat{x}\). The aim of this subsection will be to compare the mean and variance of \(y\) calculated by the UT with different choices of the orthogonal matrix \(O\). More specifically, the following choice of \(O\) will be considered: i) \(UT_Q: O = Q\), ii) \(UT_I: O = I\), iii) \(UT_U: O = U^T\), where the matrices \(Q\) and \(U\) are described in the previous section. Note that the choice \(UT_Q\) leads to the ChD.

Simplicity of the function \(g\) allows to calculate the mean \(E[y]\) and variance \(E[y]^2\) exactly. The UT-based approximations will be compared against the true values using the absolute error (AE), e.g.
\[
AE(E[x^T x]) = |E[x^T x] - E_{UT}[x^T x]|,
\]
where \(E_{UT}[x^T x]\) denotes the (approximate) value of the mean calculated by the UT. For comparison, also the values obtained by approximation of moments of \(y\) using the Taylor expansion (TE) of \(g\) up to the first order at the mean of \(\hat{x}\) will be shown.

The AE of the mean and variance was analyzed for different correlation coefficients \(\rho\). The reason is, as was mentioned above, that for \(\rho = 0\) the covariance \(P_x\) is diagonal and the three considered MDs return the same result. The position of the sigma points however differs very much for \(\rho \neq 0\). For the experiment the mean was chosen as \(\hat{x} = [10, 0]^T\) and the variances \(\sigma_1^2 = \sigma_2^2 = 1\). The results of the analysis were depicted in Figures 1 and 2. From the figures it is clear that the TE approximation provides good estimates for neither the mean nor the variance. The UT for all MDs provide an exact value of the mean which is in accordance with [4].

However, the AEs of the variance provided by the UT with different MDs are significantly distinct, especially for higher correlation coefficient.

To analyze this behavior, an analytical computation of the AE will be done. First, let \((P_x)_{(i,j)}\) denote the element of the covariance matrix \(P_x\) at the position \((i, j)\). Note that \((P_x)_{(i,j)} = (P_x)_{(j,i)}\) for \(i \neq j\). Then, as the UT provides the mean \(E[y]\) exactly, the second raw moment \(E[y]^2] = E[g(x)^2]\) will be calculated instead of the central moment \(\text{var}[y]\). Third, the Taylor series expansion (denoted as \(\hat{g}\)) of \(g^2\) at \(\hat{x} = E[x] = [m_1, m_2]^T\) will be used for the analysis:
\[
\hat{g}^2 = g^2(\hat{x}) + \frac{1}{2} (x - \hat{x})^T A + \frac{1}{2} (x - \hat{x})^T B (x - \hat{x}) + \frac{1}{3} (x - \hat{x})^T C (x - \hat{x}) \otimes^2 + \frac{1}{4} (x - \hat{x})^T D (x - \hat{x}) \otimes^3, \tag{19}
\]
where \(A \otimes B\) and \(A \otimes^k\) denote Kronecker product and power.
respectively.
\[ A = \begin{bmatrix} 4m_1^2 + 4m_1m_2^2 & 8m_1m_2 & 8m_1m_2 & 4m_1^2 & 12m_1^2 + 4m_1^2 & 4m_1m_2 \\ 4m_1m_2 & 8m_1m_2 & 8m_1m_2 & 4m_1m_2 & 8m_1m_2 & 4m_1m_2 \\ 24m_1 & 8m_2 & 8m_2 & 8m_1 & 24m_1 \\ 24 & 0 & 8 & 0 & 8 & 0 \\ 0 & 8 & 8 & 0 & 0 & 24 \end{bmatrix}, \]
\[ B = \begin{bmatrix} 12m_2 & 12m_2 & 12m_2 & 24m_1m_2 & 8m_1m_2 \\ 12m_2 & 12m_2 & 12m_2 & 24m_1m_2 & 8m_1m_2 \\ 24m_1 & 8m_2 & 8m_2 & 8m_1 & 24m_1 \\ 24 & 0 & 8 & 0 & 8 \\ 0 & 8 & 8 & 0 & 0 \end{bmatrix}. \]

Now,
\[ E[g^2] = \frac{1}{2} \text{tr}(A^T B A) + \frac{1}{4} E[(x - \hat{x})^T D(x - \hat{x})^{\otimes 3}], \]
(20)
as the mean of the linear and cubic terms of (19) are zero. After a tedious calculation of the last term in (20) the true second raw moment can be expressed as
\[ E[g^2] = \frac{1}{2} \text{tr}(A^T B A) + \frac{1}{4} E[(x - \hat{x})^T D(x - \hat{x})^{\otimes 3}] + \frac{1}{4} \sum_{i=1}^{n_x} \frac{1}{4} \sum_{i=1}^{n_x} W_i(X_i - \hat{x})^T B(X_i - \hat{x}) D(X_i - \hat{x})^{\otimes 3}. \]
(21)
Now, the UT approximation (denoted as \( E_{UT}[g^2] \)) of \( E[g^2] \) will be computed with the scaling parameter \( \kappa = 3 - n_x \) using Taylor series expansion of the function \( g \):
\[ E_{UT}[g^2] = \frac{2n_x}{4} \sum_{i=0}^{2n_x} W_i g^2(X_i) = \frac{2n_x}{4} \sum_{i=0}^{2n_x} W_i g^2(X_i) = g^2(\hat{x}) + \frac{1}{2} \sum_{i=1}^{n_x} W_i (X_i - \hat{x})^T B(X_i - \hat{x}) D(X_i - \hat{x})^{\otimes 3}. \]
(22)
Note that for the choice \( \kappa = 3 - n_x \) the parameter \( \kappa \) disappears from all terms up to the fourth one. The second RHS term of (22) equals to
\[ \frac{1}{2} \sum_{i=1}^{n_x} W_i (X_i - \hat{x})^T B(X_i - \hat{x}) = \frac{1}{2} \sum_{i=1}^{n_x} W_i (n_x + \kappa) \delta_i^T B \delta_i \]
\[ = \frac{1}{2} \sum_{i=1}^{n_x} \delta_i^T B \delta_i = \frac{1}{2} \sum_{i=1}^{n_x} \text{tr}(B \delta_i \delta_i^T) = \frac{1}{2} \text{tr}(B \sum_{i=1}^{n_x} \delta_i \delta_i^T) \]
\[ = \frac{1}{2} \text{tr}(B \delta_i \delta_i^T). \]
(23)
The third RHS term of (22) can be after a tedious calculation written as
\[ \frac{1}{4} \sum_{i=1}^{n_x} W_i (X_i - \hat{x})^T D(X_i - \hat{x})^{\otimes 3} = 3 \sum_{i=1}^{n_x} (\langle \delta_i, \delta_i \rangle)^2, \]
(24)
where \( \langle \cdot, \cdot \rangle \) denotes the inner product. Hence, \( E_{UT}[g^2] \) is given by
\[ E_{UT}[g^2] = g^2(\hat{x}) + \frac{1}{2} \text{tr}(A^T B A) + 3 \sum_{i=1}^{n_x} (\langle \delta_i, \delta_i \rangle)^2. \]
(25)
To be able to compare the true \( E[g^2] \) given by (21) and its UT approximation \( E_{UT}[g^2] \) given by (25), the true value must be expressed in terms of a MD of \( P_x \):
\[ E[g^2] = g^2(\hat{x}) + \frac{1}{2} \text{tr}(A^T B A) + 3 \sum_{i=1}^{n_x} (\langle \delta_i, \delta_i \rangle)^2 + 2 \text{det}(S_x S_x^T) + 6 (\langle \delta_1, \delta_2 \rangle). \]
(26)
Now, by direct comparison of (25) and (26) it is clear that
\[ \text{AE}(E[g^2]) = |2 \text{det}(S_x S_x^T) + 6 (\langle \delta_1, \delta_2 \rangle)|, \]
(27)
where the first term \( 2 \text{det}(S_x S_x^T) = 2 \text{det}(P_x) \) is equal for all MDs, while the second term \( 6 (\langle \delta_1, \delta_2 \rangle) \) depends on the selected MD of \( P_x \). The sigma points used by UT are orthogonal, hence the second term is zero and UT provides the smallest absolute error \( \text{AE}(E[g^2]) \) among all the MDs.

B. Numerical comparison - case II

This case is concerned with a conversion from polar to Cartesian coordinates which is common in many tracking applications [5]. A sensor provides position of the object in polar coordinates \( p_p = [r, \theta]^T \) and this position is then converted to a Cartesian position \( p_c = [x, y]^T \) as follows
\[ p_c = [x, y]^T = [r \cos(\theta), r \sin(\theta)]^T = g(p_p). \]

For the simulation the mean \( \text{E}(p_p) \) will be given by \( \text{E}(p_p) = [1m, 90^\circ]^T \), i.e. the true position is \([0, 1]\). The standard deviation of the range \( r \) and bearing \( \theta \) will be \( \sigma_r = 2cm \) and \( \sigma_\theta = 15^\circ \). To study approximation error of the UT with different MDs, a correlation of the measurements will be considered, i.e. \( \text{var}(p_p) = \begin{bmatrix} \sigma_r^2 & \sigma_r \sigma_\theta \\ \sigma_r \sigma_\theta & \sigma_\theta^2 \end{bmatrix} \). The true mean \( \text{E}(p_c) \) and covariance matrix \( \text{var}(p_c) \) were computed using numerical integration methods and their approximations using the UT with different MDs and using the TE. The results of the absolute error of the mean for different correlation coefficients \( \rho \) are given in Figure 3 and of absolute error of the trace of the covariance matrix in Figure 4. Similarly to the previous case, the mean provided by the UT with different MDs show similar absolute error while the results of the covariance matrix approximation are distinct. The highest difference is approximately for the correlation coefficient
\[ \rho = 0.7. \] Hence, for this coefficient also the \( 1-\sigma \) contour is calculated for each approximation; see Figure 5. Note that the \( 1-\sigma \) contour is the locus of points \( \{ x : (x-\hat{x})P_x^{-1}(x-\hat{x})^T \} \) and graphically represents the size, position and orientation of \( P_x \). From the \( 1-\sigma \) contour it follows that the UTU and UTI provide covariance matrix approximate very close to the true one, while UTQ corresponding to the ChD exhibits significantly worse results.

**C. Results of the analysis**

The above analysis revealed the following findings:

- For a to-be-transformed random variable with uncorrelated elements all the three considered MDs provide identical sigma points and hence they make almost no difference on quality of the UT approximation. In such a case the ChD may be preferred for its low costs.

- If the random variable contains correlated elements, the use of different MDs may significantly affect quality of the UT approximation of the mean or covariance matrix of the transformed random variable. The two cases above showed that the UTI should be preferred, which was proven for \( g(x) = x^T x \) analytically and shown for the polar to Cartesian conversion numerically.

The reason may be the largest span of the sigma points for UTI compared to other MDs, which may help to capture the nonlinearity of \( g \). This behavior was even more evident, if the diagonal elements of the covariance matrix were different by several orders of magnitude, although the results have not been shown for the sake of brevity.

This fact is closely related to another issue concerning the size of the ellipsoid on which all the sigma points lie regardless of the orthogonal matrix \( O \) choice. If the function \( g \) is close to be linear within the region defined by the ellipsoid, the MDs will make almost no difference on quality of the approximate mean and covariance matrix. It is important to note that such condition is determined not only by the covariance matrix \( P_x \) but also by the nonlinearity of \( g \) [18].

- If the elements of the to-be-transformed variable exhibit strong correlation so that the corresponding covariance matrix is nearly singular, another issue must be taken into account, which is numerical stability of the algorithm computing the MD. The SVD is much more numerically stable for nearly singular covariance matrices than the ChD, as was demonstrated in [9]. So in such cases the SVD may be preferred for computing the MD and either UTI or UTU can be used.

Note that the results of the analysis can also be applied to other techniques based on deterministically chosen weighted points such as the cubature or quadrature filters [1], [7].

**V. NUMERICAL ILLUSTRATION**

Suppose the following tracking problem motivated by [19] is given. The object follows the coordinated turn motion (CTM) model, which was chosen to ensure correlation of the states. The state of the object is defined as \( x_k = [x_1, x_2, x_3, x_4, x_5]^T = [x_k, y_k, \dot{x}_k, \dot{y}_k, \omega_k]^T \) (i.e., it consists of the positions and velocities in the \( x \) and \( y \) directions and of the turn rate \( \omega \) and thus its dimension is \( n_x = 5 \)) which evolves as

\[
\mathbf{x}_{k+1} = \begin{bmatrix}
1 & 0 & \sin(\omega T) & -\frac{1}{2}\cos(\omega T) & 0 \\
0 & 1 & \frac{1}{2}\cos(\omega T) & \sin(\omega T) & 0 \\
0 & 0 & \cos(\omega T) & -\sin(\omega T) & 0 \\
0 & 0 & \sin(\omega T) & \cos(\omega T) & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \mathbf{x}_k + \mathbf{G} \mathbf{w}_k,
\]

with

\[
\mathbf{G}^T = \begin{bmatrix}
T^2/3 & 0 & T & 0 & 0 \\
0 & T^2/3 & 0 & T & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

where \( T = 1 \) [sec] is the sampling interval and \( \mathbf{w}_k \) is the Gaussian zero-mean state noise with covariance matrix \( \mathbf{Q} \), i.e., \( p(\mathbf{w}_k) = \mathcal{N}(\mathbf{w}_k; 0, \mathbf{Q}) \), with \( \mathbf{Q} = \text{diag}(10^{-2}, 10^{-2}, 10^{-1}) \).

The object is observed from a platform located at the origin, which provides the measurement \( \mathbf{z}_k \) given by the range \( r_k \) and bearing \( \theta_k \) from the platform to the object

\[
\mathbf{z}_k = [\sqrt{x_k^2 + y_k^2}, \arctan \frac{y_k}{x_k}]^T + \mathbf{v}_k
\]

where the standard deviation of the range and bearing noise is \( R_{r_k} = 70 \) [m] and \( R_{\theta_k} = 10 \) [mrad]. The target follows the trajectory depicted in Figure 6. The target starts at \( [x_0, y_0] = [30 \cdot 10^3, 20 \cdot 10^3] \) with initial velocity \( \{ \dot{x}_0, \dot{y}_0 \} = [-200, 0] \). The turn rate of the target is

\[
\omega_k = \begin{cases}
0.05 & \text{for } k \in (20, 30) \cup (70, 80) \\
0.01 & \text{otherwise}.
\end{cases}
\]
In total $M = 10^4$ Monte Carlo (MC) simulations were carried out and for the estimation the standard UKF was used with $\kappa = 0$ and denoted as UKF$_Q$ ($O = Q$), UKF$_I$ ($O = I$) and UKF$_U$ ($O = U$).

The filters were initialized according to [19] using the first two position measurements. The initial covariance matrix for the position and velocity was calculated using the UT and for the turn-rate it was set to 0.1.

The filter results were compared using the Root Mean Square Error (RMSE) defined as

$$\text{RMSE}^\text{pos}_k = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\hat{x}_{k|i}(i) - x_k(i))^2 + (\hat{y}_{k|i}(i) - y_k(i))^2}$$

for the position error and where $x_k(i), y_k(i)$ and $\hat{x}_{k|i}(i), \hat{y}_{k|i}(i)$ denote true and estimated object positions at the $i$-th MC run. The RMSE for the velocity and turn rate were defined analogically to (32).

The results are given in Figures 7. The results demonstrate that the best estimate quality was achieved by the UKF$_I$.

For the simulation also the EKF was used, however its results were of poor quality and their depicting would make comparison of the UKF with different MDs impossible.

The results indicate, that the best estimate quality is achieved by the UKF$_I$, especially for the velocity and turn-rate. For the position estimates, the quality was very similar, with the UKF$_U$ performing slightly better than others.

In summary, the achieved results correspond with the analysis of the MDs for the UT in Section IV and indicate that the MD with the choice $O = I$ should be preferred.

**VI. CONCLUSION**

The paper dealt with an analysis of the unscented transformation, which is a major component of the unscented Kalman filter. The unscented transform is a method to provide an approximate mean and covariance matrix of a random variable transformed through a generally nonlinear function. The analysis focused on the choice of the covariance matrix decomposition and its impact on quality of the estimates provided by the UKF.

The analysis of the UT was performed using two examples whose results demonstrated that the decompositions may lead to approximations of significantly different quality. The analysis indicated that placing the sigma points on the principal axes of the ellipsoid should be preferred nevertheless a theoretical proof for a general case is not available. The results of the analysis were confirmed by a numerical example involving tracking an object undergoing a coordinated turn.

**REFERENCES**


