Abstract - The Modified Adomian Decomposition Method was utilized to study the approximate solutions of the coupled wave system analytically with the velocity feedback boundary controllers. The results were confirmed strongly by the numerical commutations.

INTRODUCTION

Stability is highly desirable for elastic systems. To analyze it, the system energy is evaluated and if its rate is dissipative, then the stability of the system could be stable. In this work, via MADM, we have investigated the approximate solution of stabilization properties of vibrating strings in parallel, i.e., Eq. (1), analytically, whose energy is damped out by boundary velocity feedback, see Eq. (4), below.

The governing equations of the system of waves in couples is [1]:

\[
\begin{align*}
\frac{\partial u}{\partial t} - c_1^2 \frac{\partial^2 u}{\partial x^2} &= \alpha (v - u), & t \in (0, \infty), \\
\frac{\partial v}{\partial t} - c_2^2 \frac{\partial^2 v}{\partial x^2} &= \alpha (u - v), & x \in (0, 1),
\end{align*}
\]

(1)

where \(c > 0\) and \(\alpha > 0\). The initial conditions are

\[
\begin{align*}
u(x, 0) &= u_0, & u_x(x, 0) = u_1, \\
v(x, 0) &= v_0, & v_x(x, 0) = v_1,
\end{align*}
\]

(2)

with the following prescribed boundary conditions:

(mixed initial–boundary value problem)

\[
\begin{align*}
u(0, t) &= 0, & u_x(1, t) = -\beta_1 u(1, t), \\
v(0, t) &= 0, & v_x(1, t) = -\beta_2 v(1, t), & t \geq 0.
\end{align*}
\]

(3)

Here \(t, x, c_1,\) and \(c_2\) are time, space, and wave propagation speeds, respectively. Also \(u\) and \(v\) are the deflections of the strings from their equilibrium positions. The spring constant, \(\alpha\), and the damping coefficients, \(\beta_i > 0, (i = 1, 2)\), depend on the control devices and they are system parameters. These parameters play important roles in the physical behavior of the system. Generally, this boundary control corresponds to a control mechanism which monitors \(u_x\) at \(x = 1\) or at \(x = 0\). This phenomenon takes place, if the system is exposed to external forces or velocity feedback boundary conditions, Eq. (3). This problem is analogous to the ordinary differential equations for coupled oscillators and has a potential application in the oscillation of objects by outside disturbances.

Associated with each solution of Eq. (1) is its total natural energy at time \(t\) [1]:

\[
E(t) = \frac{1}{2} \left[ (|u|^2 c_1^2 |u_x|^2) + (|v|^2 c_2^2 |v_x|^2) + \alpha |u - v|^2 \right] dx
\]

(4)

APPLICATION OF MADM TO LINEAR PDE

The ADM is a very powerful tool to find approximated solutions for ODE’s / PDE’s analytically, see references in [2-7]. But to solve the system in Eqs (1-3), one should use Modified Adomian Decomposition Method (MADM) [2]. Having said that, let’s considered the following linear equation in differential operators form [2]:

\[
L_t u + L_x u + \rho u = g,
\]

where,

\[
L_t = \partial^2 (.) / \partial t^2 \quad \text{and} \quad L_x = \partial^2 (.) / \partial x^2.
\]

Having considered Maclaurin series, let

\[
\begin{align*}
u &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} t^n x^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n,k} x^k \\
\rho &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \rho_{n,k} t^n x^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \rho_{n} x^n \\
g &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g_{n,k} t^n x^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} g_{n} x^n.
\end{align*}
\]

(5)

The decomposition solution using partial solution is given by:
\[ u = \phi_i + L_i^{-1} g - L_i^{-1} \rho u, \]  
\[ \phi_i = c_0 + c_1 t, \]  
(6)

where \( c_0 = u (t = 0, x) \) and \( c_1 = \partial u / \partial t (t = 0, x) \). These parameters are yet to be calculated from Eq. (2).

Also, \( L_i^{-1} = \int_{0}^{t} (t - \tau) d\tau \). Substituting, \( u, g, \) and \( \rho \) from Eq. (5) into Eq. (6) and carrying out the integrations, one gets:

\[ \begin{align*}
\sum_{n=0}^{\infty} a_n (x) t^n &= \phi_i + \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} g_{n-2} (x) \\
&\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \frac{\partial^2}{\partial x^2} a_n (x) \\
&\quad - \sum_{n=0}^{\infty} \frac{t^{n+2}}{(n+1)(n+2)} \sum_{m=0}^{n-2} \rho_m (x) a_{n-m} (x).
\end{align*} \]

(7)

Let \( n \to n - 2 \) on the right side of Eq. (7) to get

\[ \begin{align*}
\sum_{n=0}^{\infty} a_n (x) t^n &= \phi_i + \sum_{n=0}^{\infty} \frac{t^{n}}{(n)(n-1)} g_{n-2} (x) \\
&\quad - \sum_{n=0}^{\infty} \frac{t^{n}}{(n)(n-1)} \frac{\partial^2}{\partial x^2} a_{n-2} (x) \\
&\quad - \sum_{n=0}^{\infty} \frac{t^{n}}{(n)(n-1)} \sum_{m=0}^{n-2} \rho_m (x) a_{n-2-m} (x).
\end{align*} \]

(8)

Finally, equating the coefficients of like powers of \( t \) in Eq. (8), the recursive formula for the coefficients will be found as follows:

\[ \begin{align*}
a_0 &= c_0, \\
a_1 &= c_1. \end{align*} \]

(9)

For \( n \geq 2 \),

\[ \begin{align*}
a_n &= \frac{g_{n-2} (x) - \left( \frac{\partial^2}{\partial x^2} a_{n-2} (x) \right) n(n-1)}{n(n-1)} \\
&\quad - \sum_{m=0}^{n-2} \frac{\rho_m (x) a_{n-2-m} (x)}{n(n-1)},
\end{align*} \]

(10)

which leads to the final solution of the system in Eq. (1) analytically as

\[ \begin{align*}
u(x, t) &= \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} a_n (x) t^n.
\end{align*} \]

(11)

The "modified decomposition" series solutions have been found for initial-value problems by incorporating and adapting ideas of the decomposition method. Using the double decomposition technique [2], the procedure can be further generalized to treat initial-value and boundary-value problems in a similar and computationally efficient formulation with more acceleration of convergence. To this end, using double decomposition, we can write [2]:

\[ \begin{align*}
c_0 &= \sum_{m=0}^{\infty} c_0^m, \quad c_1 = \sum_{m=0}^{\infty} c_1^m, \quad a_n = \sum_{m=0}^{\infty} a_n^m. \end{align*} \]

(12)

To match the boundary conditions, we apply decomposition to the integration constants and double decomposition to the coefficients of the Maclurian series solution. Substituting Eq. (12) into the recurrence relations, Eqs. (9) and (10) leads to

\[ \begin{align*}
a_0^m &= c_0^m, \quad a_1^m = c_1^m. \end{align*} \]

(13)

And for \( n \geq 2 \),

\[ \begin{align*}
a_n^m &= \frac{g_{n-2} (x) - \left( \frac{\partial^2}{\partial x^2} a_{n-2} (x) \right) n(n-1)}{n(n-1)} \\
&\quad - \sum_{m=0}^{n-2} \frac{\rho_m (x) a_{n-2-m} (x)}{n(n-1)},
\end{align*} \]

(14)
To stagger the series, one writes
\[ u_0 = a_0^{(0)} + a_1^{(0)} x \]
\[ u_1 = a_0^{(1)} + a_1^{(1)} x + a_2^{(0)} x^2 + a_3^{(0)} x^3 \]
\[ u_2 = a_0^{(2)} + a_1^{(2)} x + a_2^{(1)} x^2 + a_3^{(1)} x^3 + a_4^{(0)} x^4 + a_5^{(0)} x^5 \]

\[ \ldots \]

\[ u_m = a_0^{(m)} + a_1^{(m)} x + \ldots + a_{2m}^{(0)} x^{2m} + a_{2m+1}^{(0)} x^{2m+1}. \]

This can be written as
\[ u_m = \sum_{n=0}^{2m+1} a_n^{(m-n/2)} x^n, \]

where \( \{n/2\} \) is the first integer greater than \( n/2 \). We now have a different decomposition for \( u \) which is suitable for boundary-value problems as
\[ u = \sum_{n=0}^{\infty} u_n = \sum_{m=0}^{\infty} a_n^{(m-n/2)} x^n. \]

Then, we derive the approximation solution as:
\[ \varphi_{m+1} \{ u \} = \sum_{n=0}^{m} u_n = \sum_{n=0}^{m} a_n^{(m-n/2)} + x \sum_{n=0}^{m} a_1^{(m-n/2)} + x^2 \sum_{n=0}^{m-1} a_2^{(m-n/2)} + x^3 \sum_{n=0}^{m-2} a_3^{(m-n/2)} + \ldots + x^{2m-2} a_{2m}^{(m-n/2)} + x^{2m+1} a_{2m+1}^{(m-n/2)}, \]

and hence,
\[ \varphi_{m+1} \{ u \} = \varphi_{m+1} \{ a_0 \} + x \varphi_{m+1} \{ a_1 \} + x^2 \varphi_{m} \{ a_2 \} + x^3 \varphi_{m} \{ a_3 \} + \ldots + x^{2m-2} \varphi_{m} \{ a_{2m} \} + x^{2m+1} \varphi_{m} \{ a_{2m+1} \}, \]

or
\[ \varphi_{m+1} \{ u \} = \sum_{n=0}^{2m+1} \varphi_{m+1-\{n/2\}} \{ a_n \} x^n. \tag{17} \]

Where, \( \varphi_{m} \{ a_n \} = \sum_{n=0}^{m-1} a_n^{(m-n/2)} \) and \( \{n/2\} \) is the first integer smaller than \( n/2 \). Here, we see that
\[ m \to \infty, \varphi_{m+1} \{ u \} \to u, \tag{17.a} \]

Therefore, using the approximate boundary conditions Eq. (15), we can compute the constants \( c_0^{(m)}, c_1^{(m)}, a_0^{(m)}, \) and \( a_1^{(m)} \). Having found the coefficients of the Maclaurin series, the final approximated solution is obtained by Eq. (17).

**APPLICATION OF MADM TO SYSTEM (1)**

Let us consider the system of coupled wave Eq. (1) with the following initial and boundary conditions, respectively:

\[ \begin{align*}
  u_{tt} - c_1^2 u_{xx} &= \alpha (v - u), & t \in (0, \infty), \\
  v_{tt} - c_2^2 v_{xx} &= \alpha (u - v), & x \in (0,1),
  \end{align*} \tag{18} \]

\[ \begin{align*}
  u(x,0) &= \sin(\pi x), & u_x(x,0) = 0, \\
  v(x,0) &= \sin(\pi x), & v_x(x,0) = 0, \\
  u(0,t) = 0, & u_x(1,t) = -\beta_1 u_x(1,t), \\
  v(0,t) = 0, & v_x(1,t) = -\beta_2 v_x(1,t), \quad t \geq 0.
  \end{align*} \tag{19, 20} \]

Without loss of generally, and to compare solutions regarding MADM with the finite difference method in [1], let

\[ c_1 = c_2 = 1, \alpha = 1, \beta_1 = 1 \text{ and } \beta_2 = 2. \]

Having considered MADM, the solution to the system (18-20), in operators form, is given by:

\[ \begin{align*}
  u &= c_0 + c_1 x + L_x^{-1} u_{tt} - L_x^{-1} (v - u), \\
  v &= d_0 + d_1 x + L_x^{-1} v_{tt} - L_x^{-1} (u - v),
  \end{align*} \tag{21} \]

where \( L_x^{-1} = \int_{0}^{x} \frac{1}{x} dx \) and \( c_0, d_0, c_1, \) and \( d_1 \) are the constant of integrations. In order to find \( c_0 \) and \( d_0 \) in Eq. (21), we apply the boundary conditions given in Eq. (20) as:

\[ \begin{align*}
  u(0,t) &= 0 \to c_0 = 0, \\
  v(0,t) &= 0 \to d_0 = 0.
  \end{align*} \]

Having considered Eq. (5), one can find that

\[ m \to \infty, \varphi_{m+1} \{ u \} \to u, \tag{17.a} \]
\[
\begin{align*}
  u &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} x^n t^k = \sum_{n=0}^{\infty} x^n \left[ \sum_{k=0}^{\infty} a_{n,k} t^k \right] \\
  &= \sum_{n=0}^{\infty} a_n(t) x^n ,
  \\
  v &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{n,k} x^n t^k = \sum_{n=0}^{\infty} x^n \left[ \sum_{k=0}^{\infty} b_{n,k} t^k \right] \\
  &= \sum_{n=0}^{\infty} b_n(t) x^n.
\end{align*}
\]

Substituting Eq. (22) into Eq. (21), to get
\[
\sum_{n=0}^{\infty} a_n(t) x^n = c_1 x + \int_{0}^{x} \int_{0}^{x} \sum_{n=0}^{\infty} \tilde{a}_n(t) x^n dx dx \\
- \int_{0}^{x} \int_{0}^{x} \sum_{n=0}^{\infty} (b_n(t) - a_n(t)) dx dx,
\]
\[
\sum_{n=0}^{\infty} b_n(t) x^n = d_1 x + \int_{0}^{x} \int_{0}^{x} \sum_{n=0}^{\infty} \tilde{b}_n(t) x^n dx dx \\
- \int_{0}^{x} \int_{0}^{x} (a_n(t) - b_n(t)) dx dx .
\]

Carrying out the integrations and letting \( n \to n - 2 \), yields
\[
\sum_{n=0}^{\infty} a_n(t) x^n = c_1 x + \sum_{n=2}^{\infty} \tilde{a}_{n-2}(t) x^n \\
- \sum_{n=2}^{\infty} (b_{n-2}(t) - a_{n-2}(t)) x^n ,
\]
\[
\sum_{n=0}^{\infty} b_n(t) x^n = d_1 x + \sum_{n=2}^{\infty} \tilde{b}_{n-2}(t) x^n \\
- \sum_{n=2}^{\infty} (a_{n-2}(t) - b_{n-2}(t)) x^n .
\]

Following the procedure in the preceding section, one gets
\[
a_0 = u(t = 0, x) = c_0 = 0 ,
  a_1 = \partial u/\partial t(t = 0, x) = c_1 ,
  
  a_0 = v(t = 0, x) = d_0 = 0 ,
  a_1 = \partial v/\partial t(t = 0, x) = d_1 ,
\]
and for \( n \geq 2 \),
\[
a_n = \frac{1}{n(n-1)}(\tilde{a}_{n-2} - (b_{n-2} - a_{n-2})) ,
  
  b_n = \frac{1}{n(n-1)}(\tilde{b}_{n-2} - (a_{n-2} - b_{n-2})).
\]

Then considering double decomposition, we get
\[
\begin{align*}
  a_n &= \sum_{m=0}^{\infty} a_n^m ,
  b_n &= \sum_{m=0}^{\infty} b_n^m ,
  
  c_i &= \sum_{m=0}^{\infty} d_i^m ,
  d_i &= \sum_{m=0}^{\infty} d_i^m \ (i = 0, 1)
\end{align*}
\]

where ,
\[
\begin{align*}
  c_0 &= 0 \Rightarrow \{ a_0^m = 0 ,
  
  d_0 &= 0 \Rightarrow \{ b_0^m = 0 ,
\end{align*}
\]

and
\[
\begin{align*}
  a_1^m &= c_1^m ,
  b_1^m &= d_1^m .
\end{align*}
\]

For \( n \geq 2 \),
\[
\begin{align*}
  a_n^m &= \frac{1}{n(n-1)}(\tilde{a}_{n-2}^m - (b_{n-2}^m - a_{n-2}^m)) ,
  
  b_n^m &= \frac{1}{n(n-1)}(\tilde{b}_{n-2}^m - (a_{n-2}^m - b_{n-2}^m)) .
\end{align*}
\]

Now, following the preceding section, we use Eq. (17) and the boundary conditions given in Eq. (20) to compute the constants \( c_i^m \) and \( d_i^m \) as follows:
\[
\begin{align*}
  \varphi_{m+1} \{ u \}(t, x = 0) &= 0 ,
  
  \frac{d}{dx} \left( \varphi_{m+1} \{ u \}(t, x) \right) \bigg|_{x=1} &= -\frac{d}{dt} \left( \varphi_{m+1} \{ u \}(t, x) \right) \bigg|_{x=1} ,
  
  \varphi_{m+1} \{ v \}(t, x = 0) &= 0 ,
  
  \frac{d}{dx} \left( \varphi_{m+1} \{ v \}(t, x) \right) \bigg|_{x=1} &= -2 \frac{d}{dt} \left( \varphi_{m+1} \{ v \}(t, x) \right) \bigg|_{x=1} .
\end{align*}
\]

Consequently, for \( m = 0 \) we have:
\[
\begin{align*}
  \varphi_{m+1} \{ u \}(t, x = 0) &= 0 ,
  
  \frac{d}{dx} \left( \varphi_{m+1} \{ u \}(t, x) \right) \bigg|_{x=1} &= -\frac{d}{dt} \left( \varphi_{m+1} \{ u \}(t, x) \right) \bigg|_{x=1} ,
  
  \varphi_{m+1} \{ v \}(t, x = 0) &= 0 ,
  
  \frac{d}{dx} \left( \varphi_{m+1} \{ v \}(t, x) \right) \bigg|_{x=1} &= -2 \frac{d}{dt} \left( \varphi_{m+1} \{ v \}(t, x) \right) \bigg|_{x=1} .
\end{align*}
\]
\[ \varphi_1 [u](t, x = 0) = \sum_{n=0}^{1} \varphi_{1_{-\{n/2\}}} (a_n) x^n = 0 \to \\
\varphi_1 [v](t, x = 0) = \sum_{n=0}^{1} \varphi_{1_{-\{n/2\}}} (b_n) x^n = 0 \to \\
a_o^0 + a_1^0 (0) = 0 \to 0 + 0 = 0,
\]
\[
\frac{d}{dx} (\varphi_1 [u](t, x)) \bigg|_{x=1} = -\frac{d}{dt} (\varphi_1 [u](t, x)) \bigg|_{t=1} \to \\
\varphi_1^0 (1) = -a_1^0 (1) \to a_1^0 = k_1 \exp(-t), \ k_1 = \text{constant.} \tag{27}
\]

Similarly,
\[
\varphi_1 [v](t, x = 0) = \sum_{n=0}^{1} \varphi_{1_{-\{n/2\}}} (b_n) x^n = 0 \to \\
b_o^0 + b_1^0 (0) = 0 \to 0 + 0 = 0,
\]
\[
\frac{d}{dx} (\varphi_1 [v](t, x)) \bigg|_{x=1} = -\frac{d}{dt} (\varphi_1 [v](t, x)) \bigg|_{t=1} \to \\
b_1^0 (1) = -2b_1^0 (1)
\]
\[
\to b_1^0 = l_1 \exp(-\frac{1}{2}t), \ l_1 = \text{constant.}
\]

Now, using Eq. (25), we can evaluate \( a_m^0 \) and \( b_m^0 \).

To obtain \( \varphi_2 [u] \) and \( \varphi_2 [v] \), substituting the above solution into Eq. (26). Similarly, we have:
\[
a^1_1 = -\frac{2}{3} k_1 \exp(-t) + \frac{5}{6} l_1 \exp(-\frac{1}{2}t) +
\]
\[
k_2 \exp(-t),
\]
\[
b^1_1 = -\frac{5}{24} l_1 \exp(-\frac{1}{2}t) - \frac{1}{6} k_1 \exp(-t) +
\]
\[
l_2 \exp(-\frac{1}{2}t),
\]

where, \( k_2 \) and \( l_2 \) are constants. Using Eq. (25), we obtain \( a^1_m \) and \( b^1_m \). Then, \( \varphi_2 [u] \) and \( \varphi_2 [v] \) are obtained as:
\[
\varphi_2 [u] = [(k_1 + k_2) \exp(-t) - \frac{2}{3} k_1 \exp(-t) - \frac{5}{6} k_1 \exp(-\frac{1}{2}t)x + \frac{1}{6}[2k_1 \exp(-t) - l_1 \exp(-\frac{1}{2}t)] x^3,
\]
\[
\varphi_2 [v] = [(l_1 + l_2) \exp(-t) - \frac{5}{24} l_1 \exp(-\frac{1}{2}t) - \frac{1}{6} k_1 \exp(-t) - \frac{1}{6} \exp(-\frac{1}{2}t)] x^3.
\]

One can continue this procedure to find \( \varphi_3, \varphi_4, \ldots \), respectively. Now, in order to find constants \( k_i \) and \( l_i (i = 1, 2, \ldots, m) \), we apply the initial conditions stated in Eq. (19). For example, when we are computing \( \varphi_1 [u] \) and \( \varphi_1 [v] \), using Maple 10, one finds 14 constants \( k_i \) and \( l_i (i = 1, 2, 3) \). Having considered the initial conditions in Eq. (19), these constants can be found by the expansion of the \( \sin (\pi x) \), and equating the coefficients of like powers of \( x \) (4 equations for \( u \) and 4 equations for \( v \)). Also, similarly, equating \( \varphi_i [u](x,0) \) and \( \varphi_i [v](x,0) \) coefficients to zero (3 equations for \( u \) and 3 equations for \( v \)) corresponding to initial conditions (see Eq. (19)). Noted that one can compute all constants with the first part of initial conditions:
\[
\varphi_1 [v](x,0) = \sin (\pi x) \text{ and } \varphi_2 [u](x,0) = \sin (\pi x).
\]

However, in order for the solution of the system in (18) to be confirmed by the second part of the initial conditions, i.e., \( \Phi_1 [u](x,0) = 0 \) and \( \Phi_2 [v](x,0) = 0 \) in Eq. (29), one needs to utilize both the first and the second parts which leads to 8 and 6 equations (first coefficients), respectively. Then the solutions follow immediately.
\[
\Phi_1 [u](x,0) \approx \sin (\pi x),
\]
\[
\Phi_2 [v](x,0) \approx \sin (\pi x),
\]
\[
\Phi_1 [u] (x,0) \approx 0,
\]
\[
\Phi_1 [v] (x,0) \approx 0.
\]

Now, we use the solutions \( \varphi_2 [u] \) and \( \varphi_2 [v] \) in the energy of the system, Eq. (4), for \( u \) and \( v \), see Eq. (17.a), to observe the stability of the system with the velocity feedback controllers. The final results are shown in Fig. 1, which is close to the finite difference solution in [1] as it shown in Fig. 2, below. Obviously, with more terms in \( \varphi (\varphi_1, \varphi_2, \ldots) \), one can obtain better confirmation. This shows that MADM is a strong method to find approximate solution to system (1-3), analytically, without using numerical computations. Note also that, the stability of the
system defined by Eqs (1-3) is assured, since $E(t) \to 0$, as $t \to \infty$, see Figs. 1 and 2, below.

**CONCLUSIONS**

We found, analytically, the approximate solutions to the coupled wave equations with their corresponding velocity feedback boundary conditions, via the Modified Adomian Decomposition Method (MADM). Maple 10 was utilized to evaluate these solutions. Having considered the solutions, we have then computed the energy of the system in order to observe the stability of the system regarding their velocity feedback controllers. The results are reasonably close to the benchmark finite difference-numerical computations’ data in [1]. Here, we can conclude that the MADM is a strong approach to solve such problems analytically without using numerical techniques. Also, it is good to mention that it is the first time, to the best of our knowledge, that the boundary valued control problems, which is governed by partial differential equations, was solved analytically and that serves the uniqueness of this research paper.

**REFERENCES**