Abstract—In this paper, we consider networked estimation where asymptotic consensus is replaced with only one fusion iteration between every two successive steps of system dynamics. With this restriction on the fusion, we show that the topology of the communication network plays a key role in the observability (and the error stability) of the estimator. For arbitrary system matrices, algebraic design of the communication topology is challenged with (i) large-scale computation, and (ii) particular fusion rules. To avoid these, structured systems theory and the notion of generic observability are implemented, which are computationally tractable and do not rely on exact fusion rules. We show the stability under weak network connectivity as compared to strong connectivity in the literature. In particular, we do not constrain the system matrix to be generically full rank compared to earlier works and show that for system matrices with rank deficiency (in the generic sense), implementing only state-estimate fusion does not recover the networked observability; thus, output fusion is required.

Keywords: Observability, Structured system theory, Generic properties, Networked estimation

I. INTRODUCTION

We consider networked estimation of linear discrete-time dynamical systems when the communication time-scale is the same as the dynamics time-scale [1]–[8]. In other words, there is only one fusion iteration per dynamical system time-step. This is in contrast to consensus-based strategies [9]–[11], where a consensus is reached within every two successive time-steps of the system dynamics. The two-time-scale approach is challenged with a large number (infinite, in general) of consensus iterations between every two steps of the dynamics, and thus, requires communication of a much faster rate. When compared to the single-time-step estimation, the two-time-scale approach is practically infeasible when the underlying system is operating under power constraints and restricted communication budget. The literature based on single time-scale (type) algorithms include [2] to more recent diffusion-based schemes [4], distributed binary detection [5], distributed moving horizon estimation [6], and information-theoretic based approach [7].

In this paper, we use a single time-scale Kalman-type estimator previously introduced in [1]. Notice that in the two-time-scale approach, the communication network is irrelevant and the performance and properties of the underlying estimator depends only on the data fusion algorithms. However, in a single time-scale method the key problem is to design the structure of the communication network. In general, in the single time-scale approach, an arbitrary communication network may not suffice to make the networked estimation error stable (e.g., see [1], [3]). In this regard, we propose sufficient inter-agent communication in order to recover the observability of the Networked Kalman-type Estimator (NKE) given that each agent may not be locally (in its neighborhood) observable. Here, we study the observability with a structural point of view, i.e., that we explore the generic properties of the system [12]–[14]. This approach is of interest when system parameters change, for example, depending on the system operating point in linearization of non-linear dynamics; it is further significant in communication network design as the approach is independent of the exact value of the weights chosen for data fusion.

Comparing with related work in the literature, we consider the system to be dynamic, unlike static estimation in [4], [5] where the target state does not evolve over time. We do not constraint the system matrix to be invertible as in [3], [7]. Our approach differs from the vanishing time-step algorithms proposed in [15] and we do not require the communication network to be strongly-connected [1], [4], [7], or for it to include a cyclic path [5], [16]. We further do not impose an agent hierarchy as in [17], [18], i.e., we assume the processing/communication duties at all agents are the same). From the aforementioned arguments, our results generalize the related works in the literature. Additionally, because of the generic approach, the methodologies presented are independent of exact system values and rely only on the zero-nonzero pattern of the system matrix. This is more robust in the sense that the analysis is not algebraic, but graph-theoretic [12], [13], [19]. Our results are applicable where precise network description is advantageous, e.g., in power networks and smart grids where networked estimator robust to system perturbations is desirable [20]. Particularly, an application to wind power plants can be envision, where the WTG dynamics are not invertible [21], and therefore, analysis of rank-deficient systems is required.

We now describe the rest of the paper. In Section II, we provide the background and preliminaries. In Section III, we state the problem formulation. We state our approach to recover the NKE observability in Section IV. Section V provides an illustrative example and simulation results, and finally, Section VI concludes the paper.

II. PRELIMINARIES

Consider a discrete-time linear dynamical system:

\[ x_{k+1} = Ax_k + v_k, \]  

where \( x_k \in \mathbb{R}^n \) is the state vector, \( A = \{a_{ij}\} \in \mathbb{R}^{n \times n} \) is the system matrix, and \( v_k \sim \mathcal{N}(0, V) \) is the system noise.
We assume the observation at each agent $i \in \{1, \ldots, N\}$ as:

$$y_k^i = C_i x_k + r_k^i,$$  \hspace{1cm} (2)

where\(^1\) $y_k^i \in \mathbb{R}^{p_i}$ is the output vector at agent $i$, \(r_k^i \sim \mathcal{N}(0, R_k)\) is the output noise, and $C_i$ is the output matrix at agent $i$. With this notation, we can write the global observation model as:

$$y_k = C x_k + r_k,$$ \hspace{1cm} (3)

where $y_k$, $C$, and $r_k$ are collections of the local variables, i.e., $r_k \sim \mathcal{N}(0, R)$ is the global observation noise with $R = \text{blockdiag}[R_1, \ldots, R_N]$, and $C = \{c_{ij}\}$ is the global output matrix.

### A. Centralized estimator

Let $\hat{x}_{k|k}^c$ be the centralized Kalman estimator [22] at time $k$ given all the observations, $y_k$, up to time $k$. It can be shown that the error in the centralized Kalman estimator, $\hat{e}_{k|k}^c = x_k - \hat{x}_{k|k}^c$, is given by

$$\hat{e}_{k|k}^c = (A - K_k CA) \hat{e}_{k-1|k-1}^c + \eta_k,$$ \hspace{1cm} (4)

where $K_k$ is the centralized Kalman gain and the vector $\eta_k$ collects the remaining terms that are independent of $\hat{e}_{k-1|k-1}$. It is well known that the centralized Kalman error, $\hat{e}_{k|k}^c$, is stable if and only if $(A, C)$ is observable\(^2\).

### B. Graph terminology

Let $X = \{x_1, \ldots, x_n\}$ denote the state set, and $Y = \{y_1, \ldots, y_N\}$ denote the output set. We define the system digraph as $G_A = (V, E)$, where $V = X \cup Y$ is the vertex set, and $E$ is the edge set. We omit the definitions of cycle, (simple) path, and Strongly Connected Component (SCC) here, and refer the readers to [23]. The definitions of parent/child SCC can be found in [3].

Here, we deal with two different graph representations: system digraph, $G_A$, representing states of the dynamic system (1) and (2), and digraph $G_W$ determining the communication network of the agents monitoring these states. Let $G_W = (V_W, E_W)$, where $V_W = \{1, \ldots, N\}$ is the vertex set, $E_W = \{(i, j) | i \leftarrow j\}$ is the edge set, and $D_i = \{i\} \cup \{j | (i, j) \in E\}$ denote the extended neighborhood of agent $i$. Notice that, in contrast to existing works we do not constrain $G_W$ to be undirected. In fact, no assumption on the topology is considered here, as designing $G_W$ is a contribution of this paper. As an example, consider Fig. 1 which shows the system digraph, $G_A$ with $n = 6$ states (encircled) and $N = 3$ agents (measurements) denoted by squares. In this work, the objective is to define the communication network between agents \{a, b, c\}.

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1Here, without loss of generality, we assume $p_i = 1$.

2Notice that detectability and observability are equivalent in the generic sense.

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### C. Structured systems theory

In this paper, we implement structural analysis as opposed to algebraic analysis. In other words, we analyze system properties which only depend on zero and nonzero pattern (structure) of the system matrix instead of dealing with the exact parameter values in the algebraic approach. Such structural property holds for almost all choices of nonzeros in the system matrix and, therefore, is called generic property. Furthermore, it can be shown that the non-admissible choices for which the generic property does not hold lie on some algebraic subspace with zero Lebesgue measure, see [12], [13] for more details. Here, we are particularly interested in generic rank and generic observability as discussed next.

**Definition 1** (S-rank): The structural rank (generic rank) of a matrix, $A$, is the maximum rank for all numerical values of the non-zero entries of the matrix $A$.

In the algebraic sense, the $S$-rank implies maximum number of non-zero elements in distinct rows and columns of a matrix [24].

**Theorem 1:** A dynamical system is generically observable if and only if in the system digraph,

- (i) every state is the begin-node of a path that ends in an output (termed as a $Y$-topped path),
- (ii) there exists a disjoint union of $Y$-topped paths and cycles that covers all the state vertices.

The proofs can be found in [19]. The following lemma is from [13], [14].

**Lemma 1:** The condition (ii) in Theorem 1 on the generic observability of $(A_{n \times n}, C_{m \times n})$ is equivalent to,

$$\text{S-rank} \left( \begin{bmatrix} A & \cdot \\ \cdot & C \end{bmatrix} \right) = n.$$ \hspace{1cm} (5)

As an example consider the system in Fig. 1. Now consider the conditions given in Theorem 1: (i) every state is a begin-vertex of a $Y$-topped path, and (ii) \{6\}, \{3, 4, 5\}, \{1, 2, a\} constitute a disjoint union of cycles and $Y$-topped paths; this implies that the system is generically $(A, C)$ observable.

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### III. Problem formulation: Networked estimator

Let $\hat{x}_{k|m}^i$ be the state estimate at time $k$ and sensor $i$ given the outputs up to time $m$, $(m \leq k)$ from sensor $i$ and its neighbors, $j \in D_i$. Each agent implements the following variant of Networked Kalman-type Estimation (NKE) as
proposed in [1], [3]:
\[
\hat{x}_{k|k-1} = \sum_{j \in D_i} w_{ij} A \hat{x}_{k-1|k-1},
\]
\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k^i \sum_{j \in D_i} C_j^T (y_j^i - C_j \hat{x}_{k|k-1}),
\]
where \( W = \{ w_{ij} \} \) is the state fusion weight matrix such that \( w_{ij} \geq 0 \) with \( \sum_{j \in D_i} w_{ij} = 1 \) (stochastic), and \( K_k^i \) is the local gain. We call Eq. (6) state fusion and Eq. (7) output fusion.

Let the estimation error at agent \( i \) and time \( k \) be
\[
e_k^i = x_k - \hat{x}_{k|k},
\]
where
\[
e_k = [(e_k^T)^T, \ldots, (e_k^N)^T]^T
\]
is the network estimation error. After some straightforward manipulations, we obtain the following networked error dynamics:
\[
e_k = (W \otimes A - K_k D_C (W \otimes A)) e_{k-1} + q_k,
\]
where
\[
K_k = \begin{bmatrix} K_k^1 \\ \vdots \\ K_k^N \end{bmatrix},
\]
\[
D_C = \begin{bmatrix} C_1^T C_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & C_N^T C_N \end{bmatrix},
\]
and \( q_k = [(q_1^1)^T, \ldots, (q_N^N)^T]^T \) collects the remaining terms—the weighted linear function of the system and output noise alone (and is independent of \( e_k \)). Comparing Eq. (10) to Eq. (4), it is straightforward to see that the networked estimation error, \( e_k \), can be stabilized if and only if, the following pair:
\[
(W \otimes A, D_C),
\]
is observable (detectable). In other words, a gain matrix, \( K_k \), exists such that
\[
\rho(W \otimes A - K_k D_C (W \otimes A)) < 1.
\]
if and only if \( (W \otimes A, D_C) \) is observable, where \( \rho \) denotes the spectral radius of a matrix. As it can be seen from Eq. (13), the communication network, \( W \), plays a major role in single time-scale distributed estimation.

**Remark 1:**
(i) The variables \( D_C \) and \( K_k \) are block diagonal matrices.
(ii) Every block diagonal, \( \sum_{j \in D_i} C_j^T C_j \), in the matrix \( D_C \), can be thought of as a representation of all the observations in the extended neighborhood, \( j \in D_i \), of agent \( i \)-extended implies that agent \( i \) is included in \( D_i \).
(iii) The diagonal entries of \( W \) are all nonzero, since every agent is in its own extended neighborhood, \( D_i \).

We refer to \( (W \otimes A, D_C) \) as the distributed system and \( G_W \otimes A \) as the system digraph associated with the (network dynamics) matrix \( W \otimes A \). For a better understanding of the structural relevance of the estimator in Eq. (7)-(6), we first consider \( W = I \) and \( D_C = \overline{D_C} \) where
\[
\overline{D_C} = \begin{bmatrix} C_1^T C_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & C_N^T C_N \end{bmatrix},
\]
implicating no information exchange among the agents. For example, consider the system in Fig. 1. This distributed system, \( (I \otimes A, D_C) \), consists of \( N \) subsystems each associated with an agent as it is shown in Fig. 2. Without any information fusion, each agent only has a partial observation of the system, and has to obtain the missing information via communicating with agents in its immediate neighbourhood. Information sharing among the agents by applying state and output fusion provides more links among the subsystems in the distributed system digraph. This extra linking, captured by the non-zeros in \( W \) and the summation in \( D_C \), has the potential to improve the generic observability of the system. In this regard, the main objective is to find a general method to make each subsystem observable.

**A. Assumptions**

In the rest of the paper, we make these assumptions:
(i) The communication between the agents is stable, i.e., the communication network is static;
(ii) The system is globally \((A, C)\)-observable;
(iii) For every agent, \( i \), we have \((A, C_i)\) and \((A, \sum_{j \in D_i} C_j^T C_j)\) are not necessarily observable.

Assumption (ii) is a typical assumption in distributed estimation implying the observability of centralized estimator; without this, no estimation scheme will work. Assumption (iii) is why our approach is more challenging and is different from current methodologies.

**B. Agent classification**

To describe our approach, we provide a novel agent classification and assume that the matrices \( A, C \) are given
such that Assumption (ii) above holds. Based on condition (i) in Theorem 1, centralized observability of the system enlist a maximal disjoint family of cycles and Y-topped paths, $\mathcal{L}$, covering all the state vertices. We term it as maximal with respect to the largest number of vertices contained in its cycles/paths. For example, from Fig. 1, two options for this set are $\{(3, 5, 3), (4, b), (1, 2, a), (6, c)\}$, and $\{(3, 4, 5, 3), (1, 1), (6, 6), (2, a)\}$, among others.

The following classification is with respect to $\mathcal{L} = \{(3, 4, 5, 3), (1, 1), (6, 6), (2, a)\}$:

**Definition 2:**

(i) Type-$\alpha$ agent is an agent that appears in the Y-topped paths in $\mathcal{L}$, e.g., agent $a$ in Fig. 1.

(ii) Type-$\beta$ agent is an agent that measures a state in the matched parent SCC in $\mathcal{L}$, e.g., agent $b$ in Fig. 1.

(iii) Type-$\gamma$ agent is an agent that is not Type-$\alpha$ nor Type-$\beta$, e.g., agent $c$ in Fig. 1, which is a child SCC.

The following lemma results from the agent classification:

**Lemma 2:** The following are crucial\(^4\) for observability:

(i) For every parent SCC, $K$, having at least one Type-$\beta$ agent observing a state in $K$.

(ii) Every Type-$\alpha$ agent.

The proof is straightforward and a direct result of Theorem 1. Removing Type-$\alpha$ agent and having a parent SCC without observation of a Type-$\beta$, respectively, violates condition (i) and condition (ii) in Theorem 1. A detailed proof is available in [3], [8], [25]. For example, in Fig. 1, both agent $a$ and agent $b$ are crucial. Notice that the given agent classification is with respect to the set $\mathcal{L}$ which is not unique. This issue can be solved by finding the counterparts of a crucial agent in other maximal sets.

We now use the agent classification and Lemma 2 for networked observability. We can recover the observability of the networked system via either $W \otimes A$ matrix (state-fusion) or $D_C$ (output-fusion). In the case of output-fusion, a link between two agents implies access to other agent’s measurement. However, the state-fusion case is more involved; a link from one agent to another (a non-zero entry in $W$) reflects a linking between agents’ subsystems in the networked system, $W \otimes A$. This will be discussed next in more details.

**IV. Recovering networked observability**

In this section, we present some helpful results for the development of the general solution for $(W \otimes A, D_C)$ observability. We discuss the role of state fusion and output fusion, separately. This is for sake of illustration and in reality agents send both state and output information over the links. The results and proofs in this section are mainly graph theoretic that is a direct consequence of our generic approach.

**A. Results on rank genericity**

The result below follows from Lemma 1 in Section II.

**Corollary 1 (Full S-rank):** System matrix, $A$, is full S-rank if and only if there exists a disjoint family of cycles covering all the state vertices in its digraph.

\(^3\)A matched SCC is an SCC that is structurally full rank

\(^4\)An observation is crucial if removing it renders the system unobservable.

**TABLE I**

**Distributed system according to different fusion levels.**

<table>
<thead>
<tr>
<th>Fusion level</th>
<th>Equivalent distributed system</th>
</tr>
</thead>
<tbody>
<tr>
<td>No data fusion</td>
<td>$(I \otimes A, D_C)$</td>
</tr>
<tr>
<td>Only state fusion</td>
<td>$(W \otimes A, D_C)$</td>
</tr>
<tr>
<td>Only output fusion</td>
<td>$(I \otimes A, D_C)$</td>
</tr>
<tr>
<td>Both measurement and state fusion</td>
<td>$(W \otimes A, D_C)$</td>
</tr>
</tbody>
</table>

From Corollary 1 and by Remark 1 (Section III) we obtain the following.

**Corollary 2:** The communication matrix, $W$, has a disjoint family of self-cycles and $S\text{-}rank(W) = N$.

This is always true because the matrix, $W$, has non-zero diagonals that can be represented as a disjoint union of self-cycles in its associated digraph. Consequently, we state the following lemma for the networked system $(W \otimes A)$. For example, in Fig. 1, both agent $a$ and agent $b$ are crucial.

**Lemma 3:** For the communication matrix, $W_{N \times N}$, and system matrix, $A_{n \times n}$, the networked system $W \otimes A$ is full S-rank if and only if $A$ is full S-rank. Mathematically,

$$S\text{-}rank(W \otimes A) = N \times n \iff S\text{-}rank(A) = n$$

**Proof:** The proof follows from the definition of the S-rank; details are available in [8].

**B. State fusion**

We now explore Eq. (6) in NKE protocol and assume that there is no output fusion. In particular, we analyze the structure of the matrix $W$ for $(W \otimes A, D_C)$ observability according to Table I. First, we provide some special cases from [3] where the system matrix, $A$, is full S-rank.

**Theorem 2:** Having a full S-rank system, $A$, the pair $(W \otimes A, D_C)$ is generically observable when for every matched parent SCC in $A$, say $K$, if agent $i$ does not have an observation of a state in $K$, then in the communication network, $G_W$, there must be a directed path from agent $i$ to an agent $j$, which has a state observation in $K$.

**Proof:** The proof can be found in [3], [8].

Consider again the three-output system in Fig. 1. Having vertices $\{3, 4, 5\}$ as parent-SCC, agent $b$ is the Type-$\beta$ agent. According to above theorem any other agent with no observation of $\{x_3, x_4, x_5\}$ must be connected to agent $b$. For example, a link from agent $a$ provides a connection from parent-SCC $\{3, 4, 5\}$ in subsystem $a$ to its counterpart SCC in subsystem $b$ in distributed system graph $G_{W \otimes A}$, and its output connectivity. A very straightforward point to mention here is that for full S-rank systems, there only exist Type-$\beta$ and Type-$\gamma$ agents.

We now provide our main result on state fusion. It turns out that if the system matrix is not full S-rank, then even using a fully-connected network (complete $G_W$ graph) does not recover observability as stated in the following theorem.

**Theorem 3:** If system, $A$, is not full S-rank, then $(W \otimes A, D_C)$ is not observable for any choice of the communication matrix $W$.

**Proof:** The proof follows from the fact that for any choice of $W$ we have

$$S\text{-}rank\left(\begin{bmatrix} W \otimes A \\ D_C \end{bmatrix}\right) < Nn.$$  \hspace{1cm} (16)
which violates the condition (i) in Theorem 1. The detailed proof is provided in [8].

The above theorem shows that for $S$-rank-deficient systems, state fusion does not guarantee the observability of the system and, thus, measurement fusion is required to recover observability. In contrast, Theorem 2 shows that when $A$ is full $S$-rank, then state fusion alone can recover observability. To the best of our knowledge, these results have not been developed before.

C. Output fusion

The other solution to make the NKE observable is output fusion as given in Eq. (7). According to the formulation, each agent shares its observation with its direct neighbours and implements this as an innovation to update its prediction. According to Table I, for output fusion the structure of the matrix $D_C$ has to be determined such that $(I \otimes A, D_C)$ is observable. Based on the definition of $D_C$, the $i$th $n \times n$ diagonal block of $D_C$ contains all measurements in the extended neighbourhood of agent $i$. In the distributed system graph $G_{(I \otimes A, D_C)}$, say for the agent $i$, this is equivalent to adding all measurements in the neighbourhood $N_i$. Based on this we now provide our main result on output fusion.

Theorem 4: The system $(I \otimes A, D_C)$ is observable if and only if in the communication network $G_W$:

(1) Every Type-$\alpha$ agent, $i$, is directly linked to every other agent $j$;

(2) For every matched parent SCC, $K$, every agent, $j$, with a state observation in $K$ is directly linked to a Type-$\beta$ agent, $i$, with a state observation in $K$.

Proof: The proof is a direct result of Lemma 2. We refer interested readers to [8] for more details.

D. NKE observability

We now consolidate the previous results. Recall that Theorem 2 sets the condition for state fusion for full $S$-rank systems, i.e., generic observability of $(W \otimes A, D_C)$. Theorem 3 states that for $S$-rank deficient systems networked observability cannot be achieved via the state fusion alone. Therefore, output fusion, i.e., generic observability of $(I \otimes A, D_C)$, is discussed in Theorem 4. Combining these results, we now provide the main theorem on generic observability of the single time-scale NKE protocol in Eqs. (6)-(7).

Theorem 5: Networked system $(W \otimes A, D_C)$ is generically observable if each agent has:

(1) A direct link from all the Type-$\alpha$ agents (output-fusion);

(2) A directed path to (at least) one Type-$\beta$ agent for every matched parent SCC of $A$. This means, if there is two or more agents observing the same parent SCC, a directed path to any one of them is sufficient (state-fusion).

Proof: The proof is a direct consequence of Theorems 2, 3, and 4.

Remark 2: In the case of Type-$\beta$ agents, every agent requires either a directed path to each Type-$\beta$ agent (as stated in the Theorem 5) or a direct link from each Type-$\beta$ agent (as stated in the Theorem 4); either one of these two conditions is sufficient for observability. However, the latter almost always requires long-distance communication links compared to the former, and therefore, is preferred in order to minimize the communication budget.

E. Local gain design

Having the agents’ network, $(G_W)$, defined as in Theorem 5, generic observability of $(W \otimes A, D_C)$ is guaranteed. This implies existence of a full gain matrix, $K$, such that $\rho(W \otimes A - K_D(W \otimes A)) < 1$. Now, according to Remark 1, the gain matrix, $K$, is needed to be block-diagonal with $N$ blocks of $n \times n$ matrices. To overcome this structural constraint of the gain matrix, $K$, we use the iterative procedure based on LMI approach proposed in [1]. It is assumed that the matrix $K$ is independent of time, $k$, and thus is denoted by $K$. We omit the details here and refer interested readers to [1] and references therein.

V. EXAMPLE AND SIMULATION

Consider the system, $(A, C)$, given in Fig. 1 with structured matrices given in the following:

\[
A = \begin{bmatrix}
x & 0 & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 \\
0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & x \\
\end{bmatrix},
\]

(17)

Based on Corollary 1, this system is $S$-rank deficient. The output matrix has the following structure:

\[
C = \begin{bmatrix}
C_a \\
C_b \\
C_c \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(18)

By Definition 2, agent $a$ is Type-$\alpha$, agent $b$ is Type-$\beta$ (crucial agents) and agent $c$ is Type-$\gamma$. We propose the following communication matrices $W_1$ and $W_2$, and their associated graphs $G_W$ in Fig. 3.

\[
W_1 = \begin{bmatrix}
x & 0 & 0 \\
0 & x & 0 \\
0 & 0 & x \\
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
x & 0 & 0 \\
0 & x & 0 \\
0 & 0 & x \\
\end{bmatrix}.
\]

(19)

The graph $G_{W_1}$ is based on Theorem 4, where agents, $(a, b)$, are directly linked among each other and both have a directed edge to agent, $c$. The network $G_{W_2}$ is based on Theorem 5, where there is a direct link from agent $a$ (Type-$\alpha$) to all other agents, and there is a path from every other agent to agent $b$ (Type-$\beta$).
with the structure in Eq. (17) and output matrix in Eq. (19) to find the block-diagonal gain matrix, normalized Mean Squared Error Estimation is bounded for all agents.

Fig. 4. Performance of the networked estimator at agents \{a, b, c\}. The normalized Mean Squared Error estimation is bounded for all agents.

For simulation, we consider random valued matrix, \( A \), with the structure in Eq. (17) and output matrix in Eq. (19) with entries equal to 1. We have \( \rho(A) = 1.09 \), which implies that system is unstable. We choose the agents’ network according to Fig. 3(Right) with random link weights. Notice that the matrix \( W \) is stochastic. We use the algorithm in [1] to find the block-diagonal gain matrix, \( K \). The system and output noise are, respectively, \( v_k \sim \mathcal{N}(0,0.05^2) \) and \( r^e_k \sim \mathcal{N}(0,0.2^2) \). As system initial state, we choose random initial values in \([0,3]\).

The Monte-Carlo evolution of the system error over 50 iterations for three agents and 1000 trials are given in Fig. 4. For the visual clarity, we show squared errors at each iteration, summed over \( n = 6 \) states, and averaged over 1000 Monte-Carlo trials. As it can be seen, the estimation error is bounded in spite of system unstability.

VI. CONCLUSION

In this paper, we study the role of the agent communication network towards error stability of the NKE protocol (6)-(7) in the context of single time-scale networked estimation. Unlike our previous work [3] (or many other works in the literature [1], [7]), here we do not impose any constraint on the system matrix, \( A \). Furthermore, we determine dynamical systems (S-rank deficient) for which no state-fusion results in an observable networked estimator and one further has to rely on output-fusion. Because of our generic approach, the link weights are free parameters and results are independent of any particular fusion rule (e.g., Metropolis-Hastings) chosen in Eqs.(6)-(7). Nevertheless, the structure of the underlying agent communication remains relevant and leads to network/infrastructure design. Furthermore, link weights can be optimized to reduce the error, which is a direction of the future work.

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