A Constructive Stabilization Approach for Open-Loop Unstable Non-Affine Systems*

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Abstract—This paper focuses on the stabilization of non-affine-in-control systems that are open-loop unstable. The main result of the paper is a general method for constructing feedback stabilization of all non-affine systems. The synthesis procedure is based on concepts of feedback passivation, and is extended for non-affine systems by deriving sufficient conditions for passivity. The developments and essential ideas of the proposed technique are validated via simulation.

I. INTRODUCTION

Consider the core problem of developing stabilizing controllers for non-affine systems of the form

\[ \dot{x} = f(x, u) \]  

(1)

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, and \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a sufficiently smooth vector field. Assume that

(A1) The unforced dynamics of \( \Sigma \) in (1), namely \( \dot{x} = f(x, 0) \) is unstable.

In general the function \( f(x, u) \) is not monotonic in the control and \( \partial f / \partial u \) may be singular at the origin. Thus \( \Sigma \) cannot be stabilized using either dynamic-inversion [1] or the modeling error compensation technique [2]. Furthermore, non-affine systems of the form given in (1) cannot be stabilized using a fixed-gain static compensator. To see this behaviour consider the system

\[ \dot{x} = x - 2u^3. \]

(2)

It is open-loop unstable and satisfies Assumption (A1). Suppose the control takes the form \( u = K(t)x \) in (2). Then the resulting closed-loop dynamics become \( \dot{x} = x - 2K^3x^3 \) that has the following equilibrium solutions:

\[ x_\ast = \begin{cases} 
0 & \text{for all } K \\
\pm \frac{1}{\sqrt{2}K^3} & \text{for } K > 0.
\end{cases} \]

(3)

The bifurcation map (Fig. 1) illustrates that the origin remains unstable except for \( K = \infty \). Furthermore, the system has three equilibrium solutions for positive values of the feedback gain that converge to the origin as \( K \to \infty \). This behaviour indicates that only an infinite control effort can stabilize the origin.

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Fig. 1. Stable (solid lines) and unstable (broken line) equilibrium solutions of (2) with \( u = Kx \)

An alternative solution to regulate (2) is to switch feedback gains in accordance with the current state. As an example, suppose the feedback gain for (2) was initialized to \( K(t = 0) = 1.5 \). Then from (3) the state would stabilize to either \( x_\text{steady} = -0.384 \) or \( x_\text{steady} = 0.384 \) depending on its initial condition. Next, if the feedback gain is switched to \( K(t > t_\text{steady}) = 2 \) the state would stabilize at \( x_\text{steady} = \pm 0.25 \). Hence, increasing the gain successively the origin can be stabilized through a finite control input. The fundamental problem with this switching scheme is that analytic determination of the switching curves for general systems of the form (1) requires substantial system knowledge and offline processing. Additionally, the switching conditions and the number of control switches depend on the initial condition and the control form, leading to a system specific design [3], [4].

In this paper the construction of an analytic state-feedback control law is pursued. The major contribution of the paper is a theoretical result which shows that under mild conditions a control law of the form \( u(x) = \alpha(x) + \nu(x) \) globally stabilizes a large class of non-affine systems. The function \( \alpha(x) \) converts an open-loop unstable system into a stable system in the Lyapunov sense, and \( \nu(x) \) is constructed to bring about the necessary energy dissipation for globally stabilizing the origin. The design procedure presented here is based on the ideas of feedback passivation introduced in [5] for control-affine systems. The general concept is to use state-feedback to render the system passive and then employ well-established results for stabilizing passive systems. Toward this end, the fundamental question to be
answered is when is a general nonlinear system passive? The well-known Kalman-Yacubovitch-Popov lemma [6] and its nonlinear counterparts derived by Hill and Moylan [7] answer this question for linear and affine-in-control systems respectively. Sufficient conditions for passivity of non-affine systems and their relationship with the existing necessary conditions [8] are derived for the first time in this paper, in Section II. The main result for stabilization of general multiple-input systems posed as sufficiency conditions is derived in Section III. The theoretical results are verified with simulation examples in Section IV. Closing remarks are discussed in Section V.

II. PASSIVE SYSTEMS

In this section the sufficiency conditions under which a nonlinear system can be considered passive are derived. Consider

\[ \Sigma_1: \begin{align*}
    \dot{x} &= f(x, u) \\
    y &= h(x, u)
\end{align*} \]  

(4)

with state-space \( X = \mathbb{R}^n \), set of input values \( U = \mathbb{R}^m \), and set of output values \( Y = \mathbb{R}^m \). The set \( \mathcal{U} \) of admissible inputs consists of all \( U \)-valued piece-wise continuous functions defined on \( \mathbb{R} \). The functions \( f(.) \) and \( h(.) \) are continuously differentiable maps defined on the open subset \( O \subset \mathbb{R}^n \). It is assumed that these vector fields are smooth mappings, with at least one equilibrium. Without loss of generality the origin is chosen as the equilibrium of \( \Sigma_1 \), that is, \( f(0,0) = 0 \) and \( h(0,0) = 0 \). In order to derive conditions for \( \Sigma_1 \) to be passive two important definitions are reviewed and presented below.

Definition II.1. The system \( \Sigma_1 \) is said to be passive if there exists a positive semi-definite storage function \( V(x) \) that satisfies \( V(0) = 0 \) and for any \( u \in \mathcal{U} \) and initial condition \( x_0 \in X \)

\[ V(x) - V(x_0) \leq \int_0^t y^T(s)u(s)ds. \]  

(5)

If the storage function is \( C^r \) times continuously differentiable with \( r \geq 1 \) then (5) is equivalent to

\[ \dot{V} \leq y^T u. \]  

(6)

Definition II.1 is the mathematical analog of stating that the amount of energy stored in a passive system is less than or equal to the energy being input. For convenience define the vector fields

\[ f_0(x) \triangleq f(x, 0) \in \mathbb{R}^n \]  

(7a)

\[ h_0(x) \triangleq h(x, 0) \in \mathbb{R}^m \]  

(7b)

where \( f_0(x) \) represents the open-loop dynamics of \( \Sigma_1 \) while \( h_0(x) \) is the output of \( \Sigma_1 \) at zero-input. Using (7) and the fact that the vector fields in \( \Sigma_1 \) are smooth, (4) is equivalently represented as

\[ \dot{x} = f_0(x) + g(x, u)u \]  

(8a)

\[ h(x, u) = h_0(x) + j(x, u)u \]  

(8b)

where the following identities have been used:

\[ f(x, u) - f_0(x) = \left( \int_0^1 \frac{\partial f(x, \gamma)}{\partial \gamma} |_{\gamma = \theta u} d\theta \right) u(x) \triangleq g(x, u)u \]  

(9)

\[ h(x, u) - h_0(x) = \left( \int_0^1 \frac{\partial h(x, \gamma)}{\partial \gamma} |_{\gamma = \theta u} d\theta \right) u(x) \triangleq j(x, u)u. \]  

(10)

The vector fields \( g(x, u) \) and \( j(x, u) \) defined above capture the effect of the control input on the motion of the dynamical system states and the output. Notice that for control-affine systems these vector fields will be independent of the control input vector. Using smoothness of the vector \( g(x, u) \), (8a) can be further decomposed as

\[ \dot{x} = f_0(x) + g_0(x, u)u + \sum_{i=1}^m u_i [R_i(x, u)u] \]  

(11)

with \( R_i(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times m} \), being a smooth map for \( 1 \leq i \leq m \) and

\[ g_i^0(x) = g_i(x, 0) \]  

(12a)

\[ g_0(x) = \frac{\partial f}{\partial u}(x, 0) = [g_1^0(x), \ldots, g_m^0(x)] \in \mathbb{R}^{r \times m} \]  

(12b)

The vector field \( g_i^0(x) \) defines the influence of the input \( u_i \) on the system about the origin and is collected for all inputs under the vector \( g_0(x) \).

The next definition gives the necessary conditions for an input/output nonlinear system \( \Sigma_1 \) to be passive. In the following and the rest of the paper, the expression

\[ \mathcal{L}_{f_0} V = \left( \frac{\partial V}{\partial x}, f_0(x) \right) \]  

(13)

represents the Lie derivative of the \( C^r (r \geq 1) \) functional \( V : \mathbb{R}^n \to \mathbb{R} \) along the vector field \( f_0(x) \). Additionally, the standard notation \( \mathcal{L}_{f_0} \) is used for Lie bracket.

Definition II.2. [8]. Let \( \Omega_1 = \{ x \in \mathbb{R}^n : \mathcal{L}_{f_0} V(x) = 0 \} \). Necessary conditions for \( \Sigma_1 \) to be passive with a \( C^2 \) positive semi-definite storage function \( V \) are

(i) \( \mathcal{L}_{f_0} V(x) \leq 0 \),

(ii) \( \mathcal{L}_{g_i} V(x) = h_i^0(x) \) \( \forall x \in \Omega_1 \),

(iii) \( \sum_{i=1}^m \frac{\partial h_i^0(x, 0)}{\partial x} \leq j^T(x, 0) + j(x, 0) \) \( \forall x \in \Omega_1 \),

where \( f_i(x, u) \) is the \( i \)-th component of the vector function \( f(x, u) \).

For a positive-definite storage function property (i) is analogous to Lyapunov’s condition \( \dot{V} \leq 0 \) for bounded stability. The other conditions in Definition II.2 follow directly from Definition II.1 by noticing that the difference \( \frac{\partial V}{\partial x} f(x, u) - \dot{h}(x, u)u \) attains its maximum at \( u = 0 \) on the set \( \Omega_1 \).

The following theorem completes Definition II.2 by presenting the sufficiency conditions required for a system \( \Sigma_1 \) to be passive.

Theorem 1. Let \( V \) be a \( C^1 \) positive semi-definite function. A system \( \Sigma_1 \) is passive if there exists some functions \( q : \mathbb{R}^n \to \mathbb{R}^n \) with \( \dot{V} \leq q(x) \) \( \forall x \in \Omega_1 \).
Theorem 1 with a positive-definite storage function. This result is an algorithm for stabilizing non-affine systems of the form
\[ \Sigma : \dot{x} = f(x, u); \quad x(0) = x_0 \] (1)
with state-space \( X = \mathbb{R}^n \) and set of input values \( U = \mathbb{R}^m \). The set \( U \) of admissible inputs consists of all \( U \)-valued piecewise continuous functions defined on \( \mathbb{R} \). The vector field \( f(.) \) is a continuously differentiable map defined on the open subset \( O \subset \mathbb{R}^n \). Without loss of generality, the origin is chosen as the equilibrium of \( \Sigma \). The control algorithm is detailed in the following four steps.

Step 1: Define vector fields for the system under study:
\[ f_0(x) = f(x, \alpha(x)) \quad \gamma \quad g(x, \nu(x)) = \int_0^1 \frac{\partial f(x, \alpha(x) + \gamma)}{\partial \gamma} \bigg|_{\gamma=\theta \nu} \ d\theta \in \mathbb{R}^{n \times m} \] (18)

Under the influence of the control in (17) and the definitions above, \( \Sigma \) becomes
\[ \dot{x} = f_0(x) + g(x, \nu(x))\nu(x). \] (19)
Notice that the vector field \( f_0(x) \) is the closed-loop dynamics with control input \( \alpha(x) \), unlike \( f_0(x) \) defined in (7a). Further, \( f_0(x) \) is independent of \( \nu(x) \) in (19). This allows separate construction of \( \alpha(x) \) independent of \( \nu(x) \) and is exploited in the following step.

Step 2: Construct \( \alpha(x) \) to ensure \( f_0(x) \) is stable in the Lyapunov sense. This step ensures the energy of the system remains bounded. Note that during construction of \( \alpha(x) \) the control function \( \nu(x) = 0 \) identically.

Step 3: Define a dummy output \( h(x, \nu(x)) \in \mathbb{R}^m \) for the system in (19) to ensure that it becomes passive through the input \( \nu(x) \), or equivalently, satisfies Definition II.1 or Theorem 1 with a positive-definite storage function.

Step 4: Finally, construct the control function \( \nu(x) \) to satisfy
\[ \nu(x) = -h(x, \nu(x)). \] (20)

From previous work on stabilization of passive systems [8] it is known that if the dynamics (19) along with the output definition in Step 3 is zero-state detectable, then the control...
given in (20) globally asymptotically stabilizes the system. Zero-state detectability states that if the output is identically zero, then the state vector approaches the origin in time. The detectability properties of $\Sigma$ can be verified through accessibility type conditions summarized below.

**Definition III.1.** [8] Suppose the system $\Sigma$ is passive with $C^r(r \geq 1)$ storage function $V$, which is positive definite and proper. Then, $\Sigma$ is zero-state detectable if $\Omega \cap S = \{0\}$ where the distribution

$$D = \text{span} \left\{ \text{ad}^k_{\mathcal{L}_0} \mathcal{L}_0^0 : 0 \leq k \leq n - 1, 1 \leq i \leq m \right\}$$

and two sets $\Omega$ and $S$, associated with $D$ be defined as

$$\Omega = \left\{ x \in \mathbb{R}^n : \mathcal{L}_0^k V(x) = 0, k = 0, 1, \ldots, r \right\}, \quad \text{(21)}$$

$$S = \left\{ x \in \mathbb{R}^n : \mathcal{L}_0^k \mathcal{L}_0^\nu V(x) = 0, \forall \tau \in D, k = 0, 1, \ldots, r - 1 \right\} \quad \text{(22)}$$

The following proposition proves that $\Sigma$ described in (1) and equivalently in (19) is asymptotically stabilized by the proposed control algorithm.

**Proposition 1.** Suppose $V$ is a $C^2$ positive-definite Lyapunov function and the functions $\alpha(x)$ and $\nu(x)$ satisfy Steps 1-4 with output $h(x, \nu(x)) = [\mathcal{L}_g V]^T$. If $\Omega \cap S = \{0\}$ then the control $u(x) = \alpha(x) + \nu(x)$ asymptotically stabilizes the system $\Sigma$. \hfill \Box

Proof. Asymptotic stabilization is shown using LaSalle’s invariant principle and Lyapunov’s direct method. The rate of change of the Lyapunov function about the trajectories of $\Sigma$ given in (19) is

$$\dot{V} = \mathcal{L}_{\mathcal{L}_0} V + \mathcal{L}_g V \nu(x).$$

Then, through construction of $\alpha(x)$

$$\dot{V} \leq \mathcal{L}_g V \nu(x).$$

By Definition II.1 and Theorem 1 (24) is passive with the output $y = (\mathcal{L}_g V)^T$. With $\Omega \cap S = \{0\}$, this passive system is zero-state detectable and $\Sigma$ is asymptotically stabilized by input $\nu(x) = -\mathcal{L}_g V$. This completes the proof. \hfill \Box

Proposition 1 is a powerful result that asymptotically stabilizes for all non-affine nonlinear systems. This method of control synthesis is general and relies upon separate construction of stiffness and damping functions $\alpha(x)$ and $\nu(x)$ respectively. The construction of $\nu(x)$ has received considerable attention in the literature under the label ‘passivity-based control’. The requirements of zero-state detectability is a consequence of employing pure output feedback for passive systems [8], [10], [11] which can be relaxed by use of other methods for control of open-loop stable systems.

**IV. NUMERICAL EXAMPLE: ONE-DIMENSIONAL NON-AFFINE UNSTABLE DYNAMICS**

The purpose of this section is to verify the theoretical developments through an open-loop unstable non-affine system. The example considered is a polynomial system of degree three given in (2). The control law for this example was developed through analytic root solving techniques in [12]. Here an alternate control law formulation is presented to globally stabilize the origin.

1) **Controller Design:** The feedback control of the form (17) is constructed in four steps.

**Step 1:** The feedback control of the form (17) is presented to globally stabilize the origin.

Using $\alpha(x)$ defined above the dynamics $f_0(x)$ become

$$f_0(x) = \begin{cases} x - x^3 & \text{if } |x| \geq 1; \\ x + 1 & \text{if } -1 < x < 0; \\ 0 & \text{if } x = 0; \\ -x - 1 & \text{if } 0 < x < 1. \end{cases}$$

Note that $f_0(x)$ described in (22) has three stable fixed points: $x = -1$, $x = 0$, and $x = 1$. Thus the dynamics of the system (2) are rendered stable for all time.

**Steps 3 & 4:** These steps construct the control input $\nu(x)$ that enforces stability of the origin. Control laws for such a class of system have been addressed by passivity-based methods. Following the formulation given in [10] control input $\nu(x)$ is constructed as

$$\nu(x) = -\frac{2}{\beta} x^2 \frac{g_0 V(x)}{1 + |g_0 V(x)|^2}$$

where $\gamma(x) = \frac{\beta}{1 + x^2 + (1 + 4\alpha^2(x))x^0},$ and $\mathcal{L}_g V(x)$ is the Lie derivative of $V(x)$ along $[0, g^0(x)]$. The design parameter $0 < \beta < 1$ bounds the control input.

Proposition 1 guarantees that the control input $\alpha(x) + \nu(x)$ asymptotically stabilizes an open-loop unstable stable system if $\Omega \cap S = \{0\}$. A routine calculation shows that $\mathcal{L}_{\mathcal{L}_0} V(x) = 0$ for $\Omega = \{-1, 0, 1\}$. Additionally,

$$0 = \mathcal{L}_{g^0} V(x) = -6x \alpha^2(x) \quad \text{(29)}$$

is satisfied for $x = 0$ so $\Omega \cap S = \{0\}$ for all $x \in \mathbb{R}$. Hence it can be concluded that the control form $\alpha(x) + \nu(x)$ globally asymptotically stabilizes the origin. Reference [12] designed $u = \sqrt{x}$ as the control law for the prescribed system using inversion, which only locally regulates the system (2).
The control is dominated by \( \alpha \) the figure at time \( t \), \( \nu \) magnitude of the control input \( \nu \). The closed-loop response is shown in Fig. 3. The behaviour of the open-loop system and the system with control input \( u = \alpha(x) \) is presented in Fig. 2. As expected the open-loop behaviour is unstable and the system with \( u = \alpha(x) \) stays at \( x = 1 \) for all time. The closed-loop response is shown in Fig. 3. The initial magnitude of the control input \( \nu (x) \) is small (specifically \( \nu(x) = 0.000293 \)) but greater than zero to ensure the state of the system becomes less than 1. It is difficult to see but in the figure at time \( t = 2 \) seconds the state is \( x(2) = 0.993 \). The control is dominated by \( \alpha(x) \) since the dynamics \( f_{\alpha}(x) \) inherently push the system toward the origin. By construction in (28) the magnitude of \( \nu(x) \) increases when the state nears the origin so as to asymptotically regulate the dynamics. This is consistent with earlier conclusions that high-gain feedback is required to stabilize the origin. Thereafter the control is turned off and the system stays at the origin for all future time. Note that the discontinuous nature of the control is an artifact of the choice of \( \alpha(x) \).

V. Conclusions

In this paper a design procedure for analytic construction of control laws for unstable non-affine systems was proposed, and sufficiency conditions for passivity were derived. This work also extended well-established control law design procedures for stable non-affine-in-control systems to unstable non-affine systems, without requiring the control influence to be non-singular throughout the domain of interest.

The proposed control laws are real-time implementable and unlike some switching schemes proposed in literature, do not require immense offline processing. The algorithm is general and can stabilize systems of the form \( \dot{x} = x - 2xu^4 \) by appropriate design of \( \alpha(x) \). Owing to the energy-based concept that is utilized for construction of the control, the results obtained are consistent with the physics of the problem and do not violate system constraints. Numerical examples illustrate that the nature of the control function \( \alpha(x) \) is continuous but not differentiable. It is interesting to note that [13]Corollary 5.8.8 proved that any nonlinear system whose linear counterparts are unstable cannot be locally \( C^1 \) stabilizable. Importantly, the control laws derived here arrive at this well-known result without making any prior assumptions about the nature of the vector fields.

References