LMI based design of a sliding mode controller for a class of uncertain fractional-order nonlinear systems*

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Abstract—In this paper, the problem of designing a sliding mode controller for a class of uncertain fractional-order nonlinear systems with $0 < \beta < 1$ is addressed by using linear matrix inequality (LMI) method. A key analysis technique is enabled by proposed a fundamental boundedness lemma, which is used for rigorous stability analysis of fractional-order systems, especially for Mittag-Leffler stability analysis of fractional-order nonlinear systems. A new switching law is given to guarantee the reachability condition. This sliding mode control law is utilized to obtain a controller capable of drawing the state trajectories onto the sliding surface and maintain the sliding motion. Numerical simulation results are presented to show the effectiveness of the proposed sliding mode control scheme.

I. INTRODUCTION

In the past few decades, much attention has been paid to the study of fractional calculus ([1]–[5]). In essence, the fractional-order differintegration operator is denoted as $D_t^\beta = d^\beta/dt^\beta$ where $\beta \in R$, just like the derivative operator for integer order. The applications of fractional calculus have been intensively investigated in many fields of science, covering engineering and automatic control. Study on the dynamics of fractional-order differential systems has attracted interest of many researchers ([6]–[9]).

Considering the fractional-order system within the context of feedback control, some fundamental definitions have been proposed in ([3]–[5]). It is well known that stability is fundamental to all control systems, certainly including fractional-order control systems. Stabilization of a fractional-order system has been investigated ([10]–[13]). Especially, the authors of [11] have proposed two advances of the fractional-order Lyapunov stability theorems. These theorems are used to prove the Mittag-Leffler (asymptotically) stability. Many results about fractional-order systems have been obtained based on these stability theorems, such as ([14]–[18]). In these papers, the Lyapunov function $V(t) = x^T(t)x(t)$ are usually chosen, and the $\beta$-order time derivative of $V(t)$, according to the Leibniz’s rule of differentiation, can be expanded as

$$D_t^\beta V = (D_t^\beta x)^T + x^T (D_t^\beta x) + 2 \Upsilon,$$

where $\sum_{k=1}^{\infty} \frac{\Gamma(1+\beta)(D_t^k x)^T(D_t^{\beta-k} x)}{\Gamma(1+k)\Gamma(1-k+\beta)}$, $\Gamma(\cdot)$ is the gamma function [19]. In order to easily obtain the results by using the stability theorems in [11], most of these papers assumed that this condition $\|\Upsilon\| \leq B_1 \|x\|$ exists. However, since $\Upsilon = \sum_{k=1}^{\infty} (\Gamma(1+\beta)/\Gamma(1+k)\Gamma(1-k+\beta))(D_t^k x)^T(D_t^{\beta-k} x)$, $\forall \beta \in (0, 1)$ converges to 0 in order to show the correctness of the condition $\|\Upsilon\| \leq B_1 \|x\|$. However, the author cannot prove that the boundedness condition $\|\Upsilon\| \leq B_1 \|x\|$, $\forall \beta \in (0, 1)$ converges to 0. As far as we know, there is no research effort reported on the existence of the boundedness condition on $\Upsilon$. In this paper, we will prove the boundedness condition on $\Upsilon$ and establish a fundamental boundedness lemma. This fundamental boundedness lemma is established to use for stability analysis of fractional-order systems, especially for Mittag-Leffler stability analysis of fractional-order nonlinear systems.

On the other hand, sliding-mode controller (SMC) is a powerful tool to robustly control incompletely modeled or uncertain systems ([22]–[26]). The main feature of SMC is to switch the control law to force the states of the system from the initial states onto some predefined sliding surface. The system on the sliding surface has desired properties such as stability, disturbance rejection capability, and tracking capability. SMC to accommodate fractional-order nonlinear systems has not yet attracted much attention, due primarily to the mathematical difficulties in stability analysis. There are limited published results which are mainly concerned about fractional-order chaotic systems under SMC. One natural problem is how to establish direct systematic methods for designing SMC for fractional-order nonlinear system.

In this paper, the problem of designing SMC for uncertain fractional-order nonlinear systems is studied, based on our proposed fundamental boundedness lemma. The central idea is firstly proving a fundamental boundedness lemma, which is utilized to stability analysis of fractional-order nonlinear systems. According to the fundamental boundedness lemma, the controller is derived to guarantee asymptotically stability of the uncertain fractional-order nonlinear systems. Furthermore, the novel stability criteria is given via LMIIs. Some simulations show the effectiveness of the proposed method. The paper is presented as follows: in section II, notations and problem statement are given. In section III, sliding surface and control scheme design is proposed and
analyzed. Numerical simulations results are shown in section IV. Finally, conclusion is addressed in section V. From this lemma, one can conclude that the assumption about boundedness condition in (14)–(17) is valid and

\[ V(t) = x^T(t)x(t). \]

According to the Leibniz’s rule of differentiation, the \( \beta \)-th order derivative of \( V(t) \) can be expanded as

\[ D_t^\beta V = (D_t^\beta x)^T x + x^T(D_t^\beta x) + 2\Upsilon, \]

where

\[ \sum_{k=1}^{\infty} \frac{\Gamma(1 + \beta)}{\Gamma(1 + k)} \frac{1}{\Gamma(1 - k + \beta)} \leq B_1 \|x\|. \]

Proof. First, one has

\[ \sum_{k=1}^{\infty} \frac{\Gamma(1 + \beta)}{\Gamma(1 + k)} \frac{1}{\Gamma(1 - k + \beta)} \leq \sum_{k=1}^{\infty} \frac{\Gamma(1 + \beta)}{\Gamma(1 + k)} \frac{1}{\Gamma(1 - k + \beta)} \|D_t^\beta x\| \|D_t^{\beta-k} x\|. \]

Since \( D_t^\beta x, (k = 1, 2, 3, \ldots) \) exist, \( D_t^\beta x, (k = 1, 2, 3, \ldots) \) are continuous. Moreover, it is obvious that \( D_t^{\beta-k} x, (k = 1, 2, 3, \ldots) \) are bounded since \( \Omega \) is the closed set. Thus, there exists \( M \) such that \( \|D_t^k x\| \leq M, (k = 1, 2, 3, \ldots) \).

On the other hand, for \( D_t^{\beta-k} x, (k = 1, 2, 3, \ldots) \), due to \( 0 < \beta < 1 \), it is apparent by using Lemma 1 that

\[ \|D_t^{\beta-k} x\| \leq K_{\text{max}} \|x\|, (k = 1, 2, 3, \ldots), \]

in which \( K_{\text{max}} > 0 \).

Furthermore, it is well known that the Gamma function is nonzero everywhere along the real line. There is in fact no complex number \( z \) for which \( \Gamma(z) = 0 \), and hence the reciprocal gamma function 1/\( \Gamma(z) \) is an entire function, with zeros at \( z = 0, -1, -2, \ldots \) in [21]. Thus, there exists a lower bound \( L_{\min} \) such that \( 0 < L_{\min} \leq \Gamma(1 - \beta + k) \) for \( k = 1, 2, 3, \ldots \).

Because \( \frac{\Gamma(k)}{\Gamma(k+1)} = \frac{1}{k}, (k = 1, 2, 3, \ldots) \), the infinite series

\[ \sum_{k=1}^{\infty} \frac{1}{\Gamma(k+1)} \]

is convergence. Therefore, there exists an upper bound \( H > 0 \) such that \( 0 < \sum_{k=1}^{\infty} \frac{1}{(1+k)^{\beta}} < H \).

According to the above analysis, the following inequality can be derived

\[ \sum_{k=1}^{\infty} \frac{\Gamma(1 + \beta)}{\Gamma(1 + k)} \frac{1}{\Gamma(1 - k + \beta)} \leq B_1 \|x\|, \]

in which \( B = \frac{\Gamma(1 + \beta)M_{\text{max}}}{H} \).

Remark 1. We have proven the fundamental boundedness lemma. From this lemma, one can conclude that the assumption about boundedness condition in ([14]–[17]) is valid and...
those results based on this boundedness condition are also valid. However, the reference [18] considers a fractional-order $\beta \in (1,2)$. Hence, the condition $||T|| \leq B_1||x||$ in [18] cannot be verified by Lemma 1, since the boundedness lemma is suitable for the case $\beta \in (0,1)$. It means that the main result based $||T|| \leq B_1||x||$ in [18] is questionable. Hence, the correctness of $||T|| \leq B_1||x||$ should be confirmed in future. The detail is listed in Remark 4.

Remark 2. It is well known that the Mittag-Leffler stability theorem is very important to deal with the stability of the fractional-order systems. However, the reference [18] considers a $\beta$th-order time derivative of the Lyapunov function cannot be easily coped. Hence, the Mittag-Leffler stability theorem was not widely used. But using our Lemma 2, it is possible to use the Mittag-Leffler stability theorem in [11] in order to get interesting results for more general fractional-order nonlinear systems.

Remark 3. The linear matrix inequality (LMI) method has been widely developed to solve the problem of the stability of integer-order systems. Now, based on the fundamental boundedness lemma proved in Lemma 2, some interesting methods based on LMI can be directly utilized to deal with the problem of the fractional-order systems.

Remark 4. We remark that if the fractional-order $\beta$ is not just limited by $\beta \in (0,1)$ and can be arbitrary finite positive non-integer, the boundedness condition on $\mathbb{T}$ may be obtained for more general fractional-order nonlinear systems. This will complicate the analysis since it appears that we will need to separate $D^{\beta-k}_t (k = 1,2,3,\ldots)$ into two part and study each part for $D^{\beta-k}_t$. Our technique are suitable for solving this issue in its full generality and this is a topic of future research.

Lemma 3. [11], [13] Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system

$$D^\beta_t x(t) = f(x,t),$$

where $f(x,t)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$. Assume that there exists a Lyapunov candidate $V(t,x(t))$ satisfying

$$\alpha_1 \|x\|^a \leq V(t,x(t)) \leq \alpha_2 \|x\|^b,$$

$$D^\beta_t V(t,x(t)) \leq -\alpha_3 \|x\|^{a+b},$$

where $\alpha_1, \alpha_2, \alpha_3, a, b$ are positive constants and $\beta \in (0,1)$. Then the equilibrium point is Mittag-Leffler stable.


Lemma 4. [26] For any matrix $X$ and $Y$ of compatible dimensions, we have $X^T Y + Y^T X \leq \varepsilon X^T X + (1/\varepsilon) Y^T Y$.

Lemma 5. [27], [28](Schur complement). Given constant symmetric matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$, and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_2^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_2^T \\ \Sigma_2 & -\Sigma_3 \end{bmatrix} < 0, \text{ or } \begin{bmatrix} -\Sigma_2 & \Sigma_3^T \\ \Sigma_3 & \Sigma_1 \end{bmatrix} < 0.$$  

Lemma 6. [12] Let $A \in \mathbb{R}^{n \times n}, 0 < \beta < 1$. The fractional-order system $d^\beta x(t)/dt^\beta = Ax(t)$ is asymptotically stable if and only if there exist two real symmetric positive matrices $P_{k1} \in \mathbb{R}^{n \times n}, k = 1,2$, and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{n \times n}, k = 1,2$, such that

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \text{Sym}(\Theta_{ij} \otimes (AP_{ij})) < 0,$$

where

$$\Theta_{11} = \begin{bmatrix} \sin(\pi \beta/2) & -\cos(\pi \beta/2) \\ \cos(\pi \beta/2) & \sin(\pi \beta/2) \end{bmatrix}, \Theta_{12} = \begin{bmatrix} \cos(\pi \beta/2) & \sin(\pi \beta/2) \\ -\sin(\pi \beta/2) & \cos(\pi \beta/2) \end{bmatrix},$$

$$\Theta_{21} = \begin{bmatrix} \sin(\pi \beta/2) & \cos(\pi \beta/2) \\ -\cos(\pi \beta/2) & \sin(\pi \beta/2) \end{bmatrix}, \Theta_{22} = \begin{bmatrix} -\cos(\pi \beta/2) & \sin(\pi \beta/2) \\ \sin(\pi \beta/2) & -\cos(\pi \beta/2) \end{bmatrix}.$$

III. SLIDING SURFACE AND CONTROL SCHEME DESIGN

The design procedure of the sliding mode control has two steps: first, constructing a sliding surface that represents a desired system dynamics. Second, developing a switching control law that makes the sliding mode possible on every point in the sliding surface [23]. In this paper, We choose the sliding surface

$$s = C_1 x + C_2 z,$$

where $D^\beta z = K x - z$, in which $C_1, C_2, K$ are constants matrices, and we define the function $\text{sgn}(s) : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\text{sgn}(s) = [\text{sgn}(s_1), \text{sgn}(s_2), \cdots, \text{sgn}(s_n)]^T.$$  

In this case, the sliding mode control law is given

$$u = -(C_1 B)^{-1}[(C_1 A + C_2 K + C_1)x + C_1 f(x,t) + \bar{u}],$$

in which $\bar{u} = w_1 - s$, with $w_1 = (\|C_1 D\|\|N x\| + B_1 + 1)\text{sgn}(s)$.

A. Reachability Analysis

In this subsection, we consider reachability of the sliding surface.

Theorem 1. Consider the fractional-order system (1) and sliding surface function (12), the trajectories of the system (1) under the controller (14) can be driven onto the sliding surface $s(t) = 0$.

Proof. Consider the following Lyapunov-Krasovskii functional candidate $V(t) = s^T(t)x(t)$. Taking the fractional differentiating with respect to time, we have

$$D^\beta_t V = (D^\beta_t s)^T s + s^T (D^\beta_t s) + 2\mathbf{T},$$

where $\mathbf{T}$ is the sliding mode controller term.
in which \( \Upsilon = \sum_{k=1}^{\infty} \frac{\Gamma(1 + \beta)(D_k^\beta s)^T(D_{k-1}^\beta s)}{\Gamma(1 + k)\Gamma(1 - k + \beta)} \). Substitution of (1), (12) and (14) into (15) yields
\[
D_k^\beta V = \{C_1[(A + \delta A)x + f(x, t) + Bu] + C_2Kx - C_2z\}^T s + s^T \{C_1[(A + \delta A)x + f(x, t) + Bu] + C_2Kx - C_2z\} + 2\Upsilon .
\]
According to (14), one has by using Lemma 2
\[
D_k^\beta V \leq \Omega^T + \Omega + 2\Upsilon \leq -2s^T \text{sgn}(s),
\]
where \( \Omega = s^T C_1 D F N x - s^T \|C_1 D\| f N x \| \text{sgn}(s) - B_1 s^T \text{sgn}(s) - s^T \text{sgn}(s) \).

Since \( \sum_{i=1}^{n} |s_i|^2 \geq \sum_{i=1}^{n} |s_i|^2 \), one can conclude that the state trajectories of the system (1) under the controller (14) will hit the sliding surface. Moreover, the trajectories of the system can be driven onto the predefined sliding surface in finite time.

**B. Sliding Mode Dynamics Analysis**

Since it has been shown that the state trajectories of the closed-loop system (1) are drawn to sliding surfaces in finite time, the stability of the sliding mode dynamics is investigated, that is, system (1) restricted on sliding surface.

*Theorem 2.* The sliding mode dynamics on the surfaces is asymptotically stable if there exist two real symmetric positive matrices \( P_{k1} \in R^{n \times n}, k = 1, 2 \) and two skew-symmetric matrices \( P_{k2} \in R^{n \times n}, k = 1, 2 \), such that
\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} < 0,
\]
\[
\begin{bmatrix}
P_{11} & P_{12} \\
-P_{12} & P_{11}
\end{bmatrix} > 0,
\]
\[
\begin{bmatrix}
P_{21} & P_{22} \\
-P_{22} & P_{21}
\end{bmatrix} > 0,
\]
where
\[
\Sigma_{11} = \sum_{i=1}^{2} \sum_{j=1}^{2} \text{Sym}\{\Theta_{ij} \otimes ((-C_1^{-1}C_2K - I)P_{ij})\},
\]
\[
\Sigma_{12} = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[\begin{array}{cccc}
\varepsilon_{11} & I_2 & I_2 & I_2 \\
I_2 & \varepsilon_{11} & I_2 & I_2 \\
I_2 & I_2 & \varepsilon_{11} & I_2 \\
I_2 & I_2 & I_2 & \varepsilon_{11}
\end{array}\right]
\]
\[
\Sigma_{22} = \sum_{i=1}^{2} \sum_{j=1}^{2} \text{Sym}\{\varepsilon_{ij}(I_2 \otimes D)(I_2 \otimes D)^T\}
\]
\[
W_{ij} = N P_{ij}, i, j = 1, 2,
\]
in which \( \Theta_{ij}, (i, j = 1, 2) \) are defined in Lemma 6.

**Proof.** Since the overall closed-loop dynamics is dependent on the control law, it is straightforward to verify that the sliding mode dynamics can be obtained
\[
D_k^\beta x(t) = (-C_1^{-1}C_2K - I)x(t) + \delta Ax(t).
\]
IV. NUMERICAL SIMULATIONS

The following example is used to show the applicability of the proposed controller.

Example: Consider the system (1) with the following parameters

\[
A = \begin{bmatrix}
-1.75 & -0.66 & 2.4 \\
-2.7 & 1.29 & 2.4 \\
-0.35 & 1.95 & 2.78
\end{bmatrix},
\]

\[
f(x, t) = \begin{bmatrix} 0 \\
-0.3x_1x_3 \\
0.1x_1x_2
\end{bmatrix},
\]

\[
B = D = N = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
F(t) = 0.1\sin(0.1t)\begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad (25)
\]

where \( \beta = 0.95 \). Let

\[
C_1 = \begin{bmatrix}
-0.6411 & 0 & 0 \\
0 & -0.6411 & 0 \\
0 & 0 & -0.6411
\end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, P_{12} = P_{22} = 0. \quad (26)
\]

According to Theorem 2 for the fractional-order system (1), a feasible solution of the symmetric matrices and scalars are found using MATLAB’s LMI Control Toolbox:

\[
K = \begin{bmatrix}
-0.6179 & 0 & 0 \\
0 & -0.6179 & 0 \\
0 & 0 & -0.6179
\end{bmatrix},
\]

\[
P_{11} = \begin{bmatrix} 0.5102 & -0.1775 & -0.0486 \\
-0.1775 & 0.0777 & 0.0375 \\
-0.0486 & 0.0375 & 1.2298
\end{bmatrix},
\]

\[
P_{21} = \begin{bmatrix} 0.5134 & -0.1801 & -0.0479 \\
-0.1801 & 0.0791 & 0.0380 \\
-0.0479 & 0.0380 & 1.2312
\end{bmatrix},
\]

\[
\varepsilon_{11} = 9.9234, \varepsilon_{12} = 5.0217, \varepsilon_{21} = 9.9234, \varepsilon_{22} = 5.0217. \quad (27)
\]

By using (14), (26) and (27), we can obtain the following SMC law:

\[
u = \begin{bmatrix}
-0.2138 & 0.6600 & -2.4000 \\
2.7000 & -3.2538 & -2.4000 \\
0.3500 & -1.9500 & -6.3036
\end{bmatrix} x
\]

\[ + \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} f(x, t)
\]

\[ + \begin{bmatrix}
-1.5598 & 0 & 0 \\
0 & -1.5598 & 0 \\
0 & 0 & -1.5598
\end{bmatrix} ar{u}. \quad (28)
\]

According to Theorem 1, the system (1) and (40), we have the parameters for convergence to sliding surface:

\[
s = \begin{bmatrix}
-0.6411 & 0 & 0 \\
0 & -0.6411 & 0 \\
0 & 0 & -0.6411
\end{bmatrix} x
\]

\[ + \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} z,
\]

\[
D_{\delta}^{0.95} z = \begin{bmatrix} -0.6179 & 0 & 0 \\
0 & -0.6179 & 0 \\
0 & 0 & -0.6179
\end{bmatrix} x
\]

\[ - \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} z. \quad (29)
\]

Figures (1-3) are the results of the simulations with the initial condition \([x(0), y(0), z(0)]^T = [4.13, -2.79, -6.21]^T\).

The fractional integration operator is approximated via Carlson method in frequency range \((0.01, 100) \text{ rad/s}\) by using MATLAB tool box called Ninteger. Fig. 1 shows the time response of the states \(x, y, z\) of the system under the controller (28) and the sliding surface (29). It shows that the sliding control guarantees the reaching to the sliding surface and final stabilization. Fig. 2 shows the sliding surface (29).
The control input (28) is depicted in Fig. 3. It is obvious that the designed controller asymptotically stabilizes the unstable fractional-order nonlinear system and the closed-loop system behavior is satisfactory.

V. CONCLUSIONS

A new sliding mode law is proposed for a class of uncertain fractional-order nonlinear systems. The fundamental boundedness lemma, which can be used for stability analysis of fractional-order systems, especially for Mittag-Leffler stability analysis of fractional-order nonlinear systems, has been presented to prove the reach condition. The controller has been designed to guarantee the state trajectories onto the sliding surface and maintain the sliding motion. The stability criteria is given via LMIs. Numerical simulation results are presented to show the effectiveness of the proposed method.

REFERENCES