Optimal Control of PDE-based Systems
by Using a Finite-Dimensional Approximation Scheme

Angelo Alessandri, Patrizia Bagnerini, Mauro Gaggero

Abstract—The design of closed-loop finite-dimensional controllers for systems described by partial differential equations is tackled by combining tools borrowed by research areas such as approximation and operator theory. The proposed paradigm is based on the idea of using operators to account for the dynamics, regulator, and measurement mappings. Specifically, we rely on a well-established setting of Banach spaces, which is well-suited to supporting the generality of the approach. First, we define a class of Lipschitz operators with finite seminorm and formulate a tracking problem in the Banach spaces of real-valued functions. Second, we search for controllers that ensure stability and minimize a given performance index. The design of such regulators is achieved by resorting to an approximation scheme based on the extended Ritz method. Such a scheme consists in constraining the regulation operator to take on a fixed structure, where a finite number of free parameters can be suitably chosen. The problem is then reduced to a mathematical programming one of nonlinear type in general, in which the values of the parameters are optimized to guarantee stability. A family of nonlinear approximators to which the most common classes of feedforward neural networks belong are employed to accomplish the design via a convenient choice of their parameters (i.e., the weights), as shown by means of simulations with the optimal control of an unstable heat equation.

I. INTRODUCTION

The design of feedback optimal controllers for systems described by ordinary differential equations (ODEs) is a complex task, especially in high-dimensional settings. Further difficulties arise when dealing with distributed parameter systems, i.e., systems modeled by partial differential equations (PDEs). In this paper, we address the construction of closed-loop optimal controllers for PDE-based systems where system and regulator are treated as operators. The design method is based on the idea of approximating the optimal controller by means of a Lipschitz operator in such a way to ensure stability according to a suitably defined notion on the Banach spaces to which the signals belong.

PDEs are widely employed for the purpose of modeling in a large number of applications: thermal convection, spatially distributed chemical reactions, flexible beams or plates, electromagnetic or acoustic waves, fluid flows, and other distributed phenomena that are of interest in both engineering and physics. Thus, a lot of attention has been devoted to the control of such systems (see, among others, [1]–[8] and the references therein). In such a context, the design of this paper is providing a contribution to feedback optimal control of PDE-based systems that is well-suited for various types of PDEs. Closed-loop control is in general more difficult to deal with than open-loop one but often preferable because of its intrinsic robustness. Unfortunately, the stability under the action of a feedback regulator may be hard to be proved [9]–[13].

Among the possible approaches to study the stability of feedback PDE-based systems, in [14] a Banach space setting is proposed that allows one to deal with general nonlinear operators. The operators can be used in such a context like transfer functions with linear systems. Based on such a theoretical background, conditions are established to design stabilizing optimal controllers for systems described by PDEs. However, the analytical solution of the optimal control problem may be difficult to find except in very few cases, and thus one has to resort to the search of solutions in approximate form.

In the literature concerning optimal control of PDEs, two types of approaches are usually considered, i.e., “discretize-then-optimize” and “optimize-then-discretize.” In the first, the PDE is discretized into an approximate finite-dimensional ODE as a preliminary step before accomplishing the design of the controllers (see, e.g., [3], [15]–[18]). Since the resulting ODE may be of high dimension, numerous model-reduction techniques for low-order approximation of various classes of PDEs were developed, such as methods based on the Karhunen-Loève expansion, empirical eigenfunctions, and proper orthogonal decomposition [19]–[22]. The advantage of the discretize-then-optimize approach is the possibility of using the well-established techniques available for the design of feedback optimal controllers for systems described by ODEs. The main limitation of such an approach lies in the difficulty of finding suitable reduced-order models that are accurate for all possible classes of PDEs, especially for nonlinear systems defined in high-dimensional domains. By contrast, the optimize-then-discretize paradigm stems from the idea of casting and solving the problem in its original infinite-dimensional setting via a variational formulation. Thus, the original optimal control problem is reduced to the problem of finding a numerical solution of a system of PDEs (made up by the original equation and by an adjoint equation), for which a number of techniques are reported in the literature (see, among others, [23]–[27]). The main limitation of such methods is the open-loop form of the resulting solution.

In this paper, in line with [28], a different approach is considered as a compromise between the discretize-
then-optimize and optimize-then-discretize paradigms. More specifically, we address the approximate solution of optimal control problems basing on the extended Ritz method (ERIM) [29], as an alternative to direct methods borrowed from the calculus of variations such as the Ritz [30], [31] and weighted residual methods (least-squares, collocation, and Galerkin) [32], [33]. The basic idea lies in constraining the regulation operator to take on fixed structures that depend on a finite number of free parameters to be tuned. As a novelty to compare with [28], here we explicitly take into account the stability conditions basing on the mathematical framework proposed in [14]. By substituting such structures into the cost functional to be minimized and the constraints given by the model equation and closed-loop stability conditions, the problem reduces to a mathematical programming one that consists in finding the optimal values of the parameters. The search of the optimal parameters is performed by using sequential quadratic programming (SQP), which is particularly well-suited to solving nonlinear programming problems [34].

The rest of the paper is organized as follows. In Section II, the proposed approach for the construction of optimal regulators is presented. Section III describes the application of such a method to the optimal control of an unstable heat equation. Concerning such a testbed, numerical results are reported in Section IV. Conclusions are drawn in Section V.

II. SYSTEM DESCRIPTION AND STABILITY

Let us consider the control loop depicted in Fig. 1, where $R$, $P$, and $H$ are given operators (nonlinear in general). $R$ represents a regulator that acts on the system described by $P$ (associated with an output $y$) depending on the tracking error between the desired reference $v$ and the actual signal $z$ given by the output of the measurement operator $H$.

Basing on the foregoing, we need to define the classes of the functions that are the domains and ranges of the various operators introduced so far. Let us denote by $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{V}$ the space of functions of one or more variables, including time, of $u$, $y$, and $v$, respectively. Moreover, let us consider the following definitions, where $\Omega_A$ is a bounded linear subspace of a Banach space $A$ of real-valued functions endowed with norm $\| \cdot \|_A$.

Definition 1: Given two Banach spaces $A$ and $B$ of real-valued functions, an operator $T : A \to B$ is stable if there exist some constants $\alpha_T > 0$, $\beta_T \geq 0$ such that

$$\|T(a)\|_B \leq \alpha_T \|a\|_A + \beta_T$$

for all $a \in A$ (see [35]).

Definition 2: Given two Banach spaces $A$ and $B$ of real-valued functions, an operator $T : A \to B$ is Lipschitz if there exists a real constant $L \geq 0$ such that $\|T(a_1) - T(a_2)\|_B \leq L\|a_1 - a_2\|_A$ for all $a_1, a_2 \in A$.

Let now consider a Lipschitz operator $T : A \to B$ and define as a norm the following:

$$\|T\| \triangleq \|T(0)\|_B + \|T\|$$

where

$$\|T\| \triangleq \sup_{a_1, a_2 \in A, a_1 \neq a_2} \frac{\|T(a_1) - T(a_2)\|_B}{\|a_1 - a_2\|_A}$$

is a seminorm. Hence, we obtain $\|T(a_1) - T(a_2)\|_B \leq \|T\| \|a_1 - a_2\|_A \leq \|T\| \|a_1 - a_2\|_A$ for all $a_1, a_2 \in A$.

Before proceeding, we need to assume the following.

Assumption 1: $\mathcal{Y}$, $\mathcal{V}$, $\mathcal{U}$, and $\mathcal{U}$ are Banach spaces of real-valued functions.

Let us denote by $\mathcal{R}$ the set of admissible regulators for the considered plant, i.e., the set of all operators $R : \Omega_v \to \mathcal{U}$ such that $H(P(R(\cdot)))$ is Lipschitz. Using the results of [14], it is easy to prove the following (see Lemma 1 in [14]).

Lemma 1: $\mathcal{R}$ with the norm $\| \cdot \|$ is a Banach space.

The property stated above is preliminary to go on with the contents of the next sections, as will be detailed in the following. Let $e \triangleq v - H(y)$ be the tracking error. Since

$$e = v - H(P(R(e)))$$

we refer to an operator $K : \Omega_v \to \Omega_y$ that maps $v \in \Omega_v$ onto $e \in \Omega_v$. We shall study the stability properties of such an operator. Toward this end, we need to assume the following.

Assumption 2: $H$ is Lipschitz.

Assumption 3: $R \in \mathcal{R}$ is such that $\|H(P(R))\| < 1$.

Then, we can state the following (see Lemmas 2 and 3 in [14]).

Theorem 1: Under Assumptions 2 and 3, $K$ is stable and

$$\|e\|_\mathcal{V} \leq \left(\|(I + H(P(R)))^{-1}(0)\|_\mathcal{V} + \frac{1}{1 - \|H(P(R))\|}\right)\|v\|_\mathcal{V}.$$  

Unfortunately, Theorem 1 provides no suggestion to proceed with the design of the stabilizing regulator. In the following, we shall address this problem by searching for controllers that ensure stability and perform as desired. Toward this end, let $J(R)$ be a generic performance index for the system to control, which is assumed to be continuous in its argument (for example, such an index may be a measure of the tracking error over time). The goal is that of finding the minimum of $J(R)$ on the set of admissible regulators that ensure stability. In other words, we would address the problem

$$\left\{ \begin{array}{l}
\inf_{R \in \mathcal{R}} J(R) \\
\text{subject to (2) and } \|H(P(R))\| < 1.
\end{array} \right.$$
Since the mapping $J$ is continuous, one can account for the
stability constraint by replacing it with $\| H(P(R)) \| \leq 1 - \varepsilon$, for some $\varepsilon \in (0, 1)$. Based on the results of [14], we
can conclude about the existence of the solution for such
a problem. Thus, we focus on the following problem from
now on:
\[
\begin{align*}
\min_{R \in \mathcal{R}} J(R) \\
\text{subject to (2) and } \| H(P(R)) \| \leq 1 - \varepsilon.
\end{align*}
\] (4)

Finding an analytical solution to problem (4) is a very
difficult task, which can be accomplished only under quite
simplified assumptions on the type of PDE, cost function, and
constraints. The aforesaid suggests to search for a numerical
technique to compute approximate solutions. Specifically,
we shall search for suboptimal solutions to problem (4)
according to the ERIM paradigm [28], [29]. In practice, the
solution of (4) will be searched in a subset of $\mathcal{R}$: let us denote
by $\mathcal{F} \subseteq \mathcal{R}$ such a subset that accounts for the constraints of
(4). A parametrized operator $\Gamma_n : \Omega_L \times \mathbb{R}^{N(n)} \to \mathcal{U}$ takes on
the place of $R$, where $N(n)$ is the number of free parameters
to be chosen ($n \in \mathbb{N}_+$ provides a measure of the complexity
of the parametrized operator $\Gamma_n$). Thus, we replace (2) with
\[
e = v - H(P(\Gamma_n(e))).
\] (5)

The problem stated in (4) is addressed by means of an ad-
hoc approximation scheme that relies on the use of operators
$\Gamma_n$ to approximate the solution. Toward this end, for any
$n \in \mathbb{N}_+$, let us define the sets of such operators as
\[
A^n \triangleq \left\{ \Gamma_n(\cdot, w) \in \mathcal{R} : w \in \mathbb{R}^{N(n)} \right\}, \quad n = 1, 2, \ldots
\] The choice of the parametrized operators $\Gamma_n$ is arbitrary
and quite vast. The only requisite is that the sequence of sets $\{A^n\}_{n=1}^{\infty}$ has an infinite nested structure, i.e.,
\[
A^1 \subset A^2 \subset \cdots \subset A^n \subset \cdots.
\] (6)

Moreover, the operators $\Gamma_n$ must be such that the sequence
$\{A^n\}_{n=1}^{\infty}$ is dense in $\mathcal{F}$. We introduce the following definitions.

**Definition 3:** A sequence $\{A^n\}_{n=1}^{\infty}$ of sets of operators
$\Gamma_n$ is dense in $\mathcal{F} \subseteq \mathcal{R}$ if, for any $\varepsilon > 0$ and $\Gamma \in \mathcal{F}$, there
exist a complexity $\bar{n}_\varepsilon$ and a parameter vector $w$ such that
$\| \Gamma_n - \Gamma \|_\mathcal{R} \leq \varepsilon$ for any $n \geq \bar{n}_\varepsilon$.

**Definition 4:** A sequence $\{A^n\}_{n=1}^{\infty}$ of sets of oper-
ators $\Gamma_n$ that is dense in $\mathcal{F} \subseteq \mathcal{R}$ is endowed with
the infinite nested structure (6) is called “approximating
sequence.” The operators $\Gamma_n$ belonging to the sets $A^n$ are
called “approximating operators.”

For each integer $n$ specifying the complexity of the approx-
imating operators $\Gamma_n$, we take into account the presence
of the constraints given by $\mathcal{F}$ by considering the minimiza-

Because of the infinite nested structure (7), when the number of
parameters increases, the set of parameterized operators
$\mathcal{F}^n$ “invade” $\mathcal{F}$. Given $n \in \mathbb{N}_+$, we deal with the following
problem:
\[
\begin{align*}
\min_{\Gamma_n \in \mathcal{F}^n} J(\Gamma_n) \\
\text{subject to (5) and } \| H(P(\Gamma_n)) \| \leq 1 - \varepsilon.
\end{align*}
\] (8)

As the set $\mathcal{F}^n$ grows, the solution of (8) will approximate
better and better (in a proper sense) the solution of problem
(4). Since $\Gamma_n$ depends on the parameter vector $w$, in practice
one has to solve the following mathematical programming
problem (nonlinear in general):
\[
\begin{align*}
\min_{w \in \mathbb{R}^{N(n)}} J(\Gamma_n(w)) \\
\text{subject to (5) and } \| H(P(\Gamma_n(w))) \| \leq 1 - \varepsilon.
\end{align*}
\] (9)

As for the choice of the class of approximating oper-
ators $\Gamma_n$, we focus on one-hidden-layer feedforward neural
networks. A vast literature is available on the capability of
such networks to approximate functions, functionals, and
operators. The interested reader can refer to, e.g., [36], [37],
and the references therein.

It is noteworthy that it may be difficult to compute the quantity $\| H(P(\Gamma_n(w))) \|$ to impose the stability constraint
in (9). However, one can exploit the properties of the
norm $\| \cdot \|$ and use $\| H \|_\mathcal{F} \| P \|_\mathcal{F} \| \Gamma_n(w) \|$ instead of
$\| H(P(\Gamma_n(w))) \| \leq 1 - \varepsilon$ since $\| H(P(\Gamma_n(w))) \| \leq
\| H \|_\mathcal{F} \| P \|_\mathcal{F} \| \Gamma_n(w) \|$. Such a condition is more conservative
but easier to be verified, as will be clearer in the next section.

For the same reasoning, one can use an upper bound on each
norm instead of the norm itself.

### III. Application to a One-Dimensional Unstable

#### Heat Equation

In this section, we focus on a case study by applying
the above-described approach to the optimal control of an
unstable heat equation [10]. More specifically, let us consider
the heat flow in a thin rod made up of some heat-conducting
material, subject to some external heat source along its length
and some boundary conditions at each end. It is assumed
that the rod is so thin that the temperature at all points of
the section may be considered to be the same. We take into
account not only the loss of heat to a surrounding medium but
also the destabilizing heat generation inside the rod. Let us
denote by $y(x, t)$ and $u(t)$ the temperature and the amount
of heat generated by an external source, respectively. The
considered system is modeled by the following equation (all
the coefficients are normalized):
\[
\begin{align*}
\frac{\partial y(x, t)}{\partial t} - \frac{\partial^2 y(x, t)}{\partial x^2} - \lambda y(x, t) &= u(t), \quad x \in \Omega, \ t \in [0, t_f] \\
y(x, 0) &= 0, \quad x \in \Omega \\
y(0, t) &= y(L, t) = 0, \quad t \in [0, t_f]
\end{align*}
\] (10)

where $L > 0$, $\lambda > 0$, and $t_f > 0$; $\Omega \triangleq [0, L]$ is the space
domain and $[0, t_f]$ is a time interval. Thus, $y : \Omega \times [0, t_f] \to \mathbb{R}$
and $u : [0, t_f] \to \mathbb{R}$ have to be regarded as the state of
the system (i.e., the temperature of the medium at time $t$ and
position $x$) and the control input, respectively. We impose a limitation on the control input, i.e., $u(t) \in [u_{\text{min}}, u_{\text{max}}]$ for all $t \in [0, t_f]$, where $u_{\text{min}}$ and $u_{\text{max}}$ are given constants.

We assume to measure the mean temperature over the entire space domain, i.e., to know the following:

$$z(t) = \frac{1}{L} \int_0^L y(\xi, t) d\xi, \ t \in [0, t_f].$$  \hspace{1cm} (11)$$

Using such a measure of the state of the system, our aim is to track a desired reheating curve, i.e., we wish the average of the temperature over the domain evolves according to a certain reference curve $v(t)$. Thus, we search for a regulator on the space of continuous real-valued functions, i.e., $R : C^0([0, t_f]) \rightarrow C^0([0, t_f])$ such that $u(\cdot) = R(e(\cdot))$, where $e(t) \triangleq v(t) - z(t)$. The goal of such a regulator is to minimize the tracking error

$$J(R) \triangleq \int_0^{t_f} e(t)^2 dt.$$  \hspace{1cm} (12)$$

Among the possible choices, we focus our attention on regulation operators that take on the structure of one-hidden-layer feedforward neural networks with hyperbolic tangent activation functions, which are dense in the space of continuous functions defined on a compact set with respect to the supremum norm [38]. Note that $C^0([0, t_f])$ with the supremum norm is a Banach space.

To simplify the computation of the norm, we set to zero the biases of the networks, i.e., we define $\Gamma_n$ as follows:

$$\Gamma_n(w, e) \triangleq \sum_{j=1}^n c_j \tanh(\alpha_j e)$$  \hspace{1cm} (13)$$

where $n$ is the number of activation functions and $w \triangleq \text{col}(c_j, j = 1, \ldots, n; \alpha_j, j = 1, \ldots, n) \in \mathbb{R}^{2n}$. Since the biases of the networks are set equal to zero, we have $\Gamma_n(\cdot, 0) = 0$. Thus, we can write

$$\|\Gamma_n\| = \sup_{e_1 \neq e_2} \frac{\|\Gamma_n(\cdot, e_1) - \Gamma_n(\cdot, e_2)\|}{\|e_1 - e_2\|} = \sup_{e_1 \neq e_2} \frac{1}{\|e_1 - e_2\|} \cdot \left(\sum_{j=1}^n c_j (\tanh(\alpha_j e_1) - \tanh(\alpha_j e_2))\right)$$

$$\leq \sup_{e_1 \neq e_2} \frac{1}{\|e_1 - e_2\|} \cdot \sup_{t \in [0, t_f], x \in \Omega} \left|\sum_{j=1}^n c_j \alpha_j \tanh(\alpha_j e_1) - \tanh(\alpha_j e_2)\right|$$

$$\leq \sum_{j=1}^n |c_j \alpha_j|.$$  \hspace{1cm} (14)$$

The solution of (10) can be analytically obtained by using separation of variables techniques via a convenient change of variables, and then substituting the boundary conditions into the general series solution (see, e.g., [39]). Specifically, using the terminology of Section II, the operator $P$ governing the dynamics of the temperature of the rod has the following form:

$$P(u) \triangleq \exp(\lambda t) \sum_{j=1}^\infty \frac{2}{L} \sin \left(\frac{j\pi x}{L}\right) \int_0^t \exp \left(-\frac{j^2\pi^2}{L^2} (t - s)\right)$$

$$\left(\int_0^L u(\xi, s) \sin \left(\frac{j\pi \xi}{L}\right) d\xi\right) ds,$$ \hspace{1cm} (15)$$

$$x \in \Omega, t \in [0, t_f].$$

Since $P(0) = 0$, we obtain

$$\|P\| = \sup_{u_1 \neq u_2} \frac{\|P(u_1) - P(u_2)\|}{\|u_1 - u_2\|} = \sup_{u_1 \neq u_2} \frac{1}{\|u_1 - u_2\|} \times \left(\sum_{j=1}^\infty \frac{2}{L} \sin \left(\frac{j\pi x}{L}\right) \exp(\lambda t) \int_0^t \exp \left(-\frac{j^2\pi^2}{L^2} (t - s)\right)$$

$$\times \left(\int_0^L |u_1(\xi, s) - u_2(\xi, s)| \sin \left(\frac{j\pi \xi}{L}\right) d\xi\right) ds\right)$$

$$\leq \sup_{u_1 \neq u_2} \frac{1}{\|u_1 - u_2\|} \sup_{t \in [0, t_f], x \in \Omega} \left(\sum_{j=1}^\infty \frac{2}{L} \sin \left(\frac{j\pi x}{L}\right) \exp(\lambda t) \int_0^t \exp \left(-\frac{j^2\pi^2}{L^2} (t - s)\right)$$

$$\times \left(\int_0^L |u_1(\xi, s) - u_2(\xi, s)| \sin \left(\frac{j\pi \xi}{L}\right) d\xi\right) ds\right)$$

$$\leq \exp(\lambda t) \sum_{j=1}^\infty \frac{2L^2}{j^2 \pi^4} \sin \left(\frac{j\pi x}{L}\right) (1 - (-1)^j) \approx \zeta L^2 \exp(\lambda t_f)$$

where $\zeta \approx 0.13564$.

The measurement operator $H$ accounts for (11), i.e., its output is the average of the temperature over the entire domain, as follows:

$$H(y) \triangleq \frac{1}{L} \int_0^L y(\xi, t) d\xi, \ t \in [0, t_f].$$  \hspace{1cm} (16)$$

Note that $H$ is such that $H(0) = 0$, thus we can write

$$\|H\| = \sup_{y_1 \neq y_2} \frac{\|H(y_1) - H(y_2)\|}{\|y_1 - y_2\|} = \sup_{y_1 \neq y_2} \frac{1}{\|y_1 - y_2\|} \left(\frac{1}{L} \int_0^L (y_1(\xi, t) - y_2(\xi, t)) d\xi\right)$$

$$\leq \sup_{y_1 \neq y_2} \frac{1}{\|y_1 - y_2\|} \frac{1}{L} \int_0^L d\xi \|y_1 - y_2\| = 1.$$
Summing up, as \( \|H(P(\Gamma_n))\| \leq \|H\| \|P\|_{\Gamma_n} \), we can write

\[
H(P(\Gamma_n)) \leq \zeta L^2 \exp(\lambda t_f) \sum_{j=1}^{n} c_j \alpha_j.
\]

Given \( \varepsilon > 0 \), the solution of the optimal control problem consists in finding the optimal parameters that minimize \( J(\Gamma_n) \) under the constraints (10), \( u(t) \in [u_{\text{min}}, u_{\text{max}}] \) for all \( t \in [0, t_f] \), and the stability constraint

\[
\zeta L^2 \exp(\lambda t_f) \sum_{j=1}^{n} c_j \alpha_j \leq 1 - \varepsilon.
\]

IV. SIMULATION RESULTS

We choose \( L = 4 \), \( \lambda = 0.5 \), \( t_f = 10 \), \( u_{\text{min}} = 0 \), \( u_{\text{max}} = 50 \), \( \varepsilon = 10^{-9} \), and a step-increasing reference curve

\[
v(t) = \begin{cases} 
10t & \text{if } t \in [0, 2) \\
20 & \text{if } t \in [2, 5) \\
20 + 10(t - 5) & \text{if } t \in [5, 7) \\
40 & \text{if } t \in [7, 10].
\end{cases}
\]

For the numerical solution of (10), we used second order standard centered finite differences in space and forward Euler differences in time (see, e.g., [40]). Let \( \Delta t \) and \( \Delta x \) be the time and space discretization steps, respectively.

All the simulations were performed by using the Optimization Toolbox of Matlab on a personal computer with a 1.8 GHz Core2 Duo CPU and 2 GB of RAM. Basing on the successful results of [28], the optimizations were performed by using the SQP algorithm via the \texttt{fmincon} routine [41].

Four case studies (i.e., case A, B, C, and D) were investigated. Table I contains the values of the optimal costs and the simulation times for some values of \( n \) together with the details of the considered case studies. In Fig. 2, the temperature of the rod in the case D with \( n = 15 \) basis functions and the corresponding optimal control input are reported. Fig. 3 shows the relationship between the optimal cost and the space discretization of the considered approaches for various numbers \( n \) of basis functions.

From the simulation results, it turns out that the reference curve is followed quite well. A convergent behavior of the optimal cost is shown by the proposed method: a fine discretization allows one to obtain better results (in terms of lower values of the cost function) than a coarse one. From the sequences of values in Table I and Fig. 3, we are able to compute the approximate order of convergence (i.e., the slope of the curves in Fig. 3) of the considered approach: for instance, the order of convergence with \( n = 15 \) basis functions turns out to be equal to 1.79. Of course, the finer the discretization, the higher the computational effort that is needed to find the solution. Furthermore, in general, once fixed the discretization grid, the optimal cost reduces if \( n \) increases. This can be explained by noting that a higher \( n \) implies also a larger number of parameters to be optimized, and thus the parametrized structures used to approximate the regulation operator have a higher approximation capability. However, the computational effort needed to obtain the results increases with \( n \) since a higher number of parameters have to be optimized.

V. CONCLUSIONS

A novel approach to the design of closed-loop controllers for systems described by PDEs is presented by using a Banach setting to define Lipschitz operators that account for the system dynamics, controller, and measurements. The design goal consists in searching for controllers that ensure stability and minimize a given performance index. Such regulators are approximated by using the extended Ritz method. The proposed approach has been fully developed for an optimal control problem of an unstable heat equation. Successful simulation results are reported to show the effectiveness of the proposed method.

REFERENCES

TABLE I
SUMMARY OF THE SIMULATION RESULTS.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\Delta t$</th>
<th>$\Delta x$</th>
<th>Optimal cost</th>
<th>Simulation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 5</td>
<td>n = 10</td>
<td>n = 15</td>
<td>n = 5</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
<td>1.0</td>
<td>3.18</td>
<td>2.82</td>
</tr>
<tr>
<td>B</td>
<td>0.1</td>
<td>0.5</td>
<td>2.22</td>
<td>2.73</td>
</tr>
<tr>
<td>C</td>
<td>0.01</td>
<td>0.25</td>
<td>0.47</td>
<td>0.10</td>
</tr>
<tr>
<td>D</td>
<td>0.005</td>
<td>0.125</td>
<td>0.11</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Fig. 2. Temperature distribution of the rod and control input obtained in the case D of Table I with $n = 15$ basis functions.

Fig. 3. Relationship between optimal cost and space discretization for various numbers $n$ of basis functions.