Fuel minimization of a moving vehicle in suburban traffic

Galpin Thomas¹ and Petros G. Voulgaris²

Abstract— In this paper we study how a driver could use traffic light information in order to adapt his speed profile to save fuel. Some analytical results are developed for simple cases using optimization theory and numerical simulations are performed for more complex cases. Dijkstra’s shortest path algorithm is a useful tool to provide optimizing policies.

I. INTRODUCTION

The major influence of road transport in global CO₂ emissions and the raise in oil prices are good motivations to find new solutions to save fuel. It is proved that the maximum fuel efficiency of a motor vehicle is realized at constant speed [7], [6], [7]. This optimal speed depends on the characteristics of the vehicle but for an average powered vehicle it is about 50 mph and the fuel efficiency decreases significantly for lower speed [1].

The influence of traffic lights is important to take into account because they may require stops, which involve more fuel consumption. [5] used traffic light information to optimize fuel economy and trip time but their primary optimization variable was trip time. [7] used optimization theory to find best speed profile to minimize work and fuel consumption, but in absence of traffic lights.

In this paper, the method proposed uses analytical results to find the optimal speed profile in the case of one traffic light. For more complex and realistic cases, we use Dijkstra’s algorithm to discretize time and determine the optimal speed profile.

II. FORMULATION

A. Problem definition

Our objective is to minimize the fuel consumption of a vehicle driving in a network of traffic lights while reaching a destination within a specific deadline. We use a very simple model of fuel consumption and do not take into account the effects of acceleration and gear shifting.

Moreover, we assume a constant speed between traffic lights.

B. Fuel consumption model

There exist a lot of models in the literature [1], [3] and most of them seem to agree on a cubic consumption model:

\[
\frac{dC}{dt} = a_0 + a_1 V + a_2 V^2 + a_3 V^3 \quad (KJ/hr) \quad (1)
\]

In this equation, \( C \) represents the fuel consumption in kilojoule (or milliliter). Therefore, along a trip we want to minimize

\[
C = \int (a_0 + a_1 V + a_2 V^2 + a_3 V^3) dt \quad (KJ) \quad (2)
\]

If \( V \) is constant, it is the same to minimize

\[
C = (a_0 + a_1 V + a_2 V^2 + a_3 V^3) \frac{d}{V} \quad (3)
\]

Here, \( d \) is the distance to travel and \( t \) is the time to cover this distance. Thus, we want to minimize \( C(V) \), where

\[
C(V) = \left( \frac{a_0}{V^2} + a_1 + a_2 V + a_3 V^2 \right) \quad (KJ/mile) \quad (4)
\]

It is also convenient to have an expression for the consumption per unit of distance, so we define \( C_d \) as

\[
C_d(V) = \frac{a_0}{V} + a_1 + a_2 V + a_3 V^2 \quad (KJ/mile) \quad (5)
\]

This is essentially the inverse of Miles Per Gallon. To determine the coefficients \( a_0, a_1, a_2 \) and \( a_3 \) we use the model of [1]. For an average powered car (AVPWR) at constant speed, [2] models the consumption as

\[
C_d(V) = \frac{\alpha_{fpwr} V_{gear} + \alpha_{acc}}{V} + \alpha_{tire} + \alpha_{air} V^2
\]

We note that in this model the coefficient in front of \( V \) is null (\( a_2 = 0 \)).

- The first term \( \alpha_{fpwr} V_{gear} \) takes into account the friction of the engine, \( V_{gear} \) can be seen as the average speed in gear used. Although we do not take into account gear shifting, a good approximation of \( V_{gear} \) is \( V_{gear} = 55 \text{ mph} \) (miles per hour).

- The second term \( \alpha_{air} \) takes into account the consumption of all the accessories (such as air conditioning, lights, audio system, brakes system, etc.).
The third term $\alpha_{tire}$ takes into account the resistance of tires which induces drag.

The last term $\alpha_{air} V^2$ takes into account the drag generated by the particular shape of the car.

Typical values of constants are shown in table (6).

<table>
<thead>
<tr>
<th>$\alpha_{fpwr}$</th>
<th>$\alpha_{acc}$</th>
<th>$\alpha_{tire}$</th>
<th>$\alpha_{air}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1692</td>
<td>6750</td>
<td>316</td>
<td>0.403</td>
</tr>
</tbody>
</table>

We can see in Fig. 1 that for a given vehicle (with the above characteristics), there exists an optimal speed which minimizes the fuel consumption $C_d$; we will call it $V_{opt}$. In this case, $V_{opt}$ is roughly equal to 50 mph, which is consistent with the reality.

![Fig. 1. Consumption $C_d$ as a function of speed](image)

### III. ZERO AND ONE TRAFFIC LIGHT PROBLEMS

#### A. No traffic light

Before dealing with the single traffic light problem, let us see what happens in the simplest possible case without any traffic light. We want to reach a certain distance $D$ in a maximum time $t_f$ while minimizing the fuel consumption. This is obvious, nonetheless we have to notice that there are two different cases. First, if $V_{opt}$ is such that $V_{opt} \geq D/t_f$, the driver should drive at the optimal speed $V = V_{opt}$ to minimize the fuel consumption. The second case to consider is when $V_{opt}$ is not large enough to reach the distance $D$ before the final time $t_f$. In this case, the driver should go faster than $V_{opt}$ but at the slowest possible speed (to minimize fuel consumption), i.e., $V = D/t_f$. In summary, either one has enough time to reach the distance $D$ with $V_{opt}$ or one has to drive at speed $D/t_f$ (Fig. 2).

![Fig. 2. Summary of the two different cases for no traffic light](image)

#### B. One traffic light

Now, we consider the case of one traffic light (red between $t_1$ and $t_2$) which is situated at the distance $d$ of the starting point (Fig. 3).

Once again, there are two different cases depending on the value of $V_{opt}$. The easiest case to deal with is when $V_{opt}$ is “large enough”. If fact, if $V_{opt} \geq d/t_1$, then $V_{opt}$ is roughly equal to 50 mph, which is consistent with the reality.

![Fig. 3. One traffic light at distance d](image)

Let us call $V_1$ the speed from the origin to distance $d$ of the starting point. $V_{opt} < d/t_1$ is more complicated and needs to be broken in three sub-cases. In the first sub-case, we determine the optimal speed profile in a situation in which the driver will be fast enough not to hit the traffic light. In the second sub-case, the driver will hit the red traffic light and will have to stop until it turns green. In the last sub-case the driver will go beyond the red traffic light (in other words, he will hit a green traffic light). Let us call $V_1$ the speed from the origin to distance $d$ of the starting point. $V_{opt} < d/t_1$ is more complicated and needs to be broken in three sub-cases. In the first sub-case, we determine the optimal speed profile in a situation in which the driver will be fast enough not to hit the traffic light. In the second sub-case, the driver will hit the red traffic light and will have to stop until it turns green. In the last sub-case the driver will go beyond the red traffic light (in other words, he will hit a green traffic light). Let us call $V_1$ the speed from the origin to distance $d$ of the starting point.
and $V_2$ the speed from $d$ to $D$. The three different cases described above are $V_1 \geq d/t_1$, $d/t_2 \leq V_1 \leq d/t_1$ and $V_1 \leq d/t_2$.

1) First sub-case, $V_1 \geq d/t_1$: First we deal with the case when one does not hit the traffic light. We take a random speed profile (with constant speed between the starting point and the traffic light and between the traffic light and $D$) where we choose not to cross the traffic light (Fig. 5). We can show by a simple analysis that the optimal speed profile (with the same shape as in Fig. 5) is the one shown in Fig. 6. In fact, in the first section (from the origin to distance $d$), the optimal strategy is to hit the corner of the traffic light because $V_{opt} \leq d/t_1$. In the second section (from distance $d$ to distance $D$), there are two possibilities: either the driver should drive at $V_{opt}$ or he should reach the final distance $D$ at the final time $t_f$. In Fig. 6, $V_{opt}$ is not large enough so that the driver should adopt the strategy of “case 2” illustrated in Fig. 2. Thus, the optimal speed profile is described in Fig. 6.

2) Second sub-case, $d/t_2 \leq V_1 \leq d/t_1$: Now, we deal with the second sub-case when one hits the traffic light (we remind to the reader that we are still in the case where $V_{opt} < d/t_1$). We take a random speed profile where we choose to hit the traffic light at $t = t_s$ (Fig. 7). The optimal speed profile from $C$ to $D$ is known: according to the section III-A, it is $V = V_{opt}$ if $V_{opt} \geq \frac{D-d}{t_f-t_2}$ and $V = \frac{D-d}{t_f-t_2}$ otherwise. Therefore, it remains to determine the best speed profile from A to C.

To do so, we break the objective function (the function to minimize) into two parts. The first will take into account the consumption from A to B and the second the consumption from B to C. To determine the value of these consumption, we use (2):

$$C_{AC} = C_{AB} + C_{BC}$$

$$= (a_0 + a_1V_1 + a_2V_1^2 + a_3V_1^3)t_s + a_0(t_2 - t_s)$$

$$= (a_2V_1 + a_3V_1^2)d + a_1d + a_0t_2$$

In these equations, $V_1$ represents the speed from A to B. Since $a_1d$ and $a_0t_2$ are constant, the function to minimize is:

$$f(V_1) = a_2V_1 + a_3V_1^2$$

We want to minimize $f(V_1)$ which is subject to some constraints. In fact, we are in the case where one hits the traffic light so $V_1$ must be greater than $d/t_2$ and less than $d/t_1$. Finally, the minimization problem to solve is:

$$\begin{align*}
\min & \quad a_2V_1 + a_3V_1^2 \\
\text{subject to} & \quad d/t_2 \leq V_1 \leq d/t_1
\end{align*}$$

Since $a_2$ and $a_3$ are positive, the function $f(V_1)$ is increasing on the interval $(\frac{d}{t_2}, \frac{d}{t_1})$ and so it is minimized in $V_1 = d/t_2$. In other words, we have to hit the end of the red traffic light (Fig. 8).

3) Third sub-case, $V_1 \leq d/t_2$: In the last sub-case, one goes beyond the red traffic light and reaches the distance $d$ during a green traffic light at $t = t_3$. From now on, we consider a random speed profile with $V_1 \leq d/t_2$ (Fig. 9). This time we will take $a_2 = 0$ (like in the model of [2]) to simplify equations but the results would be unchanged even if $a_2$ were different from 0. First, let us note that the optimal speed $V_{opt}$ verifies

$$\frac{dC_{AC}(V_{opt})}{dV}(V_{opt}) = 0.$$  

Therefore, using (4):

$$-a_0 + 2a_3V_{opt}^3 = 0$$

(8)
To determine the fuel consumption to minimize, we break the objective function into two parts and use (3).\n
\[ C_{AC} = C_{AB} + C_{BC} \]

\[ = \left( \frac{a_0}{V_1} + a_1 + a_3 V_2^2 \right) d + \left( \frac{a_0}{V_1} + a_1 + a_3 V_2^2 \right) d' \]

But \( V_1 = \frac{d}{t_3} \) and \( V_2 = \frac{d'}{t_4 - t_3} \) so, after simplifications,

\[ C_{AC} = a_1 d + a_1 d' + a_0 t_4 + a_3 \left( \frac{d^3}{t_3^3} + \frac{d'^3}{(t_4 - t_3)^2} \right) \]

But the quantity \( a_1 d + a_3 d' \) is constant and positive, so minimizing \( C_{AC} \) is the same as minimizing

\[ C = a_3 \left( \frac{d^3}{t_3^3} + \frac{d'^3}{(t_4 - t_3)^2} \right) + a_0 t_4 \tag{9} \]

We want to minimize (9) with respect to \( t_3 \) and \( t_4 \), but there are some constraints on \( t_3 \) and \( t_4 \). Thus, the minimization problem to solve is:

\[
\begin{cases}
\min \quad a_3 \left( \frac{d^3}{t_3^3} + \frac{d'^3}{(t_4 - t_3)^2} \right) + a_0 t_4 \\
t_2 - t_3 \leq 0 \quad (\lambda_1) \\
t_3 - t_4 \leq 0 \quad (\lambda_2) \\
t_4 - t_f \leq 0 \quad (\lambda_3)
\end{cases}
\]

To solve this system, we use Karush-Kuhn-Tucker conditions (KKT conditions) which are necessary conditions for optimality [8]. In our case there will be sufficient conditions because the objective function and the constraints are convex. In this system, \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the Lagrange multipliers associated to the constraints \( t_2 - t_3 \leq 0, t_3 - t_4 \leq 0 \) and \( t_4 - t_f \leq 0 \). The first step is to form the Lagrangian:

\[
L(t_3, t_4, \lambda_1, \lambda_2, \lambda_3) = a_3 \left( \frac{d^3}{t_3^3} + \frac{d'^3}{(t_4 - t_3)^2} \right) + a_0 t_4 \\
+ \lambda_1 (t_2 - t_3) + \lambda_2 (t_3 - t_4) \\
+ \lambda_3 (t_4 - t_f)
\]

Therefore, KKT conditions for system (10) are (following the method described in [8]):

\[
\begin{align*}
-2a_3(V_2^3 - V_1^3) &- \lambda_1 + \lambda_2 = 0 \\
-2a_3 V_2^3 + a_0 - \lambda_2 + \lambda_3 = 0 \\
\lambda_1 (t_2 - t_3) &- \lambda_2 (t_3 - t_4) = 0 \\
\lambda_2 (t_3 - t_4) &- \lambda_3 (t_4 - t_f) = 0 \\
\lambda_i &\geq 0 \quad i = 1, 2, 3 \\
t_2 - t_3 &\leq 0 \\
t_3 - t_4 &\leq 0 \\
t_4 - t_f &\leq 0
\end{align*}
\tag{11}
\]

Before solving (11), we remind to the reader that if we can find \((V_1, V_2, \lambda_1, \lambda_2 \text{ and } \lambda_3)\) which verify all the equations of (11), then \((V_1, V_2)\) is the optimal speed profile to minimize fuel consumption.

First we note that \( t_3 \neq t_4 \) (otherwise \( V_2 \) would be infinite) so that the fourth equation of system (11) implies \( \lambda_2 = 0 \). Then the first equation becomes

\[
\lambda_1 = 2a_3(V_2^3 - V_1^3)
\tag{12}
\]

The non negativity of \( \lambda_1 \) requires \( V_2 \geq V_1 \), in other words there does not exist optimal speed profile with \( V_2 < V_1 \). From now on, we assume \( V_2 \geq V_1 \).

Let us determine the value of \( V_1 \) and \( V_2 \). If we look at the fifth equation of (11), there are two cases: \( \lambda_3 = 0 \) or \( t_4 = t_f \). If \( \lambda_3 = 0 \), the second equation implies \( -2a_3 V_2^3 + a_0 = 0 \), which means that \( V_2 = V_{opt} \) (using (8)). Now, if we look at (12), there are again two cases: either \( V_2 > V_1 \), or \( V_2 = V_1 \). If \( V_2 > V_1 \), then \( \lambda_1 \neq 0 \) and so the third equation of (11) tells us \( t_2 = t_3 \).
4) Summary of the one traffic light problem: As previously seen, there exist many possible configurations depending on timing data, distances data and $V_{opt}$. Essentially, if $V_{opt} \geq d/t_1$, the optimal strategy is to drive at speed $V_{opt}$. If $V_{opt} < d/t_1$, unless we can reach the final distance without hitting the traffic light, the best strategy is either to pass the traffic light just before it turns to red, or to drive slowly enough to hit the end of the red traffic light (the choice between those two depends on the values of $t_1$, $t_2$, $d$ and $t_f$). We remark that waiting at a traffic light is never the best solution.

IV. USING DIJKSTRA’S ALGORITHM TO FIND THE OPTIMAL SPEED PROFILE

For more than one traffic light, the situation becomes too complex and we cannot use analytical expressions anymore. That is why we use Dijkstra’s algorithm to discretize the decision problem.

A. Definition

Dijkstra’s algorithm is an algorithm which finds the shortest path (from a source to every other nodes) for a graph with non-negative edge costs [4]. In the way we model our problem, we use Dijkstra’s algorithm between a source to a destination. Moreover, Dijkstra’s algorithm gives the total cost of the shortest path.

B. Method

In this section, we show how we use Dijkstra’s algorithm to determine the optimal speed profile through a set of traffic lights (with respect to fuel minimization). We still consider the same problem as before but with many traffic lights. We call $d_i$ the distance of each traffic light from the origin. The method we use is to discretize the time by setting nodes at each $d_i$’s. We also set a node at the origin (source) and at the distance D (destination). The cost between two nodes is the value of fuel consumption to go from one node the other node. Let us illustrate this idea with a simple example: one traffic light (red between $t_1$ and $t_2$) situated at distance $d_1$ from the origin (Fig. 10). In Fig. 10, the circles represent the nodes we decided to set. Node 1 is the origin and node 7 is the destination. The key in Dijkstra’s algorithm is to set up the relations (nodes cannot be linked to every other nodes) and the costs between the different nodes. For example, in this situation, node 1 can be directly linked to every other nodes except node 7. Nodes 2, 5 and 6 can only go to node 7. Node 3 (which is at the beginning of the red traffic light) can go to node 5 (in this case the driver waits at the traffic light) or node 7 (in this case the driver passes just before the traffic light turns to red). Node 4 can only go to node 5 (if one hits a red traffic light, he or she has to wait till it turns green). The cost between two nodes (which can be linked) is set using (3). For example, the cost between node 1 and node 4 is $C(V_{14})$, where $V_{14}$ is the speed to go from node 1 to 4. Then, by considering these relations between nodes, we can draw a graph with non negative edge costs and use Dijkstra’s algorithm to find the shortest path between the origin and the destination. The equivalent graph of our example is illustrated in Fig. 11. In Fig. 11, the labels $C_{ij}$ represent the costs from node i to node j: $C_{ij} = C(V_{ij})$. For numerical simulations (with more than one traffic light), we put many nodes at each $d_i$’s to better discretize the problem and have more precise results.

C. Results for the two traffic lights problem

To illustrate the method on simple situations, we took the case of two sets of traffic lights. We set hundred nodes at $d_1$ and $d_2$ (the distances of the two set of traffic lights), the results of two different cases are presented in Fig. 12, and 13. The straight dot lines represents $V_{opt}$, the plain horizontal lines represent the red traffic light and the plain non horizontal lines represents the optimal speed profile calculated with Dijkstra’s algorithm. We can see that the results from section III-B can be extended to these cases: the best strategy is always not to cross the traffic lights, either we ride at the optimal...
speed or we hit a "corner" at a traffic light. For example, in case 2, the calculated strategy is to hit the end of the first red traffic light and then to drive at $V_{\text{opt}}$ during the second and third section. We also note that the best strategy in the the third section is always to drive at $V_{\text{opt}}$, which makes sense because there are not constraints anymore generated by traffic lights.

D. Results for a realistic case

We used exactly the same method on a realistic situation. For the repartition (lenghth and distance) of the traffic lights we used the data of [5]. Data timing and distance represent a real situation in the city of Greenville, SC. The distance D is equal to 5 kilometers to reach in less than 400 seconds. We put two hundred of nodes per sections (distances where there is a set of traffic lights), the calculated optimal path is shown in Fig. 14.

Once again we remark that we never cross a single traffic light. The overall strategy is to drive the closest possible to $V_{\text{opt}}$ while avoiding to stop at traffic lights. This means that at each sections, the driver should drive at $V_{\text{opt}}$ if it does not make him hit a red traffic light, and if it does, he should hit one of the corners of the traffic light.

V. CONCLUSIONS

We used Dijkstra’s algorithm to find the optimal speed profile through a set of traffic lights. The numerical simulations done in section IV confirm the analytical results of section III-B. The best strategy to minimize fuel consumption is to drive the closest possible to $V_{\text{opt}}$ while avoiding to stop at a traffic light. Although this is only proved analytically for the one traffic light case, it remains to be proved in general. This is the subject of future works and publications. Furthermore, a more accurate analysis should take into account the effects of the acceleration, gear shifting and traffic queues.

REFERENCES