Passivity-based feedback stabilization for Bernoulli jump nonlinear systems

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Abstract—We study feedback stabilization of a Bernoulli jump nonlinear system. First, the relationship between stochastic passivity and stochastic stability is established. Then, by using passivity-based tools, a state feedback controller design method that stabilizes a wide class of Bernoulli jump nonlinear systems is proposed. Specifically, the class of Bernoulli jump nonlinear systems considered includes nonlinear dynamics that may not be affine in the control input.

I. INTRODUCTION

In this paper, we study feedback stabilization of Bernoulli jump nonlinear systems (BJNSs) using passivity-based tools. A Bernoulli jump system (BJS) is a special class of hybrid systems. The system consists of $m$ modes that describe the possible dynamics according to which the system state evolves. The particular mode that is active at any time is described by a discrete variable that takes value in the set $\mathbb{M} = \{1, 2, \ldots, m\}$ stochastically. Specifically, the discrete variable chooses any particular value according to a given probability distribution, and independently from one time step to the next.

BJS is a simplified version of a Markovian jump system (MJS), in which the system mode is chosen according to a Markovian random process. MJSs have been used to characterize random abrupt changes in system dynamics caused by, for instance, changes of subsystem interconnections, random component failures or repairs, and change of operating points [19]. A vast literature is available for the stability and stabilization of MJSs [6], [8], [18], [27]. However, most of these results deal either with Markovian jump linear systems (MJLSs), in which the dynamics of every mode is linear, or MJSs in which every mode has dynamics that is linear with additive unknown nonlinearities. These nonlinearities are assumed to be Lipschitz and norm-bounded and usually considered as fictitious uncertainties, so that the robust control framework applies [6], [18]. Despite significant work for particular MJNSs [2], [17], [25], to our best knowledge, the feedback stabilization problem for general MJNSs is still open.

In this paper, we use a passivity based approach to solve the feedback stabilization problem of a particular, but broad, class of MJNSs – systems in which the mode selection is done by a memoryless process and the dynamics of every individual mode is arbitrary except for the constraint that passivity indices [3] exist for the dynamics (although the dynamics need not be passive). This approach, in particular, allows us to consider dynamics that are not affine in the control input, which seems to be a major assumption in works such as [2], [17], [23].

Dissipativity theory, in general, and passivity, in particular, have now become an important analysis and design tool, especially for nonlinear systems. For a thorough overview of passivity, we refer the readers to [24]. Two properties are of particular interest for passive systems. First, Lyapunov function candidates can be obtained naturally from the storage functions induced by passivity [11], [12]. As a result, stability and stabilization problems can often be solved immediately once passivity is guaranteed [5], [14], [15]. Second, passivity is preserved under negative feedback and parallel interconnections, thereby making it an effective approach for the analysis and design of large-scale systems.

The classical concept of passivity has been generalized in various directions and applied to, e.g., networked control systems [10], [13], [26], switched systems [21], [28], and stochastic systems [9]. Of particular interest to the present paper is the work of Lin et al. [17], who studied the stabilization problem for interconnected continuous time MJNSs, with dynamics for every mode being affine in the control input, by introducing a stochastic passivity concept. However, much fewer work has been done for discrete-time stochastic systems with regard to passivity [4], [22]. The results for continuous-time systems cannot be immediately applied to discrete-time systems since it is well known that passivity may not be preserved under sampling. In this paper, we borrow the definition of stochastic passivity from Wang et al. [23], which can be viewed as a generalization of the classical passivity for discrete time MJSs. The main contribution of [23] is the generalization of the KYP lemma for discrete-time MJNSs, which gives a necessary and sufficient condition for a MJNS to be passive. In this paper, we extend the theory of stochastic passivity by using it to solve the feedback stabilization problem of BJNSs in which passivity indices exist for each mode. We also establish the relation between the stochastic passivity concept proposed in [23] and various notions of stochastic stability.

The contributions of this paper are now summarized. First, we adopt the stochastic passivity concept from [23] for BJNSs, and provide the relation between stochastic passivity and various notions of stochastic stability. Second, we propose a passivity-based feedback stabilization method for the class of BJNSs for which the dynamics in every mode has passivity indices. Unlike existing methods, our approach does not require the dynamics to be affine in the control input.

The rest of the paper is organized as follows. The problem formulation is presented in Section II, where the basic definitions of stochastic passivity and stochastic
stability are also given. Section III establishes the connection between stochastic passivity and stochastic stability. Section IV focuses on the feedback stabilization problem of BJNSs using passivity indices. Section VI concludes the paper.

Notations: Throughout this paper, we denote random variables using boldface letters, e.g. \( x \). For any \( 0 \leq k \leq j \) we use the notation \( x^t_k = (x(k), x(k+1), \ldots, x(j)) \) to denote a finite segment of a sequence \( x = \{x(0), x(1), \ldots\} \) and we omit the subscript \( k \) when it is equal to 0. The space of square-summable vector functions over \( \mathbb{Z}^+ \) is denoted by \( l^2(0, \infty) \), and for a sequence \( x = \{x(0), x(1), \ldots\} \in l^2[0, \infty) \), we denote the norms \( ||x(k)|| = \sqrt{x^T(k)x(k)} \) and \( ||x|| = \left( \sum_{k=0}^{\infty} x^T(k)x(k) \right)^{\frac{1}{2}} \). For a real matrix \( A \), \( \sigma_{\text{max}}(A) \) and \( \sigma_{\text{min}}(A) \) denote its largest and smallest singular values, respectively. Moreover, if \( A = A^T \), then we use \( \lambda_{\text{max}}(A) \) and \( \lambda_{\text{min}}(A) \) to denote its largest and smallest eigenvalues, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for the stated algebraic operations. A class \( K \) function is a function \( \beta(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \), which is continuous, strictly increasing and vanishes at zero.

II. PROBLEM FORMULATION

Consider the discrete-time Bernoulli jump nonlinear system

\[
\begin{align*}
x(k+1) &= f_{\theta(k)}(x(k), u(k)), \\
y(k) &= h_{\theta(k)}(x(k), u(k)), \\
& \quad k \geq 0,
\end{align*}
\]

(1)

where \( x(k) \in \mathbb{R}^n \) is the system state, \( u(k) \in \mathbb{R}^q \) is the control input and \( y(k) \in \mathbb{R}^r \) is the output. The random variable \( \theta(k) : \mathbb{Z}^+ \to \mathcal{M} \cong \{1, 2, \ldots, m\} \) is referred to as the system mode. We assume that the random process \( \theta = \{\theta(k)\} \) is a time homogeneous stationary process that consists of independent and identically distributed (i.i.d.) random variables taking values in the finite set \( \mathcal{M} \).

System (1) evolves as \( x(k+1) = f_i(x(k), u(k)) \) and \( y(k) = h_i(x(k), u(k)) \) when \( \theta(k) = i \). It is assumed that functions \( f_i \) and \( h_i \) are smooth maps and the origin \((x^*, u^*) = (0, 0)\) is an isolated equilibrium for all \( i = 1, 2, \ldots, m \). The function \( h_i(\cdot), i \in \mathcal{M} \) is assumed to be locally Lipschitz near the equilibrium, satisfying

\[
\begin{align*}
\tilde{h}_i(\cdot)(||x(k)|| + ||u(k)||) &\leq h_i(x(k), u(k)) \leq \tilde{h}_i(\cdot)(||x(k)|| + ||u(k)||),
\end{align*}
\]

(2)

where \( \tilde{h}_i \geq \tilde{h}_i \geq 0 \) are scalar constants.

The following definitions are standard.

Definition 1: Consider the discrete-time Bernoulli jump nonlinear system (1) and an open set \( \mathbb{D} \) in the state space containing the origin \( x = 0 \). Let \( u(k) = 0 \) for all \( k \in \mathbb{Z}^+ \). The equilibrium of system (1) is said to be locally

1) Lyapunov stable in probability, if \( \forall \epsilon > 0 \), there exists a class \( K \) function \( \beta(\cdot) \) such that for any \( x(0), \theta(0) \in \mathbb{D} \setminus \{0\} \times \mathcal{M} \) and all \( k \in \mathbb{Z}^+ \)

\[
\Pr\{||x(k)|| < \beta(||x(0)||)\} \geq 1 - \epsilon.
\]

(3)

2) asymptotically stable in probability if it is Lyapunov stable in probability and additionally

\[
\Pr\{\lim_{k \to \infty} ||x(k)|| = 0\} = 1.
\]

(4)

3) stochastically stable, if for every initial state \( (x(0), \theta(0)) \in \mathbb{D} \times \mathcal{M} \),

\[
\mathbb{E}(||x||^2(x(0), \theta(0)) < \infty.
\]

Definition 2: System (1) is said to be locally zero-state detectable if for any \( (x(0), \theta(0)) \in \mathbb{D} \setminus \{0\} \times \mathcal{M} \) and \( u(k) = 0 \) for all \( k \in \mathbb{Z}^+ \), \( ||y(k)|| \to 0 \) in probability as \( k \to \infty \), i.e.,

\[
\Pr\{\lim_{k \to \infty} ||y(k)|| = 0\} = 1 \implies \mathbb{E}||y||^2 < \infty.
\]

Definition 3: System (1) is said to be input-output \( l^2 \) detectable if \( \forall u \in l^2[0, \infty), \exists \gamma \in \mathbb{R}^+ \) and \( \beta \in \mathbb{R} \) such that the inequality

\[
\mathbb{E}||y||^2 \leq \gamma^2||u||^2 + \beta
\]

holds. The infimum of \( \gamma \) is called the \( l^2 \) gain of the system and is denoted by \( \gamma^* = \inf \gamma \).

The concept of passivity as defined originally pertained to single mode deterministic systems. Recently there have been attempts to generalize this definition to stochastic systems \([1], [16] \) and MJSSs, in particular \([23] \). We borrow the following definition from \([23] \).

Definition 4: The discrete-time Bernoulli jump nonlinear system (1) is said to be locally stochastically passive if there exists a set of positive definite functions \( V(x(k), i) : \mathbb{D} \times \mathcal{M} \to \mathbb{R}^+ \) corresponding to every mode \( i \in \mathcal{M} \), called the storage functions, such that \( V(0, i) = 0 \) and

\[
\alpha_i(||x(k)||) \leq V(x(k), i) \leq \alpha_i(||x(k)||)
\]

(5)

where \( \alpha_i(\cdot) \) and \( \tilde{\alpha}_i(\cdot) \) are class \( K \) functions, such that for all \( k \in \mathbb{Z}^+ \), \( \alpha_i(\cdot) \in \mathbb{M} \) and all \( (x(k), u(k)) \) in a neighborhood of the equilibrium point \((0, 0)\),

\[
\mathbb{E}[V(x(k+1), \theta(k+1))|x(k), \theta(k)] - V(x(k), \theta(k)) \leq y^T(k)u(k).
\]

(6)

Further if there exists a set of positive definite functions \( S(x(k), i) : \mathbb{D} \times \mathcal{M} \to \mathbb{R}^+ \) such that for all \( k \in \mathbb{Z}^+ \),

\[
\mathbb{E}[V(x(k+1), \theta(k+1))|x(k), \theta(k)] - V(x(k), \theta(k)) \leq y^T(k)u(k) - S(x(k), \theta(k)),
\]

(7)

then system (1) is said to be locally state strictly stochastically passive.

Note that when the BJNS has only one mode, (6) and (7) recover the traditional definition of passivity and state strict passivity, respectively. Due to the stochasticity inherent in the system dynamics, instead of requiring the increase of the storage function to be upper bounded by the supply rate \( y^T(k)u(k) \), we add an expectation operator to the storage function with respect to the system mode. The implication is that the average increase of the storage function is now upper bounded by the supply rate \( y^T(k)u(k) \). We can also generalize the concept of passivity indices (see \([3, \text{Equation}\ (2.85)]\)) from deterministic systems to BJNSs.
Definition 5: System (1) is said to have mode dependent passivity indices \((\nu_i, \rho_i)\) if there exists a set of storage functions \(V(x(k), i) : \mathbb{R}_+ \times \mathbb{M} \to \mathbb{R}_+\) such that for all \(k \in \mathbb{Z}^+, \theta(k) \in \mathbb{M}\) and all \((x(k), u(k))\) in a neighborhood of the equilibrium point,

\[
E[V(x(k + 1), \theta(k + 1))] - V(x(k), \theta(k)) \leq (1 + \rho_\theta(k)\nu_\theta(k))y^T(k)u(k) - \rho_\theta(k)y^T(k)y(k) - \nu_\theta(k)u^T(k)u(k), \tag{8}
\]

In the sequel, we use passivity indices \((\nu_i, \rho_i)\) to refer to the entire set of mode dependent passivity indices \(\{(\nu_i, \rho_i) | i \in \mathbb{M}\}\) with a slight abuse of notation. Note that passivity indices may exist even for a non-passive system.

III. STOCHASTIC PASSIVITY AND STABILITY

In this section, we show that the close relation between passivity and stability for deterministic systems continues to hold for BJNSs with the definition of stochastic passivity given in Section II. In particular, the relations given in Table I hold as summarized in the following results without proof.

**Theorem 1:** If the discrete-time Bernoulli jump nonlinear system (1) is locally stochastically passive, then it is locally Lyapunov stable in probability.

**Theorem 2:** If system (1) is locally state strictly stochastically passive with \(c_i \geq \|x(k)\|^2 \leq V(x(k), i) \leq \alpha_i \|x(k)\|^2\) and \(S(x(k), i) \geq c_i \|x(k)\|^2\) for all \(x(k) \in \mathbb{D}\) and some \(\alpha_i > \alpha_x > 0, c_i > 0\), then it is locally stochastically stable.

IV. FEEDBACK STABILIZATION OF BERNULLI JUMP NONLINEAR SYSTEMS WITH KNOWN PASSIVITY INDICES

The design of a stabilizing feedback controller for a BJNS is known to be a difficult problem. Some results based on LMI approaches and T-S fuzzy models are available in [7], [25]. However, a key assumption in these approaches is that the nonlinear jump system can be approximated by T-S fuzzy models. We now propose a method to design the feedback controller that ensures input-output \(l^2\) stability of the closed-loop system if the passivity indices of the BJNS are known. We first present the following result.

**Theorem 3:** Suppose that the discrete-time Bernoulli jump nonlinear system (1) has passivity indices \((\nu_i, \rho_i)\). Then the system is input-output \(l^2\) stable if

\[
E[\rho_\theta(k)h^2_\theta(k)] > 0, \tag{9}
\]

where

\[
h^2_\theta(i) = \begin{cases} h^2_1, & \text{if } \rho_i \leq 0, \\ h^2_2, & \text{if } \rho_i > 0, \end{cases}, \quad i \in \mathbb{M},
\]

and \(h_1\) and \(h_2\) have been defined in (2). Moreover, an upper bound of the \(l^2\) gain \(\gamma^*\) can be obtained by solving the following optimization problem.

\[
\min_{\alpha_i, \rho_i > 0} \gamma(a), \quad \text{s.t. } E[\gamma_y(\theta(k))h_\theta^2(\theta(k))] < 0, \tag{10}
\]

where

\[
\gamma(a) = \left(\frac{E[\gamma_u(\theta(k))]E[h_\theta^2(\theta(k))]}{E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]}\right)^{\frac{1}{2}},
\]

\[
h^2_\theta(i) = \begin{cases} h^2_1, & \text{if } \gamma(i) \geq 0, \\ h^2_2, & \text{if } \gamma(i) < 0, \end{cases},
\]

\[
\gamma_y(i) = \begin{cases} -\nu_i + 0.5\alpha_i^{-1} + \rho_i\nu_i, & \text{if } \gamma(i) \geq 0, \\ -\rho_i + 0.5\alpha_i[1 + \rho_i\nu_i], & \text{if } \gamma(i) < 0, \end{cases}, \quad i \in \mathbb{M}.
\]

**Proof:** First, we show that if \(E[\rho_\theta(h_\theta^2(\theta))] > 0\), then there exist \(\alpha\)'s such that \(E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]<0\). In particular, we let \(\alpha = 2[1 + \rho_i\nu_i]^{-1}\epsilon\) and obtain

\[
E[\gamma_u(\theta(k))] = E[-\nu_\theta + (4\epsilon)^{-1}[1 + \rho_\theta(\nu_\theta)^2]],
\]

where \(0 < \epsilon < E[\rho_\theta(h_\theta^2(\theta))]((E[h^2_\theta(\theta)])^{-1})\) ensures that

\[
E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]<0.
\]

Next, by using Definition 5 and taking expectation on both sides of (8) with respect to \(x(k), \theta(k)\), we obtain that for any \(k \in \mathbb{Z}^+\), and any \(u(k) \in \mathbb{R}_m\) near the origin,

\[
E[V(x(k + 1), \theta(k + 1)) - V(x(k), \theta(k))] \leq E[(1 + \rho_\theta(k)\nu_\theta(k))y^T(k)u(k) - \rho_\theta(k)y^T(k)y(k)]
\]

\[
- \nu_\theta u^T(k)u(k).
\]

Sum over \(k\) ranging from 0 to \(N - 1\),

\[
E[V(x(N), \theta(N)) - V(x(0), \theta(0))]
\]

\[
\leq \sum_{k=0}^{N-1} E[(1 + \rho_\theta(k)\nu_\theta(k))y^T(k)u(k) - \rho_\theta(k)y^T(k)y(k)]
\]

\[
- \nu_\theta u^T(k)u(k).
\]

Let \(N \to \infty\), and from the assumption that \(V(\cdot, \cdot)\) is a positive definite function, we obtain

\[
- E[V(x(0), \theta(0))]
\]

\[
\leq \sum_{k=0}^{\infty} E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]||x(k)||^2 + ||y(k)||^2]
\]

\[
= E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]\sum_{k=0}^{\infty} E(||x(k)||^2 + ||y(k)||^2)
\]

\[
= E[\gamma_y(\theta(k))h^2_\theta(\theta(k))][||x(k)||^2 + ||y(k)||^2]
\]

\[
\leq E[h^2_\theta(\theta(k))]^{-1} E[\gamma_\theta(\theta(k))h^2_\theta(\theta(k))] E[||y||^2]
\]

\[
+ E[\gamma_\theta(\theta(k))][||u||^2],
\]

where \((a)\) holds because \(\gamma_\theta(\theta(k))h^2_\theta(\theta(k))\) and \(||x(k)||^2 + ||y(k)||^2\) are mutually independent, \((b)\) follows from the assumption that \(E[\gamma_y(\theta(k))h^2_\theta(\theta(k))] < 0\). It is also worth noticing that the assumption of the mode random variables \(\theta(k), k \in \mathbb{Z}^+\) being i.i.d. makes \(E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]\) and \(E[\gamma_y(\theta(k))h^2_\theta(\theta(k))]\) time-invariant functions of \(a_i\).
TABLE I

<table>
<thead>
<tr>
<th>(Stochastically) Passive</th>
<th>Lyapunov stable</th>
<th>Asymptotically stable</th>
<th>Stochastically stable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic system</td>
<td></td>
<td></td>
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<tr>
<td>Lyapunov stable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bernoulli jump system</td>
<td></td>
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</tr>
</tbody>
</table>

Moreover, notice that for every \( i \in \mathbb{M} \), that \( \gamma_y(i) < 0 \) implies that \( \gamma_x(i) > 0 \). As a result, \( \mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))] < 0 \) implies \( \mathbb{E}[\gamma_x(\theta(k))] < 0 \) and \( \mathbb{E}[\gamma_x(\theta(k))] < 0 \). Therefore,

\[
\mathbb{E}[||y||^2] \leq \gamma^2(a)||u||^2 - (\mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))])^{-1} \cdot \mathbb{E}[V(x(0), \theta(0))] \mathbb{E}[h^2_{\theta(k)}],
\]

which means that the system is input-output stable with the \( l^2 \) gain smaller than or equal to \( \gamma(a) \).

**Remark 1:** For the special case where \( \nu_i = \nu \) and \( \rho_i = \rho \) for all \( i \in \mathbb{M} \), the condition in (9) reduces to requiring \( \rho > 0 \). Furthermore, it can be verified that

\[
\gamma(a) = \left( \frac{2\alpha + |1 + \nu|}{\alpha + |1 + \nu| - 2\rho} \right)^{\frac{1}{2}}
\]

achieves its minimum \( \gamma^* = |\rho|^{-1} \) at \( a^* = \rho \) if \( |\nu| < 1 \); or \( \gamma^* = |\nu| \) at \( a^* = \nu^{-1} \) if \( |\nu| > 1 \). In other words, if the BJNS (1) has passivity indices \( (\nu, \rho) \) that are independent of the mode index and \( \rho > 0 \), then it is input-output \( l^2 \) stable with \( l^2 \) gain \( \gamma^* \leq \max(|\rho|^{-1}, |\nu|) \).

Theorem 4 can be used to obtain a passivity-index based controller design method for BJNSs which ensures the input-output \( l^2 \) stability of the closed-loop system.

**Theorem 4:** Consider the feedback interconnection of two discrete-time BJNSs with passivity indices \( (\tilde{\rho}_i, \tilde{\nu}_i) \) \((i \in \mathbb{M}_1 = \{1, 2, \ldots, m_1\})\) and \((\bar{\rho}_j, \bar{\nu}_j) \((j \in \mathbb{M}_2 = \{1, 2, \ldots, m_2\})\), respectively. Assume that the random process \( \theta = (\theta_1, \theta_2) \) is Bernoulli. The interconnection is input-output \( l^2 \) stable if

\[
\mathbb{E}[\lambda_{\max}(Q(\theta(k))h^2(\theta(k)))] < 0,
\]

where for \( \theta(k) = (i, j) \in \mathbb{M}_1 \times \mathbb{M}_2 \),

\[
h^2_{(i, j)} = \begin{cases} 
\tilde{h}^2_{(i, j)}, & \text{if } \lambda_{\max}(Q(i,j)) \geq 0, \\
\bar{h}^2_{(i, j)}, & \text{if } \lambda_{\max}(Q(i,j)) < 0.
\end{cases}
\]

Furthermore, an upper bound on the \( l^2 \) gain can be obtained by solving the following problem

\[
\min \gamma(a), \text{ s.t. } \mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))] < 0,
\]

where

\[
\gamma(a) = \left( \frac{\mathbb{E}[\gamma_y(\theta(k))] \mathbb{E}[h^2_{\theta(k)}]}{\mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))]^{-1}} \right)^{\frac{1}{2}},
\]

\[
h^2_{(i, j)} = \begin{cases} 
\tilde{h}^2_{(i, j)}, & \text{if } \gamma_y(i,j) \geq 0, \\
\bar{h}^2_{(i, j)}, & \text{if } \gamma_y(i,j) < 0.
\end{cases}
\]

Then, by defining \( x \triangleq [x_1^T \ x_2^T]^T \), \( y \triangleq [y_1^T \ y_2^T]^T \), \( r \triangleq [r_1^T \ r_2^T]^T \) and

\[
V(x(k), \theta(k)) = V_1(x_1(k), \theta_1(k)) + V_2(x_2(k), \theta_2(k)),
\]

and using the signal relations due to interconnection, which is \( u_1 = r_1 - y_2, u_2 = r_2 + y_1 \), we have for any \( x(k) \) near the origin and any \( \theta(k) \in \mathbb{M}_1 \times \mathbb{M}_2 \).

\[
\mathbb{E}[V(x(k+1), \theta(k+1))|x(k), \theta(k)] - V(x(k), \theta(k)) \leq (1 + \rho_{\theta(i)}\nu_{\theta(i)})y_1^T(k)u_2(k) - \rho_{\theta(i)}y_1^T(k)u_2(k) - \nu_{\theta(i)}u_1^T(k)u_1(k).
\]

Then, take expectation on both sides with respect to \( x(k), \theta(k) \), and sum over \( k \) ranging from 0 to \( N-1 \), letting \( N \to \infty \),

\[
-\mathbb{E}[V(x(0), \theta(0))]
\]

\[
\leq \sum_{k=0}^{N-1} \mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))] \mathbb{E}[||x(k)||^2 + ||r(k)||^2]
\]

\[
+ \mathbb{E}[\gamma_y(\theta(k))]|r(k)|^2
\]

\[
\leq \mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))](\mathbb{E}[h^2_{\theta(k)}])^{-1} \mathbb{E}[||y||^2
\]

\[
+ \mathbb{E}[\gamma_y(\theta(k))]|r(k)|^2,
\]

where (a) holds because \( \gamma_y(\theta(k))h^2(\theta(k)) \) and \( ||x(k)||^2 + ||u(k)||^2 \) are mutually independent, (b) follows from the assumption that \( \mathbb{E}[\gamma_y(\theta(k))h^2(\theta(k))] < 0 \).
Therefore,
\[
E \|y\|^2 \leq \gamma^2(a)\|u\|^2 - (E[\gamma y(\theta(k))h^2(\theta(k))])^{-1} \cdot E[V(x(0),\theta(0))] \cdot \bar{h}^2(k),
\]
which means that the system is input-output stable with the \(l^2\) gain smaller than or equal to \(\gamma(a)\).

Theorem 4 can be used to design a state feedback stabilizing controller. In particular, system \(G_1\) can be viewed as the plant to be controlled, and system \(G_2\) as the controller to be designed. Suppose that the plant \(G_1\) is unstable. Using Theorem 4, we can obtain passivity indices of the controller \(G_2\) such that inequality (11) is satisfied, which guarantees that the closed-loop system is input-output \(l^2\) stable with a guaranteed upper bound on the \(l^2\) gain.

Note that this method assumes \(\theta_2(k) = \theta_1(k)\) for all \(k \in \mathbb{Z}^+\). However, Theorem 4 is more general. If we assume that the intersection set \(\Lambda \triangleq \bigcap_{k \in \mathbb{Z}} \Lambda(\bar{\nu}, \bar{\rho})\) is not empty, then each element of \(\Lambda\), denoted by \((\bar{\nu}, \bar{\rho})\), defines a mode-independent controller with passivity indices \((\bar{\nu}, \bar{\rho})\) that feedback stabilizes the BJNS \(G_1\). In this case, the controller does not need to know the mode of the plant in real time.

Remark 2: It needs to be mentioned that the well-posedness of the problem requires that \(\rho_i \nu_i \neq 1\), \(\forall i \in \mathcal{M}\). This is not a stringent constraint. We refer the reader to [20, Assumption A1] for a detailed discussion. Although the main result in [20] is for continuous time deterministic systems, the assumption that \(\rho_i \nu_i \neq 1\) continues to be a natural choice for the setup in this paper, since \(\rho_i \nu_i = 1\) renders the input-output mapping independent of \(H\), and thus, trivial.

V. Example

Consider the following Bernoulli jump nonlinear system with two modes and assume that \(\Pr\{\theta(k) = 1\} = P\) and \(\Pr\{\theta(k) = 2\} = 1 - P\). When \(\theta(k) = 1\), the system evolves according to the following dynamics,
\[
\begin{align*}
\dot{x}_1(k+1) &= 2.5x_1^2(k) + 0.1x_2(k), \\
\dot{x}_2(k+1) &= 0.4x_2(k) - 0.5u(k), \\
y(k) &= 5x_1(k) + 5x_2(k) - 5u(k),
\end{align*}
\]
when \(\theta(k) = 2\), the system evolves as
\[
\begin{align*}
\dot{x}_1(k+1) &= 0.5x_1(k) + 0.2u(k), \\
\dot{x}_2(k+1) &= 0, \\
y(k) &= x_1(k) + 0.8u(k).
\end{align*}
\]
The storage function is chosen as
\[
V(x(k), \theta(k)) = \begin{cases} 
|x_1(k)| + |x_2(k)|, & \text{if } \theta(k) = 1, \\
x_1^2(k), & \text{if } \theta(k) = 2,
\end{cases}
\]
For \(\theta(k) = 1\) it holds that
\[
\begin{align*}
V(x(k+1), 1) - V(x(k), 1) - u(k)y(k) - 0.1y(k)^2 &
\leq -2.5(|x_2(k)| + |u(k)| + 0.2)(|x_2(k)| - |u(k)|) \\
&- |x_1(k)|.
\end{align*}
\]
If \(|u(k)| \leq |x_2(k)|\) for all \(k \in \mathbb{Z}^+\), then it holds globally in the state space that
\[
V(x(k+1), 1) - V(x(k), 1) \leq u(k)y(k) + 0.1y(k)^2,
\]
which implies that the dynamics in mode 1 is not passive, but has passivity indices \((0, -0.1)\).

For \(\theta(k) = 2\) it holds that
\[
\begin{align*}
V(x(k+1), 2) - V(x(k), 2) - u(k)y(k) + 0.5y(k)^2 &
= -0.25x_1^2(k) - 0.44u^2(k) \leq 0,
\end{align*}
\]
which implies that the dynamics in mode 2 is passive with passivity indices \((0, 0.5)\).

According to the definition in (2), we have \(h_1 = \bar{h}_1 = 5\) and \(h_2 = 0.8, \bar{h}_2 = 1\). By Theorem 3, a sufficient condition for the BJNS to be input-output \(l^2\) stable is given by
\[
-0.1 \cdot 5 \cdot P + 0.5 \cdot 0.8 \cdot (1 - P) > 0,
\]
which is equivalent to \(P < 4/9\).

Fig. 1. State trajectories of the Bernoulli jump nonlinear system.

![State trajectory](image1)

Fig. 2. Modes of the Bernoulli jump nonlinear system.

![Modes](image2)

Simulation results are provided for a particular control law
\[
u(k) = \begin{cases} 
0.7x_2(k), & \text{if } \theta(k) = 1, \\
-x_1(k), & \text{if } \theta(k) = 2,
\end{cases}
\]
with \(P = 0.33\) and the initial condition \((5, 1)\). The state trajectories are plotted in Fig. 1 with the corresponding mode realization depicted in Fig. 2. The increase in the storage function \(V(x(k+1), \theta(k+1)) - V(x(k), \theta(k))\) is compared.
with the supply rate in Fig. 3, where it can be seen that the BJNS is stochastically dissipative with the given mode-dependent supply rate.

VI. CONCLUSION

In this paper, we studied the problem of feedback stabilization for discrete time Bernoulli jump nonlinear systems using a passivity-based method. We borrowed the concept of stochastic passivity for BJNSs from [23]. First, the relation between stochastic passivity and stochastic stability was established. Then, using passivity-based techniques we proposed a controller design such that the closed-loop system is input-output $l^2$ stable.

This work can be extended in many directions. More general dissipativity concepts such as QSR dissipativity, which includes passivity (indices) as a special case can be considered. Second, communication networks in the feedback loop that introduce delays or noises can be considered. Third, Markovian jump nonlinear systems may be studied where the jumping modes of the system form a Markovian process instead of a Bernoulli process.

REFERENCES


