Abstract—This paper presents a gain scheduled, full-order, anti-windup compensation strategy for LTI systems. The paper makes two contributions. Firstly it shows that, given two point-design anti-windup compensators with differing performance levels and associated regions of attraction, it is possible to interpolate between them to generate a family of compensators with intermediate performance levels and regions of attraction. Secondly, it is shown how this family of compensators can be used to construct a gain-scheduled anti-windup compensator that guarantees stability of the closed-loop system providing the initial states of the system are contained within a certain positively invariant set. The resulting gain scheduled anti-windup compensator is simple to construct and straightforward to implement. The effectiveness of the proposed technique is shown through a simulation example.

I. INTRODUCTION

Anti-windup compensators are additional control system elements which assist a nominal controller in the event of control signal saturation; they are inactive until saturation occurs and, ideally, inactive after saturation has ceased. The majority of anti-windup schemes available in the literature today take the form of linear-time-invariant (LTI) systems that are activated upon saturation of the control signal. Their goal is to ensure that some degree of stability and/or performance is maintained during and after saturation has occurred. Surveys of existing anti-windup schemes can be found in [13], [2] and more comprehensive information found in the books [4], [5], [12], [24]. Linearity of the anti-windup compensator makes its construction and closed-loop analysis relatively straightforward and for this reason, linear anti-windup schemes have attracted much of the research community’s attention. In addition, linear anti-windup schemes are straightforward to implement, making them appealing to the practitioner.

Despite this, it has been known for more than a decade that linear anti-windup schemes are unable to festoon a given saturated closed-loop system with "optimal" performance in all circumstances. The reason for this is that, because control saturation introduces nonlinearity in an otherwise linear system, the arising nonlinear system may have markedly different “small” and “large” signal behavior. If the anti-windup compensator is purely linear and the open-loop plant stable, it is typical for it to be designed to enforce various “large signal” (global) stability and performance properties. The downside to this is that the small-signal performance may suffer as a result.

For this reason, researchers have proposed various nonlinear anti-windup schemes to remedy this problem. One way of constructing a nonlinear anti-windup scheme is to activate different anti-windup compensators depending on the saturation level; this is the approach taken in [15], [20], [10]. Closely related to this approach is the deferred-action anti-windup approach of [9] in which the anti-windup compensator is not activated until the control signal exceeds a value beyond that of the saturation level. Similarly the work of [19], [22] studies the alternative strategy of activating the compensator before saturation is reached. However, perhaps a more general approach is the one of scheduled anti-windup that was first introduced in [23] (see also [24], Chapter 8). Roughly speaking the idea in [23] was to construct a discrete number of anti-windup compensators each associated with an ellipsoidal invariant set and then to switch anti-windup gain as the state traversed successive ellipsoids. This idea was further refined into a continuous scheduling approach in [1] and [3]. In addition, other scheduling techniques based on similar ideas have been presented in the literature (see for example [20]). It must also be mentioned that these approaches to anti-windup design take inspiration from the earlier “nonlinear” approaches to regulation and tracking of constrained input linear systems (e.g. [21], [18], [7], [25]), which the interested reader is encouraged to consult.

The main issue with some of the above scheduling techniques is that the construction of the scheduled anti-windup compensator and its corresponding scheduling function tends to be quite complicated and, arguably, some of the intuition (a watchword for the early anti-windup schemes) and elegance is lost. This paper proposes a scheduled anti-windup strategy that is effectively based on the construction of two point-design linear anti-windup compensators: one demonstrating good large signal properties (large region of attraction) and one demonstrating good small signal properties (small $L_2$ gain with small region of attraction). These anti-windup compensators are then interpolated in a way similar to that given in [11] in order to generate a family of anti-windup compensators. For each fixed value of the scheduling parameter, we then get an anti-windup compensator with properties that are mid-way between the two original compensators. Finally, it is shown that under certain choices of design parameters, these compensators can be scheduled with guaranteed stability; the resulting scheduled anti-windup compensator is exceptionally simple in both design and implementation.

A. Notation

The notation is standard throughout the paper. The set of real (complex) matrices of dimension $n \times m$ is denoted by $\mathbb{R}^{n \times m}$ ($\mathbb{C}^{n \times m}$). For a matrix $F \in \mathbb{R}^{n \times m}$, $F'$ denotes its transpose, and $F_i$ denotes the $i$th row of $F$. If $n = m$ and $F = F'$, the matrix is called symmetric; if in addition $F = F' > 0$, then it is called symmetric positive definite (s.p.d.). We will use the notation $H e \{ F \} = F' + F$ for $F \in \mathbb{R}^{n \times n}$.

The vector $x \in \mathbb{R}^n$ has Euclidean norm defined as

$$
\|x\| := \sqrt{x'x}
$$
The $L_2$ norm of a signal is defined as
\[ \|x(t)\|_2 := \sqrt{\int_0^\infty \|x(t)\|^2 dt} \]

For compactness, the dependence on time of all signals and time varying parameters will be dropped unless considered necessary to map them explicitly. Throughout the paper the following ellipsoidal sets, centered at the origin, are used:
\[ \mathcal{E}^n(P) = \{ x \in \mathbb{R}^n : \|x\|^2 \leq 1 \} \]
where $P > 0$ is some s.p.d matrix. The boundary of $\mathcal{E}^n(P)$ is denoted $\partial \mathcal{E}^n(P)$ and its interior $\text{int} \mathcal{E}^n(P)$. The induced $L_2$ norm (or $L_2$ gain) of a system $T : w \mapsto z$ is defined as
\[ \|T\|_{1,2} := \inf \{ \gamma : \|z\|_2 < \gamma \|w\|_2 + \beta, \; \gamma, \beta > 0 \} \]

II. PROBLEM STATEMENT

Consider the LTI plant
\[ P \sim \begin{cases} \dot{x}_p &= \ A_p x_p + B_p u_m + B_p d \; d \; d \\ y &= C_p x_p \end{cases} \tag{1} \]
where the system states are given by $x_p(t) \in \mathbb{R}^{n_p}$, the control inputs are $u_m(t) \in \mathbb{R}^{n_m}$, and the disturbance signal is $d(t) \in \mathbb{R}^{n_d}$. The system's measured outputs are given by $y(t) \in \mathbb{R}^{n_y}$ and the matrices $A_p$, $B_p$, $B_{pd}$, and $C_p$ are all constant, real and of suitable dimensions. The nominal controller designed for the unsaturated plant (linear plant) is given by
\[ C \sim \begin{cases} \dot{x}_c &= \ A_c x_c + B_c r + B_c (y + y_{aw}) \\ y_c &= C_c x_c + D_c r + D_c (y + y_{aw}) \end{cases} \tag{2} \]
where the $x_c(t) \in \mathbb{R}^{n_c}$ is the controller state vector and $r(t) \in \mathbb{R}^{n_r}$ is the reference signal. The matrices $A_c$, $B_c$, $B_{cr}$, $C_c$, $D_c$, and $D_{cr}$ are again constant, real and of suitable dimensions; $y_{aw}$ is a signal generated by the anti-windup compensator. It is assumed that in the absence of saturation (that is $u_m = y_c$ and $y_{aw} = 0$), the nominal linear controller is stable and has desirable performance properties. Formally, it is assumed that the matrix
\[ A_{CL} = \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_c C_p & A_p \end{bmatrix} \]
is Hurwitz.

In reality the plant input is saturated (component-wise) such that $u_m = \text{sat}(u)$ where
\[ \text{sat}(u) := [\text{sat}(u_1), \ldots, \text{sat}(u_{n_u})] \]
and $\text{sat}(u_i) := \min\{|u_i|, 0\} \times \text{sgn}(u_i)$. Taking inspiration from [17] this paper considers two forms of anti-windup compensator. Firstly, we consider the linear anti-windup compensator given by
\[ \mathcal{AW}_{lin} \sim \begin{cases} \dot{x}_{aw} &= (A_p + B_p F)x_{aw} + B_p \phi \\ u_{aw} &= F x_{aw} \\ y_{aw} &= C_p x_{aw} \\ \phi &= D z(y_c - u_{aw}) \end{cases} \tag{3} \]
where the matrix $F$ is the parameter available to the designer and $D z(u) = u - \text{sat}(u)$. We also consider the nonlinear anti-windup compensator given by
\[ \mathcal{AW}_{nonlin} \sim \begin{cases} \dot{x}_{aw} &= (A_p + B_p F(x_{aw})) x_{aw} + B_p \phi \\ u_{aw} &= F(x_{aw}) x_{aw} \\ y_{aw} &= C_p x_{aw} \\ \phi &= D z(y_c - u_{aw}) \end{cases} \tag{4} \]
where in this case the matrix valued function $F(x_{aw})$ is available for the designer to choose. Our reason for considering such specific forms of anti-windup compensator is because it was shown in [17] that with the compensator structure (3), the overall closed-loop nonlinear system exhibits an attractive decoupling property which is useful for analysis and design purposes (see also [16]). This approach is also very close to the $L_2$ anti-windup approach or the MRAW approach developed by Teel and colleagues (e.g. [14], [24]). The approach in this paper is to blend together two linear anti-windup compensators of the form (3) to obtain a family of linear anti-windup compensators with properties which are mid-way between the point-design compensators. We then seek a scalar (piece-wise) scheduling function $\beta(x_{aw})$ such that a compensator of the form (4), with $F(x_{aw}) = F(\beta(x_{aw}))$, guarantees stability for the closed-loop nonlinear system. Since compensator scheduling is done by changing the slope of the saturation function, the region of attraction will contract. In general, guaranteeing stability for some continuous scalar scheduling function $\beta(x_{aw})$ may be difficult since it must be ensured that the system states remain within the contracting region of attraction, hence it is necessary that the scheduling parameter changes with sufficiently small velocity; we take a different and simpler approach that uses nested ellipsoidal sets to schedule the parameter $\beta(x_{aw})$ in a piece-wise constant fashion.

III. A FAMILY OF LINEAR ANTI-WINDUP COMPENSATORS

A. Preliminary result

The family of linear anti-windup compensators proposed has, as indicated above, the form given in equation (3). It has been noted, for example in [17], that the combination of the plant (1), controller (2) and anti-windup compensator (3) with $u_m = \text{sat}(u)$ and $u = y_c - u_{aw}$ can be represented as shown in Figure 1 which shows a cascade interconnection of two systems. The first of these systems is the nominal linear system given by
\[ G_{CL} = \begin{cases} \dot{x}_{CL} &= A_{CL} x_{CL} + B_{CL} w \\ y_c &= C_{CL} x_{CL} + D_{CL} w \\ y_l &= C_{yCL} x_{CL} \end{cases} \tag{8} \]
\[ \begin{bmatrix} He\{ ApQ(\beta) + B_p((1 - \beta)L_1 + \beta L_2) \} & B_p - (1 - \beta)\epsilon_1 L_1 W_1 - \beta \epsilon_2 L_2 W_2 & 0 & Q(\beta) C_p \\ * & * & * & \star \\ * & * & * & * \\ -2(1 - \beta)W_1 - 2\beta W_2 & (1 - \beta)\epsilon_1 W_1 + \beta \epsilon_2 W_2 & - (1 - \beta)\gamma_1 I - \beta \gamma_2 I & 0 \\ \end{bmatrix} < 0 \] (5)

\[ \begin{bmatrix} A_p Q(\beta) + Q(\beta) A_p' + B_p L(\beta) + L(\beta)' B_p' & B_p - \epsilon_2 L_1 W_2 & 0 & Q(\beta) C_p \\ * & * & * & \star \\ * & * & * & * \\ -2W(\beta) & \epsilon_2 W_2 & - (1 - \beta)\gamma_1 I - \beta \gamma_2 I & 0 \\ \end{bmatrix} < 0 \] (6)

\[ \begin{bmatrix} A_p Q(\beta) + Q(\beta) A_p' + B_p L(\beta) + L(\beta)' B_p' & B_p U(\beta) - \epsilon(\beta) L(\beta) & 0 & Q(\beta) C_p \\ * & * & * & \star \\ * & * & * & * \\ -2U(\beta) & \epsilon(\beta) I & - \gamma(\beta) I & 0 \\ \end{bmatrix} < 0 \] (7)

where

\[ B_{CL} = \begin{bmatrix} B_p D_{cr} & B_{pd} \\ B_{cr} & 0 \end{bmatrix} \quad D_{CL} = \begin{bmatrix} D_{cr} & 0 \end{bmatrix} \] (9)

\[ C_{CL} = \begin{bmatrix} C_c & D_c C_p \end{bmatrix} \quad C_{qCL} = \begin{bmatrix} C_p & 0 \end{bmatrix} \] (10)

and \( w = [r' \quad d']' \).

The second of these systems contains the linear anti-windup compensator dynamics, (3). Note that desirable anti-windup compensator performance can be achieved by keeping \( T_p : y_c \rightarrow y_{aw} \) small in some appropriate sense. The following theorem is used to construct a full-order anti-windup compensator (3) which provides local stability and performance guarantees.

**Theorem 1 (Full-order AW compensation [16]):** Consider the feedback interconnection of (1), (2) and (3), with \( u_m = \text{sat}(u) \) and \( u = y_c - u_{aw} \). Assume there exist matrices \( Q = Q' > 0 \), diagonal \( U > 0 \) and unstructured \( L \), and scalar \( \gamma \) such that the following matrix inequality conditions are satisfied

\[ \begin{bmatrix} A_p Q + Q A_p' + B_p L + L' B_p' & B_p U - \epsilon(\beta) L(\beta) & 0 & Q C_p \\ * & * & * & \star \\ * & * & * & * \\ -2U(\beta) & \epsilon(\beta) I & - \gamma(\beta) I & 0 \\ \end{bmatrix} < 0 \] (11)

\[ \begin{bmatrix} -\bar{u}^2 & \frac{1}{\gamma} \\ \star & -\gamma I \end{bmatrix} \begin{bmatrix} F_1 Q & F_1 Q \\ -Q & -Q \end{bmatrix} < 0 \quad \forall i \in \{1, \ldots, n_u\} \] (12)

Then constructing \( F = L Q^{-1} \) ensures that

1. \( V(x_{aw}) = x_{aw}' Q^{-1} x_{aw} \) is a Lyapunov function for the system (3) and it is strictly decreasing in the set \( \mathcal{E}^{\nu}(Q^{-1}) \) (i.e. \( \mathcal{E}^{\nu}(Q^{-1}) \) is in the region of attraction of the origin).

2. The \( L_2 \) gain of the map \( T_p \) is bounded by \( \gamma \) for all \( u = y_c - u_{aw} \) such that \( \phi \in \text{Sector}[0, \epsilon] \).

**B. An interpolated compensator**

The approach presented in this paper is based on the bumpless transfer approach given in [11], where it was shown that it is possible to schedule between two (static) state-feedback controllers via a single parameter, thus a scheduled feedback gain \( F(\beta) \) could be found with \( \beta \) being the scheduling parameter. This work used quadratic Lyapunov functions to prove stability for each individual controller, and proved that under suitable choices of \( F(\beta) \), a linear combination of the Lyapunov functions could prove stability of the scheduled closed-loop. The first contribution of this paper is to extend this formulation to the actuator saturation setting. The main complication of doing this is that while we change between low and high performance AW compensators, the region of attraction of the nonlinear feedback system contracts, and hence we need to ensure that the AW states remain within this contracting set. Unlike the work presented in [11], in the AW compensator design process it is required that the solution of the individual compensators share certain parameters, which means that there is some (weak) coupling between the LMI’s used to design the compensators.

**Proposition 1:** Consider the system interconnection (1), (2) and (3) with \( u_m = \text{sat}(u) \) and \( u = y_c - u_{aw} \). Define constant scalars \( 0 < \epsilon_2 < \epsilon_1 < 1 \) and assume that there exist matrices \( \{ F_1, Q_1, U_1 \} \) and \( \{ F_2, Q_2, U_2 \} \) which are solutions to matrix inequalities (11)-(12) for sector bounds \( \epsilon_1, \epsilon_2 \) and performance indices \( \gamma_1, \gamma_2 \) respectively. Let \( Q_1 > Q_2 \) and \( U_1 = (\epsilon_1/\epsilon_2) U_2 \). Then, for any (static) \( \beta \in [0, 1] \), with \( F = F(\beta) \) constructed as

\[ F(\beta) := F_1 - \beta (F_1 - F_2) Q_2 Q(\beta)^{-1} \] (13)

where

\[ Q(\beta) := (1 - \beta)Q_1 - \beta Q_2 \] (14)

1. \( V_0(x_{aw}) = x_{aw}' Q(\beta)^{-1} x_{aw} \) is a Lyapunov function for the system (3) and it is strictly decreasing in the set \( \mathcal{E}^{\nu}(Q(\beta)^{-1}) \) (i.e. \( \mathcal{E}^{\nu}(Q(\beta)^{-1}) \) is in the region of attraction of the origin).

2. The \( L_2 \) gain of the map \( T_p \) is bounded by \( \gamma(\beta) \) for all \( u = y_c - u_{aw} \) such that \( \phi \in \text{Sector}[0, \epsilon] \). \( \epsilon(\beta) := \frac{\epsilon_1 \epsilon_2}{(1 - \beta) \epsilon_2 + \beta \epsilon_1} \) (15)

\[ \gamma(\beta) := (1 - \beta) \gamma_1 + \beta \gamma_2 \] (16)

**Proof:** The proof consists of two parts: (i) the first part will show that if LMI (11) is solved for two separate (but not independent) AW compensator designs, a linear combination of the two LMI’s is a solution to the parametrized AW compensator problem; (ii) the second part will provide sufficient conditions that show that the sector condition LMI’s in (12) of the new combined solution are satisfied if these conditions
are satisfied at its vertices, i.e. \( \{F(\beta) = F_1, Q(\beta) = Q_1, \epsilon(\beta) = \epsilon_1\} \) and \( \{F(\beta) = F_2, Q(\beta) = Q_2, \epsilon(\beta) = \epsilon_2\} \).

(i) Local Asymptotic Stability

The proof follows similar ideas to those presented in [11]. Notice that if matrices \( F_1, Q_1, U_1 = W_1^{-1} \) satisfy LMI (11) for some \( \epsilon_1 \), then

\[
H_1 = \begin{bmatrix}
He(A_\beta Q_1 + B_\beta L_1) & B_\beta - \epsilon_1 L_1 W_1 & 0 & Q_1 C_p \\
* & -2W_1 & \epsilon_1 W_1 & 0 \\
* & * & -\gamma I & 0 \\
* & * & * & -\gamma I \\
\end{bmatrix} < 0
\]

Similarly, if \( F_2, Q_2, U_2 = W_2^{-1} \) satisfy LMI (11) for some \( \epsilon_2 \), then

\[
H_2 = \begin{bmatrix}
He(A_\beta Q_2 + B_\beta L_2) & B_\beta - \epsilon_2 L_2 W_2 & 0 & Q_2 C_p \\
* & -2W_2 & \epsilon_2 W_2 & 0 \\
* & * & -\gamma I & 0 \\
* & * & * & -\gamma I \\
\end{bmatrix} < 0
\]

This means that for any \( \beta \in [0, 1] \),

\[
(1 - \beta)H_1 + \beta H_2 < 0
\]

which after defining \( Q(\beta) \) as in equation (14), renders LMI (5). Note that \( L_1 = F_1 Q_1, (i = \{1, 2\}) \), and that

\[
(1 - \beta)L_1 + \beta L_2 = (1 - \beta)F_1 Q_1 + \beta F_2 Q_2 = \frac{F_1 Q_1 + \beta Q_2 + \beta(F_2 - F_1)Q_2}{\beta}
\]

If we define \( F(\beta) \) as in equation (13), then

\[
L(\beta) = (1 - \beta)L_1 + \beta L_2 = F(\beta)Q(\beta)
\]

Next, if we set \( W_1 = (\epsilon_2/\epsilon_1)W_2 \), equation (5) may be rewritten as in equation (6) with \( U(\beta) = W^{-1}(\beta) \) where

\[
W(\beta) = \frac{(1 - \beta)\epsilon_1 + \beta \epsilon_2}{\epsilon_1} W_2
\]

Finally, defining \( \gamma(\beta) \) and \( \epsilon(\beta) \) as in equations (16) and (15) respectively, and applying the congruence transformation

\[
\Pi = \text{diag}(I, U(\beta), I, I)
\]

inequality (6) holds if and only if inequality (7) holds.

Note that LMI (7) has the same form as that of LMI (11) in Theorem 1 which guarantees that, for any fixed \( \beta \in [0, 1] \), the origin of the interconnection is locally asymptotically stable and ensures (locally) that \( \|T_{y \rightarrow y_c}\|_{\text{II}} < \kappa(\gamma) \).

(ii) Sector Condition

If \( x_{aw} \in E^{\nu_r}(Q(\beta)) \), then \( F_i(\beta)x_{aw} \leq \frac{1}{1 - \epsilon(\beta)} \tilde{u}^2 \) for all \( i \) and hence the sector condition \( \phi \in \text{Sector}[0, \epsilon(\beta)I] \) holds, if the following inequality holds:

\[
S(\beta) = F_i(\beta)Q(\beta)F_i(\beta) - \tilde{u}^2/(1 - \epsilon(\beta)) < 0 \quad \forall \ i \in [1, \ldots, n_u]
\]

which is exactly the inequality (12) in Theorem 1 for the interpolated compensator. In order to prove that inequality (20) holds, we will make use of the following fact.

Fact 1: A scalar function \( S(\beta) \) is guaranteed to be negative in \( \beta \in [\beta_1, \beta_2] \) if:

1. \( S(\beta_1) < 0 \) and \( S(\beta_2) < 0 \)
2. \( \text{sign}(\frac{\partial S(\beta)}{\partial \beta}) \) is constant for all \( \beta \in [\beta_1, \beta_2] \)

Condition 1 in Fact 1 is assumed to be satisfied since LMI (12) in Theorem 1 is satisfied for \( \beta = 0 \) and \( \beta = 1 \). It is possible to show after tedious algebraic manipulation, that \( \frac{\partial^2 S(\beta)}{\partial \beta^2} = 0 \) for \( F(\beta) \), \( Q(\beta) \) and \( \epsilon(\beta) \) as given in equations (13), (14) and (15), which in turn implies that \( \frac{\partial S(\beta)}{\partial \beta} \) is constant. Then, Condition 2 in Fact 1 is satisfied and we can conclude that the sector condition inequality (20) is satisfied for any \( \beta \in [0, 1] \). This completes the proof.

Remark 1: Notice that if \( \beta \) is increased (i.e. \( \beta \) is set closer to one), the performance index \( \gamma(\beta) \) will decrease and enhanced performance is expected. However, it is easy to observe that if this is the case then \( Q(\beta) \) will approach \( Q_2 \) which by definition is assumed to be less than \( Q_1 \), and the estimate of the ROA \( E^{\nu_r}(Q^{-1}(\beta)) \) will in fact shrink.

IV. GAIN SCHEDULED AW COMPENSATION

A more ambitious goal is to schedule the anti-windup gains \( F(\beta) \) dynamically such that they provide large region of attraction size and, as the state converges towards the origin, good \( L_2 \) performance properties. Nonetheless, if we want to “blend” the family of AW compensators parametrized by \( \beta \), it is necessary to choose some appropriate scheduling strategy such that \( \beta(.) : \mathbb{R}^n \rightarrow [0, 1] \) is now a time varying function that depends on the size of the anti-windup compensator state. The main complication is that if the velocity of the scheduling parameter \( \beta(.) \) is too high, then the AW compensation states may leave the ROA, hence stability and performance, will not be guaranteed. Hence we need to construct \( \beta(x_{aw}) \) as a function of \( x_{aw} \), such that if \( x_{aw}(0) \in E^{\nu_r}(Q^{-1}(0)) \), then

\[
x_{aw}(t) \in E^{\nu_r}(Q^{-1}(\beta(x_{aw}(t)))) \quad \forall t \geq 0
\]

Furthermore, the states must converge asymptotically to the origin. In the next section, we show that a simple choice of scheduling function enables some positive invariance condition to hold.

A. A coordinate transformation

The proof of stability in the case that \( \beta(.) \) is a function of the system states rather than a fixed value is complicated by the fact that allowing \( F(.) \) to be a function of the system states, causes the anti-windup compensator to become nonlinear. The approach used for the linear anti-windup design, based on that in [17], then breaks down because linearity of the compensator is crucial for that work and also for the derivation of Theorem 1. Instead here we follow a similar approach to that used in Chapter 8 in reference [24].

Defining \( e = x + x_{aw} \), the closed-loop state equations of the interconnection defined by (1), (2) and (4) can be written as:

\[
\begin{bmatrix}
\dot{e} \\
x_c
\end{bmatrix} =
\begin{bmatrix}
A_p + B_p D_c C_p & B_p C_c \\
B_c p & A_c
\end{bmatrix}
\begin{bmatrix}
e \\
x_c
\end{bmatrix}
\]

\[
x_{aw} = A_p + B_p F(\beta)x_{aw} + B_p D_c(y_c - F(\beta)x_{aw})
\]

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Note that equation (21) describes the dynamics of an unforced linear system (in fact, it describes the linear nominal closed-loop system) and exhibits stability by virtue of the stabilizing effect of the linear controller; that is $A_{CL}$ is Hurwitz. Due to the decoupling observed in equations (21)-(22), stability reduces to examining stability of equation (22) which is driven by $y_c = D_c x_e + C_c x_c$. Therefore it is sufficient to establish asymptotic stability of the system

$$\dot{x}_{aw} = (A_p + B_p F(\beta)) x_{aw} + B_p \phi \tag{23}$$

$$\phi = -D x(F(\beta)x_{aw}) \tag{24}$$

Obviously, because $F(\beta)$ varies as a function of the states $x_{aw}$, one has to choose $\beta(\cdot) : \mathbb{R}^n \mapsto [0, 1]$ carefully in order for stability to hold.

B. Stability proof

The main result provides a way of scheduling the anti-windup gain $F(\beta)$, based on the family of linear anti-windup compensators derived earlier. The proposed scheduling structure is similar to the contractively invariant set approach proposed in [6, 7], where it is shown that under suitable constructions of the scheduling parameter, the gain of the linear (state feedback) controller could be scheduled in a piecewise fashion such that performance is improved and the stability is preserved. Our proposition differs from the work in [6] in the sense that we are implementing a stabilizing AW compensator rather than a stability preserving state feedback law, and that we use special constructions of the s.p.d. Lyapunov matrix $P(\beta)$ and the compensator gain $F(\beta)$ (rather than $\beta(\cdot)$ itself) such that it is possible to interpolate between to frozen-point compensators in a stable fashion.

Finally, we consider a gain-scheduling strategy where $\beta(\cdot)$ is chosen such the AW states $x_{aw}$ are ensured to belong to the smallest invariant set from a family of nested ellipsoidal sets; this will guarantee both stability and enhanced performance of the system.

In order to provide a proof of our main proposition, it is necessary to state the following fact and lemma which define the nested ellipsoidal set considered in this paper.

Fact 2: Given the ellipsoids $E_i := E^{\mu}(Q_i^{-1})$ then if $Q_i > Q_{i+1}$ it follows that $E_{i+1} \subset E_i$.

This immediately leads to the following lemma.

Lemma 1: Let the scalars $\beta_i \in [0, 1]$, $i \in \{1, \ldots, N\}$ be such that $\beta_{i+1} > \beta_i$, and define s.p.d matrices $Q_1$, $Q_2$ such that $Q_1 > Q_2$. Then with

$$Q_i = (1 - \beta_i)Q_1 + \beta_i Q_2, \tag{25}$$

it follows that $E_{i+1} \subset E_i$ for all $i \in \{1, \ldots, N\}$.

Proof: From Fact 2, $E_{i+1} \subset E_i$ is true if $Q_i > Q_{i+1}$, or equivalently $Q_i - Q_{i+1} > 0$. Using the expression for $Q_i$ in (25), we have

$$Q_i - Q_{i+1} = (\beta_{i+1} - \beta_i)(Q_1 + Q_2)$$

which is positive definite by virtue of $\beta_{i+1} > \beta_i$ and $Q_1, Q_2 > 0$. \qed

Proposition 2: Let $\{F_1, Q_1, U_1\}$ and $\{F_2, Q_2, U_2\}$ be solutions to Proposition 1 for given $\epsilon_1 < \epsilon_2 \in (0, 1)$ and $\gamma_1 > \gamma_2 > 0$. Then the ellipsoidal sets $E^{\mu}(Q_i^{-1})$ are guaranteed to be nested and positively invariant if Lemma 1 holds and $\beta$ is chosen such that

$$\beta(x_{aw}) = \begin{cases} 
\beta_1 & \text{if } x_{aw} \in E_1/E_2 \\
\beta_2 & \text{if } x_{aw} \in E_2/E_3 \\
\vdots & \\
\beta_{N-1} & \text{if } x_{aw} \in E_{N-1}/E_N \\
\beta_N & \text{if } x_{aw} \in E_N 
\end{cases} \tag{26}$$

where $\beta_i \in [0, 1]$, $i \in \{1, \ldots, N\}$, $\beta_{i+1} > \beta_i$, $\beta_1 = 0$ and $\beta_N = 1$. Then, the origin of the nonlinear closed-loop system (24) is locally asymptotically stable for all $x_{aw}(0) \in E^{\mu}(Q_1^{-1})$.

Remark 2: Proposition 2 allows a nonlinear scheduled anti-windup compensator to be constructed via a very simple mechanism given two linear anti-windup compensators satisfying Proposition 1. The scheduling parameter $\beta(\cdot)$ is a piecewise constant function, where the designer only has to determine a suitable number of points $N$; the switching point between different $\beta(\cdot)$’s may be easily calculated online. No additional conditions are involved in making this construction of nonlinear anti-windup compensator, hence its simplicity may be very attractive for practical systems. This is in contrast to other scheduled anti-windup techniques where more complex conditions are required (see for example [23]).

Proof of Proposition 2: First assume that $x(0) \in E_i = \{x_{aw} \in \mathbb{R}^n : x^T Q_i^{-1} x \leq 1\}$. Then in this case, from the definition of $\beta(\cdot)$ given in the statement of Proposition 2, it follows that $x_{aw}(0) \in E_i/E_{i+1}$. In this set it follows that $\beta = \beta_i$ and $Dz \in [0, \epsilon(\beta_i)]$. Hence, by Proposition 1, we know that $V_{\beta_i}(x_{aw}) < 0$. This then implies that, after finite time $t_i$, $x_{aw}(t_i) \in \partial(E_{i+1})$.

Once, $x_{aw}(t) \in int(E_{i+1})$, $\beta$ then switches to $\beta_{i+1}$ and, in this case, $Dz \in [0, \epsilon(\beta_{i+1})]$. Hence, by Proposition 1, we know that $V_{\beta_{i+1}}(x_{aw}) < 0$ and that after finite time $t_i$, $x_{aw}(t_i) \in E_{i+2}/E_{i+3}$. An induction argument then allows us to conclude that $x_{aw}$ enters $E_N$ in finite time and, once in this set, from the statement of Proposition 2, $\beta = 1$ and $V_1 < 0$; hence the system state then approaches the origin asymptotically. \qed

Remark 3: The number of switching points $N$, and the way in which the interval $\beta \in [0, 1]$ is gridded (i.e. interval sizes $\beta_{i+1} - \beta_i$ are not obliged to be equal for all $i \in \{1, \ldots, N\}$) may influence the overall performance of the AW compensator. Although it is intuitive that a large $N$ will enhance performance, this fact is not explored explicitly in this paper and is left for future work.

V. Example

In order to demonstrate the effectiveness of the proposed AW compensator, a simulation example taken from [8] is presented. The LTI plant is stable with actuator constraint equal to $\bar{u} = 10.5$ and the severe and mild saturation levels are chosen such that the associated slopes are $\epsilon_1 = 0.99$ and $\epsilon_2 = 0.5$ respectively. The nonlinear closed-loop without AW compensator remains stable, but presents a noticeable degradation of performance (see Figure 2).
The associated performance index of each of the AW compensators is $\gamma_1 = 14.11$ and $\gamma_2 = 0.033$.

A scheduled AW compensator is constructed based on Proposition 2, and choosing a total of $N = 100$ nested ellipsoidal sets; this choice allows an appropriate “blending” of the point-design compensators, with low computational burden. Recall that in the architecture proposed, only the state-feedback term $F(\beta)$ is scheduled and $F$ is constructed from a simple convex combination of two linear compensators. Figure 2 shows how the scheduled AW compensator enhances performance for severe saturation levels, and its performance is superior to either one of the frozen-point linear compensators.

VI. CONCLUSIONS

This paper has shown how two full-order anti-windup compensators may be interpolated in a simple manner, in order to obtain a family of anti-windup compensators. It has further shown how this family of compensators may be continuously scheduled in order to obtain a gain-scheduled anti-windup compensator which can combine large signal and small signal stability and performance properties. A simple example has illustrated the effectiveness of the approach. It should be emphasized that the appealing feature of the approach is its simplicity and intuition. It is hoped that this may find favor amongst practical engineers and perhaps help the theory and practice of anti-windup design to converge.

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Fig. 2. Large signal, saturated system response: (a) No AW - reference (solid line), saturated system response (dashed line); (b) Point-design AW - high performance AW response (dashed line), low performance AW response (solid line); (c) Gain Scheduled AW - unsaturated (linear) system response (solid line), Gain Scheduled AW response (dashed line)