Output Feedback Economic Model Predictive Control of Parabolic PDE Systems

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Abstract—We proposed an economic model predictive control (EMPC) system for parabolic partial differential equation (PDE) systems in [17]. The EMPC system assumed the knowledge of the complete state spatial profile at each sampling period. From a practical point of view, measurements of the state variables are available only at a finite number of spatial positions. To address this practical consideration, an output feedback EMPC system that accounts for both manipulated input and state constraints is developed for a quasi-linear parabolic PDE system. The EMPC system is applied to a non-isothermal tubular reactor. Two EMPC systems, each utilizing a different number of measurement sensors and formulated with various degrees of accuracy (i.e., number of modes retained from the infinite-dimensional model), are presented and compared on the basis of model accuracy, and input and state constraint satisfaction.

I. INTRODUCTION

Model predictive control (MPC) is a popular optimal control technique that has gained widespread popularity within the process control industries. In the past decade, significant work has been done on MPC of partial differential equation (PDE) systems (e.g., [8], [7], [6], [10], [15], [18], [20], [21]). To guide the PDE system to the desired steady-state profile, conventional MPC schemes are formulated with the sum of squared differences between the state and inputs from their corresponding steady-state values. More recently, economic model predictive control (EMPC), a MPC scheme that is formulated with a general cost function accounting directly for the process economics, has been proposed for lumped-parameter systems described by ordinary differential equations (ODEs) which operates systems in a dynamically optimal fashion (e.g., [5], [11], [13], [12], [14]).

The significant challenge to applying the proposed EMPC systems directly to PDE systems is deriving a finite-dimensional ODEs which operates systems in a dynamically optimal fashion (e.g., [4], [9]), a system of finite-dimensional ODEs that accurately describe the dynamics of the dominant (slow) modes of the PDE system is derived for the synthesis of low-order controllers (e.g., [1] and the book [3]). Order reduction using Galerkin’s method and analytical eigenfunctions was employed in [17] where we proposed an EMPC scheme for parabolic PDE systems.

While, the EMPC system of [17] demonstrated improved closed-loop economics (greater production rate of the desired product over steady-state operation), the EMPC system assumed the knowledge of the complete state spatial profile at each sampling period. Owing to the practical relevance of formulating an EMPC for parabolic PDE system based on measurements at a finite number of spatial points, an output feedback EMPC for PDE systems is presented. The EMPC system is applied to a non-isothermal tubular reactor where a second-order chemical reaction takes place. Two EMPC systems, each utilizing a different number of measurement sensors and formulated with different number of modes retained from the infinite-dimensional ODE model, are presented and compared on the basis of model accuracy, and input and state constraint satisfaction.

II. PRELIMINARIES

A. Parabolic PDEs

We consider quasi-linear parabolic PDEs with measured outputs of the form:

\[
\frac{\partial \bar{x}}{\partial t} = A \frac{\partial \bar{x}}{\partial z} + B \frac{\partial^2 \bar{x}}{\partial z^2} + W u(t) + f(\bar{x}(z, t))
\]

\[
y_j(t) = \int_0^1 c_j(z) \bar{x}(z, t) dz, \quad j = 1, \ldots, p
\]

subject to the boundary conditions:

\[
\frac{\partial \bar{x}}{\partial z} = g_0 \bar{x}, z = 0; \quad \frac{\partial \bar{x}}{\partial z} = g_1 \bar{x}, z = 1;
\]

and the initial condition:

\[
\bar{x}(z, 0) = \bar{x}_0(z)
\]

where \(\bar{x}(z, t) = [\bar{x}_1(z, t) \cdots \bar{x}_n(z, t)]'\) denotes the vector of the system state variables, the notation \(\bar{x}'\) denotes the transpose of \(\bar{x}\), \(f(\bar{x}(z, t))\) denotes a nonlinear vector function, \(y_j(t)\) is the \(j\)-th measured output, \(c_j(z)\) are known smooth functions of \(z\) \((j = 1, \ldots, p)\) whose functional form depends on the type of the measurement sensors, \(z \in [0, 1]\) is the spatial coordinate, \(t \in [0, \infty)\) is the time, and the notation \(A, B, W, g_0\) and \(g_1\) are used to denote (constant) matrices of appropriate dimensions. The control input vector

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is denoted as \( u(t) \in \mathbb{R}^{n_u} \) and is subject to the following constraints:

\[
\text{umin} \leq u(t) \leq \text{umax} \quad (4)
\]

where \( \text{umin} \) and \( \text{umax} \) are the lower and upper bound vectors of the manipulated input vector, \( u(t) \). Moreover, the system states are also subject to the following state constraints:

\[
x_{i,\text{min}} \leq r_{x_i}(z)x_i(z, t)dz \leq x_{i,\text{max}}
\]

where \( x_{i,\text{min}} \) and \( x_{i,\text{max}} \) are the lower and upper state constraint for the \( i \)-th state, respectively. The function \( r_{x_i}(z) \in L_2(0, 1) \) where \( L_2(0, 1) \) is the space of measurable, square-integrable functions on the interval \([0, 1]\) is the state constraint distribution function.

III. METHODOLOGICAL FRAMEWORK FOR OUTPUT FEEDBACK EMPC BASED ON A REDUCED-ORDER MODEL

A. Galerkin’s Method

We first formulate the PDE system of Eqs. 1-3 as an infinite-dimensional system in the Hilbert space \( \mathcal{H}([0, 1]; \mathbb{R}^{n_x}) \), with \( \mathcal{H} \) being the space of measurable vector functions defined on \([0, 1]\), with inner product and norm:

\[
\langle \omega_1, \omega_2 \rangle = \int_0^1 \langle \omega_1(z), \omega_2(z) \rangle_{\mathbb{R}^{n_x}}dz,
\]

\[
\|\omega_1\|_2 = (\omega_1, \omega_1)^{1/2}
\]

where \( \omega_1, \omega_2 \) are two elements of \( \mathcal{H}([0, 1]; \mathbb{R}^{n_x}) \) and the notation \( (\cdot, \cdot)_{\mathbb{R}^{n_x}} \) denotes the standard inner product in \( \mathbb{R}^{n_x} \). The state function \( x(t) \) on the state-space \( \mathcal{H} \) is defined as

\[
x(t) = \bar{x}(z, t), \quad t > 0, \quad 0 \leq z \leq 1,
\]

and the operator \( A \) is defined as

\[
Ax = A\frac{d\bar{x}}{dz} + B\frac{d^2\bar{x}}{dz^2}, \quad 0 \leq z \leq 1.
\]

and the measured output operator is defined as:

\[
C\dot{x}(t) = [(c_1(\cdot), \bar{x}(\cdot, t)) \cdots (c_p(\cdot), \bar{x}(\cdot, t))]'
\]

Then, the system of Eqs. 1-3 takes the following form:

\[
\dot{x}(t) = Ax(t) + Bu(t) + F(x(t)), \quad x(0) = x_0
\]

\[
y(t) = Cx(t)
\]

where \( x_0 = \bar{x}_0(z), Bu(t) = W_0u(t) \) and \( F(x(t)) \) is a nonlinear vector function in the Hilbert space. The eigenvalue problem for \( A \) takes the form

\[
A\phi_k = \lambda_k\phi_k, \quad k = 1, \ldots, \infty
\]

subject to

\[
\frac{d\phi_k}{dz}(0) = g_0\phi_k(0); \quad \frac{d\phi_k}{dz}(1) = g_1\phi_k(1)
\]

where \( \phi_k \) is an eigenfunction corresponding to the \( k \)-th eigenvalue \( \lambda_k \) and \( \phi_k \) is an adjoint eigenfunction of the operator \( A \).

Assumption 1 below characterizes the class of parabolic PDEs considered in this work and states that the eigenspectrum of operator \( A \) can be partitioned into a finite part consisting of \( m \) slow eigenvalues which are close to the imaginary axis and a stable infinite complement containing the remaining fast eigenvalues which are far in the left-half of the complex plane, and that the separation between the slow and fast eigenvalues of \( A \) is large. We also note that the large separation of slow and fast modes of the spatial operator in parabolic PDEs ensures that a controller which exponentially stabilizes the closed-loop ODE system, also stabilizes the closed-loop infinite-dimensional system [2]. This assumption is satisfied by the majority of diffusion-convection-reaction processes [3].

Assumption 1:

1. \( Re(\lambda_1) \geq Re(\lambda_2) \geq \cdots \geq Re(\lambda_k) \geq \cdots \), where \( Re(\lambda_k) \) denotes the real part of the eigenvalue, \( \lambda_k \).

2. The eigenspectrum of \( A, \sigma(A) \), is defined as the set of all eigenvalues of \( A \), i.e. \( \sigma(A) = \{ \lambda_1, \lambda_2, \cdots \} \). \( \sigma(A) \) can be partitioned as \( \sigma(A) = \sigma_1(A) \cup \sigma_2(A) \), where \( \sigma_1(A) \) consists of the first \( m \) finite eigenvalues, i.e. \( \sigma_1(A) = \{ \lambda_1, \cdots, \lambda_m \} \), and \( |Re(\lambda_1)|/|Re(\lambda_m)| = O(1) \).

3. \( Re(\lambda_{m+1}) < 0 \) and \( |Re(\lambda_1)|/|Re(\lambda_{m+1})| = O(\epsilon) \) where \( \epsilon < 1 \) is a small positive number.

Next, we apply standard Galerkin’s method [19] to the infinite-dimensional system of Eqs. 10 to derive a finite-dimensional subsystem. Let \( \mathcal{H}_s \) and \( \mathcal{H}_f \) be modal subspaces of \( A \) defined as \( \mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \cdots, \phi_m\} \) and \( \mathcal{H}_f = \text{span}\{\phi_{m+1}, \phi_{m+2}, \cdots\} \), where \( \phi_k, k = 1, 2, \cdots \) are the eigenfunctions of \( A \) which are orthogonal to each other. Using the orthogonal projection operators, \( P_s \) and \( P_f \), which project the state \( x \) onto the subspaces \( \mathcal{H}_s \) and \( \mathcal{H}_f \) of \( A \), respectively \( (x_s = P_s x \in \mathcal{H}_s \) and \( x_f = P_f x \in \mathcal{H}_f \) ), the state \( x \) of the system of Eq. 10 can be written as

\[
x = x_s + x_f = P_s x + P_f x
\]

Applying \( P_s \) and \( P_f \) to the system of Eq. 10 and using the above decomposition for \( x \), Eq. 10 can be re-written as:

\[
\dot{x}_s(t) = A_s x_s(t) + A_s x_f(t) + F_s(x_s(t), x_f(t)) + B_s u(t), \quad x_s(0) = P_s x_0
\]

\[
\dot{x}_f(t) = A_f x_f(t) + A_f x_s(t) + F_f(x_s(t), x_f(t)) + B_f u(t), \quad x_f(0) = P_f x_0
\]

where \( A_s = P_s A, B_s = P_s B, A_f = P_f A, B_f = P_f B, F_f = P_f F, F_s = P_s F, C_s = C P_s, \) and \( C_f = C P_f \). Specifically, \( A_s = \text{diag}\{\lambda_j\}, j = 1, \cdots, m \) is a diagonal matrix of dimension \( m \times m \) and may contain unstable eigenvalues (i.e., \( Re(\lambda_j) > 0 \) for some \( j \)). Note that since the subspace of \( A \) are spanned by eigenfunctions of \( A, A_s x_f(t) = 0 \) and \( A_f x_s(t) = 0 \) based on the fact that the eigenfunctions are orthogonal to each other. The operator \( A_f \) is an unbounded exponentially stable differential operator. The first subsystem (i.e., first equation) of Eq. 14 is referred to as the slow subsystem; while, the second subsystem is referred to as the fast subsystem. Neglecting the fast subsystem, we obtain the
ODE system describing the dominant dynamics of the PDE:
\[
\begin{align*}
\dot{x}_s(t) &= A_s x_s(t) + F_s(x_s(t); 0) + B_s u(t) \\
x_s(0) &= P_s x_0 \\
y(t) &= C_s x_s(t)
\end{align*}
\] (15)

**B. State Estimation Using Output Feedback Methodology**

The objective of this section is to propose the state estimation technique used in the state estimation-based EMPC formulations that make use of a finite number, \( p \), of measured outputs \( y_j(t) \) \( (j = 1, \cdots, p) \) to compute estimates of \( x_s \) and \( x_f \). The state estimation scheme is based on a direct inversion of the measured output operator to obtain estimates of the slow modes, \( \hat{x}_s(t) \) and the concept of the approximate inertial manifolds to obtain estimates of the fast modes, \( \hat{x}_f(t) \) in the system of Eq. 14. To develop this estimation scheme, we must impose an assumption on the number of measured outputs. We assume that the number of measured outputs is equal to the number of slow modes (i.e., \( p = m \)) and the distribution functions of the measured outputs are chosen such that \( C_s^{-1} \) exists. Under this assumption, an estimate of the slow subsystem state, \( \hat{x}_s(t) \) can be obtained as follows:
\[
\hat{x}_s(t) = C_s^{-1} y(t)
\] (16)

where \( \hat{x}_s(t) \) is an estimate of \( x_s(t) \).

Since the accuracy of the estimated modes through the reconstruction of the spatially distributed PDE states is limited by the number of available measurement points, we introduce the derivation of the estimation for the fast subsystem state, \( x_f(t) \) to achieve additional accuracy of the state estimation scheme. In the infinite-dimensional system described by Eq. 14, the fast dynamics, \( \dot{x}_f(t) \) can be ignored compared with that of the slow dynamics, \( \dot{x}_s(t) \) given that \( A_f \) includes eigenvalues with large negative real part \([3\) \( (A_f \) is exponentially stable). Thus, the equation of the fast state, \( x_f(t) \) can be approximately expressed as the following equality:
\[
A_f x_f(t) + B_f u(t) + \mathcal{F}_f(x_s(t), x_f(t)) = 0
\] (17)

The fast state \( x_f \) is equal to zero at its quasi-steady-state (we note that \( x(z, t) = 0 \) is a steady-state of the nominal PDE system \( u(t) \equiv 0 \)). Accounting for the fact that \( A_f \) includes eigenvalues with large negative real parts, we can neglect the fast subsystem state, \( x_f(t) \) in the nonlinear term, \( \mathcal{F}_f(x_s, x_f) \). Using the estimated slow subsystem state, \( \hat{x}_s \) to calculate the approximate fast subsystem state, Eq. 17 becomes
\[
A_f \hat{x}_f(t) + B_f u(t) + \mathcal{F}_f(\hat{x}_s(t), 0) = 0
\] (18)

where \( \hat{x}_f(t) \) is the estimated fast state. An explicit form for the estimated fast state can be derived:
\[
\hat{x}_f(t) = -A_f^{-1}[B_f u(t) + \mathcal{F}_f(\hat{x}_s(t), 0)]
\] (19)

**Remark 1:** The accuracy of the finite-dimensional ODE model with \( m \) slow modes is of order \( \epsilon = \frac{|Re\{\lambda_1(A)\}|}{|Re\{\lambda_{m+1}(A)\}|} (O(\epsilon)) \). This means under state feedback the closeness of the closed-loop system PDE state to the closed-loop system ODE state is \( O(\epsilon) \).

Closed-loop stability under output feedback works not only for the slow and fast modes but also the real state as long as the estimation error is negligible. This occurs when \( m \) is chosen to be sufficiently large such that \( \epsilon \) is sufficiently small. Specifically, to achieve the same level of closeness for the output feedback case \( (O(\epsilon)) \), the number of measurements must be equal to the number of slow modes (i.e., \( p = m \)). From a practical standpoint, the discrepancy between the closed-loop PDE state under output feedback with more than \( m \) measurements and the one of the closed-loop PDE state under output feedback with \( m \) measurements will be indistinguishable since the achieved closed-loop system performance is limited by the number of slow modes \( (m) \) used in the design of the EMPC. Therefore, if there are more available measurement points than the slow modes \( (p > m) \), one can pick any \( m \) measurement points from the set of \( p \) points as long as the assumption that \( C_s^{-1} \) exists is satisfied. In this work, from both the open-loop and closed-loop system simulation, we have good state estimation accuracy by picking \( m \) sufficiently large and choosing the measurement points such that the inverse of the matrix \( C_s \) exists.

**IV. APPLICATION TO A TUBULAR REACTOR**

**A. Reactor Description**

We consider a non-isothermal tubular reactor where an irreversible second-order reaction of the form \( A \rightarrow B \) takes place. The process model in dimensionless variable form consists of two quasi-linear parabolic PDEs (process details and model notation can be found in [17] and [19]):
\[
\begin{align*}
\frac{\partial \bar{x}_1}{\partial t} &= -\frac{\partial \bar{x}_1}{\partial z} + \frac{1}{P e_1} \frac{\partial^2 \bar{x}_1}{\partial z^2} + \delta(z - 0)T_i + B_T B_C \exp \left( \frac{\gamma \bar{x}_1}{1 + \bar{x}_1} \right) (1 + \bar{x}_2)^2 \\
&\quad + \beta (T_s - \bar{x}_1)
\end{align*}
\] (20)
\[
\begin{align*}
\frac{\partial \bar{x}_2}{\partial t} &= -\frac{\partial \bar{x}_2}{\partial z} + \frac{1}{P e_2} \frac{\partial^2 \bar{x}_2}{\partial z^2} + \delta(z - 0)u \\
&\quad - B_C \exp \left( \frac{\gamma \bar{x}_1}{1 + \bar{x}_1} \right) (1 + \bar{x}_2)^2
\end{align*}
\]

where \( T_i \) is the inlet temperature, \( T_s \) is the jacket temperature, the states \( \bar{x}_1 \) and \( \bar{x}_2 \) are temperature and reactant concentration in the reactor, respectively, the input \( u \) is the inlet reactant concentration, \( \delta \) is the standard Dirac function, and the remaining parameters are the process parameters. The tubular reactor is subject to the following transformed boundary conditions:
\[
\begin{align*}
z = 0 : \frac{\partial \bar{x}_1}{\partial z} = P e_1 \bar{x}_1, \quad \frac{\partial \bar{x}_2}{\partial z} = P e_2 \bar{x}_2; \\
z = 1 : \frac{\partial \bar{x}_1}{\partial z} = 0, \quad \frac{\partial \bar{x}_2}{\partial z} = 0;
\end{align*}
\] (21)

The following typical values are given to the process parameters: \( P e_1 = 7, P e_2 = 7, B_T = 2.5, B_C = 0.1, \)
\[ \beta_T = 2, T_s = 0, T_i = 0 \text{ and } \gamma = 10. \] In all simulations reported below, second-order finite-difference method was used to discretize, in space, the two parabolic PDEs of Eq. 20 to derive two 101th-order set of ODEs (further increase on the order of discretization led to identical open-loop and closed-loop results).

**B. Implementation of Output Feedback EMPC**

Since the PDE system of Eq. 20 consists of two PDEs, the index \( i \) \((i = 1, 2)\) is used to denote the \( i \)-th PDE of Eq. 20. We assume the tubular reactor has \( p_1 + p_2 = p \) sensors where the first \( p_1 \) sensors measure the temperature (i.e., the state corresponding to the first PDE of Eq. 20) at measurement points \( z_{s,1j} \in [0, 1] \) for \( j = 1, 2, \ldots, p_1 \), and the next \( p_2 \) sensors measure the concentration of A (i.e., the second state) at measurement points \( z_{s,2j} \in [0, 1] \) for \( j = 1, 2, \ldots, p_2 \). Thus, the output measurements consist of the state measurements at a finite number of points in the spatial domain (i.e., \( x_i(t_k) = \{ \bar{x}_i(z_{s,1i}, t_k) \cdots \bar{x}_i(z_{s,ip_i}, t_k) \} \)) and can be written as

\[ y_{ij}(t_k) = \bar{x}_i(z_{s,ij}, t_k), \quad i = 1, 2, \quad j = 1, 2, \ldots, p_i \quad \text{(22)} \]

where the output measurement vector, \( y'(t_k) = [y'_1(t_k) \ y'_2(t_k)] \) and

\[ y'_i(t_k) = [y_{i1}(t_k) \cdots y_{ip_i}(t_k)], \quad i = 1, 2. \quad \text{(23)} \]

We assume the number of measurements satisfies \( p_i = m_i \) where \( m_i \) refers to the number of total slow modes retained from the \( i \)-th PDE in the construction of the model of Eq. 14. Since point-wise measurements are considered, the following measurement distribution function is used:

\[ c_{ij}(z) = \delta(z - z_{s,ij}), \quad i = 1, 2, \quad j = 1, 2, \ldots, p_i \quad \text{(24)} \]

where \( \delta \) is the standard Dirac function. Each measurement point, \( z_{s,ij} \) in the spatial domain is assumed to be at \( z_{s,ij} = (j - 1)/(p_i - 1) \). The choice of measurement points satisfies the assumption that \( C_{-1}^{-1} \) exists.

The state \( \bar{x}_i(z, t) \) can be decomposed into the sum of the amplitudes and the eigenfunctions of the first \( l_i \) eigenmodes:

\[ \bar{x}_i(z, t) \approx \sum_{i=1}^{l_i} a_{ij}(t) \phi_{ij}(z), \quad i = 1, 2. \quad \text{(25)} \]

where \( a_{ij}(t) \) and \( \phi_{ij}(z) \) are the amplitude and eigenfunctions associated with the \( j \)-th eigenvalue of the spatial operator \( A \).

Utilizing the decomposition of Eq. 25, the estimated slow mode vector, \( a'_{s,i}(t_k) = [a_{s,i1}(t_k) \cdots a_{s,ip_i}(t_k)] \) (recall that \( m_i = p_i \) can be written in the following form:

\[ y_i(t_k) = C_{s,i} x_i(t_k) = C_{s,i} a_{s,i}(t_k) = \begin{pmatrix} a_{s,i1}(t_k) \\ a_{s,i2}(t_k) \\ \vdots \\ a_{s,ip_i}(t_k) \end{pmatrix} \times \begin{pmatrix} \phi_{i1}(z_{s,1i}) & \phi_{i2}(z_{s,1i}) & \cdots & \phi_{ip}(z_{s,1i}) \\ \phi_{i1}(z_{s,2i}) & \phi_{i2}(z_{s,2i}) & \cdots & \phi_{ip}(z_{s,2i}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i1}(z_{s,ip_i}) & \phi_{i2}(z_{s,ip_i}) & \cdots & \phi_{ip}(z_{s,ip_i}) \end{pmatrix} \quad \text{(26)} \]

Under the assumption of that \( C_{-1}^{-1} \) exists, the estimated slow modes can be achieved by the following form:

\[ \hat{a}_{s,i}(t_k) = C_{s}^{-1} y_i(t_k) = \begin{pmatrix} \phi_{i1}(z_{s,1i}) & \phi_{i2}(z_{s,1i}) & \cdots & \phi_{ip}(z_{s,1i}) \\ \phi_{i1}(z_{s,2i}) & \phi_{i2}(z_{s,2i}) & \cdots & \phi_{ip}(z_{s,2i}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i1}(z_{s,ip_i}) & \phi_{i2}(z_{s,ip_i}) & \cdots & \phi_{ip}(z_{s,ip_i}) \end{pmatrix}^{-1} \times \begin{pmatrix} x_1(z_{s,1i}, t_k) \\ x_1(z_{s,2i}, t_k) \\ \vdots \\ x_1(z_{s,ip_i}, t_k) \end{pmatrix} \quad \text{(27)} \]

Since the decomposition of Eq. 25 provides a simplified method for describing the temporal evolution of the PDE system, the reduced-order model of Eqs. 14 is written in terms of the temporal evolution of the amplitudes of each eigenmode. After applying the decomposition of Eq. 25 to Eq. 14, multiplying both sides by the adjoint eigenfunction, and making a similar approximation as in Eq. 19, the resulting reduced-order model has the following form (the \( \tilde{ } \) notation is dropped for simplicity of presentation):

\[ \hat{a}_{s,i}(t) = A_{s,i} a_{s,i}(t) + F_{s,i} a_f(t) + B_{s,i} u(t) \]

\[ a_f(t) = A_{f,i}^{-1} [B_{f,i} u(t) + F_{f,i} a_f(t)] \]

\[ y(t) = C_{s,1} a_{s,1}(t) + C_{s,2} a_{s,2}(t) + C_{f,1} a_{f,1}(t) + C_{f,2} a_{f,2}(t) \]

where \( a_{s,i}(t) = [a_{s,i1}(t) \ a_{s,i2}(t) \cdots a_{s,ip}(t)]' \) with elements \( a_{s,ij}(t) \in \mathbb{R} \) associated with the amplitudes of the \( j \)-th eigenmodes and \( i = 1, 2 \). The vector \( a_{f,ij}(t) \) is a vector of similar structure to \( a_{s,ij}(t) \) with elements associated with the next \( m_i + 1 \) to \( l_i \) eigenmodes. The vector \( a_{s,ij}(t) \) and \( a_{f,ij}(t) \) are defined as \( a_{s,ij}(t) = [a_{s,1i}(t) \ a_{s,2i}(t)]' \) and \( a_{f,ij}(t) = [a_{f,1i}(t) \ a_{f,2i}(t)]' \). Using this notation, the matrix \( A_{s,i} \) is defined as \( A_{s,i} = \text{diag}\{\lambda_{ij}\} \), \( j = 1, \ldots, m_i \) (i.e., a diagonal \( m_i \times m_i \) matrix where diagonal entries are equal to \( \lambda_{ij} \) and the indices \( j \) and \( i \) are used to denote the \( j \)-th eigenmode of the \( i \)-th PDE state) and the matrix \( A_{f,ij} \) is defined as \( A_{f,ij} = \text{diag}\{\lambda_{ij}\} \), \( j = m_i + 1, \ldots, l_i \).

A quadratic Lyapunov function is imposed to the EMPC formulation and written in terms of the amplitudes:

\[ V(a(t)) = a'(t)P a(t) \quad \text{(29)} \]

where \( a(t) \) denotes a vector consisting of the amplitudes of all retained eigenmodes (i.e., both \( a_{s,ij}(t) \) and \( a_f(t) \)) for each PDE, \( P \) is an \( (l_i + l_2) \times (l_i + l_2) \) identity matrix and \( \tilde{p} = 3 \) is used in the formulations of the EMPC systems below.

The cost function that we consider is to maximize the overall reaction rate along the length of the reactor. The
economic cost that the EMPC works to maximize over the prediction horizon is
\[
L(x; u) = \int_0^1 r(z; t) \, dz \quad (30)
\]
where
\[
r(z; t) = B_C \exp \left( \frac{\gamma \bar{x}_1}{1 + \bar{x}_1} \right) (1 + \bar{x}_2)^2 \quad (31)
\]
is the reaction rate.

Regarding input and state constraints, the manipulated input, \( u \) is subject to constraints as follows: \(-1 \leq u \leq 1\). Owing to economic considerations, the amount of reactant material available over the period \( t_f \) is fixed. Specifically, the input trajectory should satisfy:
\[
\frac{1}{t_f} \int_0^{t_f} u(\tau) \, d\tau = 0.5 \quad (32)
\]
where \( t_f = 1.0 \). To simplify the notation, we use the notation \( u \in g(t_k) \) to denote this constraint. Constraints on the minimum and maximum temperatures along the length of the reactor are considered as state constraints. Namely, the temperature along the length of the reactor must satisfy the following inequalities:
\[
x_{1,\text{min}} \leq \bar{x}_1(z; t) \leq x_{1,\text{max}} \quad (33)
\]
for all \( z \in [0, 1] \) where \( x_{1,\text{min}} = -1 \) and \( x_{1,\text{max}} = 3 \) are the lower and upper limits, respectively.

Based on the above settings, a high-order output feedback EMPC system of the form Eq. 28 is formulated for the tubular reactor with the economic cost function of Eq. 30, the input constraint of Eq. 32 and the state constraint of Eq. 33 and has the form:
\[
\begin{align*}
\text{MIN} \quad 1 & \quad \frac{1}{N\Delta} \int_{t_k}^{t_k+N} \left( \int_0^1 r(z, \tau) \, dz \right) \, d\tau \quad (34a) \\
\text{s.t.} \quad & \quad \hat{a}_{s,i}(t) = A_{s,i} \hat{a}_{s,i}(t) + F_{s,i}(\hat{a}_s(t), \hat{a}_f(t)) + B_{s,i} u(t) \quad (34b) \\
& \quad \hat{a}_f(t) = -A_{f,1}^{-1} [B_{f,1} u(t) + F_{f,1}(\hat{a}_s(t), 0)] \quad (34c) \\
& \quad \hat{a}_{s,i}(t_k) = C_{s,i}^{-1} y_i(t_k), \quad i = 1, 2 \quad (34d) \\
& \quad -1 \leq \sum_{j=1}^{1} \hat{a}_{1j}(t) \hat{p}_{1j}(z) \leq 3 \quad (34e) \\
& \quad -1 \leq u(t) \leq 1, \quad \forall t \in [t_k, t_{k+N}) \quad (34f) \\
& \quad u(t) \in g(t_k) \quad (34g) \\
& \quad \hat{a}(t) P \bar{a}(t) \leq \bar{p} \quad (34h)
\end{align*}
\]
where \( \hat{a}_s \) is the predicted temporal evolution of the amplitudes of the slow modes, \( \hat{a}_f \) is the predicted temporal evolution of the amplitudes of the fast modes, and \( \hat{a} \) is a vector consisting of both \( \hat{a}_s \) and \( \hat{a}_f \).

C. Low-order EMPC System With Both State and Input Constraints

In this set of simulations, a low-order output feedback EMPC system is formulated by neglecting the fast modes and
\[
\begin{align*}
\text{MAX} \quad \frac{1}{N\Delta} \int_0^1 x_{1,\text{max}} \, dz \quad (35a) \\
\text{s.t.} \quad & \quad \hat{a}_s\bar{z}(t) = A_{s,i} \hat{a}_s(t) + F_{s,i}(\hat{a}_s(t), 0) + B_{s,i} u(t) \quad (35b) \\
& \quad \hat{a}_s(t_k) = C_{s,i}^{-1} y_i(t_k), \quad i = 1, 2 \quad (35c) \\
& \quad -1 \leq u(t) \leq 1, \quad \forall t \in [t_k, t_{k+N}) \quad (35d) \\
& \quad u(t) \in g(t_k) \quad (35e) \\
& \quad \hat{a}(t) P \bar{a}(t) \leq \bar{p} \quad (35f)
\end{align*}
\]
where \( \bar{a}_s \) is the predicted temporal evolution of the amplitudes of the slow modes, \( \bar{a}_f \) is the predicted temporal evolution of the amplitudes of the fast modes, and \( \bar{a} \) is a vector consisting of both \( \bar{a}_s \) and \( \bar{a}_f \).

\( a_f \) in the Eq. 34. The prediction horizon and sampling time of the EMPC are \( N = 3 \) and \( \Delta = 0.01 \). The tubular reactor is initialized with a transient state profile (i.e., not the steady-state profile corresponding to the steady-state input \( u_s = 0.5 \)).

For the reactor, two EMPC systems are formulated with the following low-order model of the PDE system:

1. The low-order model based on 11 slow modes only (i.e., \( m_1 = m_2 = 11 \)).
2. The low-order model based on 21 slow modes only (i.e., \( m_1 = m_2 = 21 \)).

where the measured output points consists of the state measurements at \( m_1 = 11 \) or \( m_1 = 21 \) points which are evenly spaced in the spatial domain. Additionally, the reactor under uniform in time distribution of the reagent material over \( t_f = 1.0 \) is also considered for comparison purposes.

The closed-loop state profiles of the reactor over one period \( t_f = 1.0 \) under the output feedback EMPC formulated with the low-order model based on 21 slow modes is displayed in Figs. 1-2. The computed manipulated input profiles from the low-order output feedback EMPC systems formulated based on 11 and 21 slow modes, respectively, over one period are shown in Fig. 3. From Fig. 3, the output feedback EMPC system based on 21 slow modes computes a smoother manipulated input profile than that of the output feedback EMPC system based on 11 slow modes owing to the increased accuracy of the reduced-order model with 21 slow modes compared to the accuracy of the reduced-order.
is 21 process under the EMPC system based on the low-order output feedback EMPC systems of Eq. 34 based on a systems of Eq. 34 based on Fig. 3. Manipulated input profiles of the low-order output feedback EMPC system economic model predictive control (EMPC) system. Over one period accurately compute the state profile, the output feedback low-order model based on 21 slow modes is able to more of the error associated with the low-order models. Since the operating strategy is to operate the reactor at the maximum temperature profiles of the tubular reactor under the EMPC systems are shown in Fig. 4. Since the temperature directly influences the reaction rate, the optimal operating strategy is to operate the reactor at the maximum allowable temperature. From Fig. 4, both EMPC systems operate the tubular reactor with a maximum temperature less than the maximum allowable which is a consequence of the error associated with the low-order models. Since the low-order model based on 21 slow modes is able to more accurately compute the state profile, the output feedback EMPC system formulated with this low-order model operates the reactor at a greater temperature than the other EMPC system. Over one period $t_f = 1$, the total reaction rate of the process under the EMPC system based on 21 slow modes is 5.13% greater than that of EMPC system based on 11 slow modes and 8.45% greater than that of the system under uniform in time distribution of the reactant material.

Remark 2: More simulation results including applying a high-order EMPC (i.e., retaining the fast modes $\alpha_t$ in the reduced-order model) and considering the effect of measurement noise can be found in [16].

V. CONCLUSION

In this work, we proposed an output feedback EMPC system economic model predictive control (EMPC) system for parabolic partial differential equation (PDE) systems that accounts for both manipulated input and state constraints. The EMPC system is applied to a non-isothermal tubular reactor. Two EMPC systems, each utilizing a different number of measurement sensors and formulated with various degrees of accuracy (i.e., number of modes retained from the infinite-dimensional model), are presented and compared.

REFERENCES