Rotation-Matrix-Based Attitude Control Without Angular Velocity Measurements

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Abstract—Rigid-body attitude control without angular velocity measurements is considered. The error rotation matrix describing the attitude error is used directly within the control algorithm. A controller composed of a proportional control term and an angular velocity control term is first considered. The proportional control term is a function of the error rotation matrix, while the angular velocity control term is realized by a strictly positive real system. Asymptotic stability of a desired closed-loop equilibrium point is shown, and a derivative free implementation of the controller is identified for use when angular velocity measurements are not available. Numerical simulation results demonstrate successful set-point regulation and noise mitigation.

Keywords: attitude control; rotation matrix; passivity-based control.

I. INTRODUCTION

Rigid-body attitude control is the control of the orientation and angular velocity of a rigid body. It is integral to the proper operation of robotic vehicles, such as spacecraft and underwater vehicles. Without a well designed attitude control system, mission objectives may not be met.

A rotation matrix uniquely and globally describes the orientation of a rigid body. Rotation matrices constitute the special orthogonal group, denoted $SO(3)$, the set of all orthonormal matrices with determinant equal to $+1$ [1], [2]. Historically, using a parameterization of the rotation matrix rather than the rotation matrix itself within a control algorithm has been favored. Common parameterizations are Rodrigues parameters, modified Rodrigues parameters (MRPs), and quaternions. All three-parameter parameterizations, such as Rodrigues parameters and MRPs, suffer from singularities, while some continuous feedback control laws that use quaternions, a parameterization consisting of four parameters, suffer from the unwinding phenomena [3]. In an effort to avoid parameterizations [4]–[7] have considered attitude control using the rotation matrix directly.

Typically, angular velocity measurements are needed for attitude control, however these measurements may not always be available. In [8], a passivity-based approach is taken to construct a angular-velocity-free attitude controller. Proportional control is realized by linear feedback of the error quaternion with constant gain, while the derivative control is realized by a strictly positive real (SPR) dynamic compensator that acts on the error quaternion.

Motivated by [4]–[7], this paper builds upon the results of [8] by considering an angular velocity free control implementation using the error rotation matrix directly. The result, which is the novel contribution of this work, is an attitude control law composed of a proportional control term and an SPR dynamic compensator, both based on the error rotation matrix. Like [8] an angular velocity free form of the controller is identified.

The remainder of this paper is organized as follows. Attitude kinematics and dynamics, as well as SPR systems, are reviewed in Sec. II. In Sec. III, the control structure is presented and asymptotic stability is proven. Two numerical examples are given in Sec. IV and closing remarks are given in Sec. V.

II. PRELIMINARIES

Let $F_a$ and $F_b$ denote the inertial and body frames of a rigid body such as a spacecraft or underwater vehicle, where the body frame is located at the center of mass of the rigid body. The rotation matrix $C_{ba} \in SO(3)$, where $SO(3) = \{ C \in \mathbb{R}^{3 \times 3} \mid C^T C = I, \det C = +1 \}$ is the special orthogonal group, uniquely and globally describes the attitude of $F_b$ relative to $F_a$ [1], [2].

The kinematics of a rigid body are given by Poisson’s equation [1], [9],

$$\dot{C}_{ba} + \omega \times C_{ba} = 0,$$

where $\omega$ is the angular velocity of $F_b$ relative to $F_a$, and the “cross” map $(\cdot) \times : \mathbb{R}^3 \rightarrow so(3)$ transforms a $3 \times 1$ column matrix to a $3 \times 3$ skewsymmetric matrix:

$$\nu^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}, \quad \forall \nu = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3,$$

where the vector space of all $3 \times 3$ skewsymmetric matrices is $so(3) = \{ S \in \mathbb{R}^{3 \times 3} \mid S^T = -S \}$ [1], [2].
The dynamics governing the motion of a rigid body are given by Euler’s equation,
\[ I\dot{\omega} + \omega \times I\omega = \tau_c, \tag{2} \]
where \( I = I^T > 0 \) is the moment of inertia, \( \tau_c \) is the control torque, and all quantities have been expressed in \( \mathcal{F}_b \).

The “uncross” map \((\cdot)^\# : \mathfrak{so}(3) \rightarrow \mathbb{R}^3 \) transforms a \( 3 \times 3 \) skewsymmetric matrix into \( 3 \times 1 \) column matrix such that \( \mathbf{v}^\times = \mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^3 \). A useful identity that employs both \((\cdot)^\times \) and \((\cdot)^\# \) is
\[ \frac{1}{2} \tr (\mathbf{v}^\times \mathbf{U}) = -\mathbf{U}^T \mathcal{P}_a(\mathbf{U})^\#, \forall \mathbf{v} \in \mathbb{R}^3, \forall \mathbf{U} \in \mathbb{R}^{3 \times 3}, \tag{3} \]
where \( \mathcal{P}_a(\mathbf{U}) = \frac{1}{2}(\mathbf{U} - \mathbf{U}^T) \) is the skewsymmetric projection operator.

**Lemma 1 (Kalman-Yakubovich-Popov (KYP) Lemma) [10].** Consider the LTI system
\[ \dot{x}_c = A_c x_c + B_c u_c, \quad y_c = C_c x_c, \tag{4} \]
where \( x_c \in \mathbb{R}^{n_c}, u_c, y_c \in \mathbb{R}^{m_c}, \) the matrices \( A_c, B_c, \) and \( C_c \) are appropriately dimensioned real matrices that form a minimal state-space realization, and \( A_c \) is Hurwitz. The system is SPR if and only if there exists \( P_c \in \mathbb{R}^{n_c \times n_c} \) and \( Q_c \in \mathbb{R}^{n_c \times n_c} \) where \( P_c = P_c^T > 0 \) and \( Q_c = Q_c^T > 0 \) such that
\[ P_c A_c + A_c^T P_c = -Q_c, \tag{5a} \]
\[ P_c B_c = C_c^T. \tag{5b} \]

### III. Control Formulation

This paper will focus on regulation of a rigid body’s attitude to a constant desired set-point. Let \( C_{da} \) be the rotation matrix describing the desired attitude. The attitude-error rotation matrix, simply called the error rotation matrix, is
\[ E = C_{bd} = C_{ba} C_{da} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}, \]
where \( e_i, i = 1, 2, 3 \) are the columns of \( E \). The attitude-error kinematic equation
\[ \dot{E} + \omega^\times E = 0 \tag{6} \]
is found by differentiating \( E \) with respect to time and substituting (1). Equation (6) can equivalently be written as
\[ \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} -\omega^\times e_1 \\ -\omega^\times e_2 \\ -\omega^\times e_3 \end{bmatrix} \begin{bmatrix} e_1^\times \\ e_2^\times \\ e_3^\times \end{bmatrix} \omega. \tag{7} \]

Consider the following control law:
\[ \tau_c = k \mathcal{P}_a(E)^\# + u, \tag{8} \]
where \( 0 < k < \infty, k \mathcal{P}_a(E)^\# \) is the proportional control term based on orientation error [4], and \( u \) is the angular velocity control term. In this paper \( u = -\frac{1}{2} \Gamma^T y_c \), where \( y_c \) is the output of the SPR controller
\[ x_c = A_c x_c + B_c \dot{e}, \quad y_c = C_c x_c, \]
with input \( u_c = \dot{e}_c = \Gamma \omega \). Note that the dimensions of \( B_c \) and \( C_c \) are \( n_c \times 9 \) and \( 9 \times n_c \), respectively. The control law in (8) is therefore
\[ \tau_c = k \mathcal{P}_a(E)^\# - \frac{1}{2} \Gamma^T y_c. \tag{9} \]

**Theorem 1.** Consider a rigid body described by the dynamics of (2). The attitude-error kinematics are given by (6) where \( E = C_{bd} = C_{ba} C_{da} \) and \( C_{da} \) is the rotation matrix associated with the desired and constant attitude. The control law is given by (9) where \( y_c = G_c u_c, u_c = \dot{e}_c = \Gamma \omega \). The operator \( G_c \) is realized by the SPR transfer matrix \( G_c(s) = C_c(sI - A_c)^{-1} B_c \) where \( A_c \) is Hurwitz, \( B_c \) has full column rank, and \( (A_c, B_c, C_c) \) is a minimal state-space realization satisfying the KYP Lemma. If the initial conditions are such that
\[ V(0) = \frac{1}{2} \omega^T (0) \Iomega(0) + \frac{k}{2} \tr (1 - E(0)) + \frac{1}{2} x_c^T(0)P_c x_c(0) < 2k, \tag{10} \]
then the equilibrium point \((E, \omega, x_c) = (1, 0, 0)\) of the closed-loop system is asymptotically stable.

**Proof** Consider the following positive definite Lyapunov function candidate:
\[ V = \frac{1}{2} \omega^T I \omega + \frac{k}{2} \tr (1 - E) + \frac{1}{2} x_c^T P_c x_c. \]

Taking the derivative and simplifying using (2), (3), (6), (9), and the KYP Lemma gives
\[ \dot{V} = \omega^T (-\omega^\times I \omega + \tau_c) - \frac{k}{2} \tr (\dot{E}) + \frac{1}{4} x_c^T (P_c A_c + A_c^T P_c) x_c + \frac{1}{2} x_c^T P_c B_c \dot{e} \]
\[ = \omega^T (k \mathcal{P}_a(E)^\# - \frac{1}{2} \Gamma^T y_c) - \frac{k}{2} \tr (-\omega^\times E) - \frac{1}{4} x_c^T Q_c x_c + \frac{1}{2} x_c^T C_c^T \Gamma \omega \]
\[ = k \omega^T \mathcal{P}_a(E)^\# - \frac{1}{2} \omega^T \Gamma^T y_c - k \omega^T \Gamma^T \mathcal{P}_a(E)^\# + \frac{1}{2} x_c^T Q_c x_c + \frac{1}{2} y_c^T C_c^T \Gamma \omega \]
\[ = -\frac{1}{4} x_c^T Q_c x_c, \tag{11} \]
which implies \( V(t) \leq V(0) \) for all \( t > 0 \). Combining this result with (10) it can be seen that \( \frac{1}{2} \tr (1 - E(t)) \leq V(t) \leq V(0) < 2k \). On route to showing asymptotic stability consider the following. First, let \( \mathcal{E} = \{(E, \omega, x_c) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^{n_c} \mid \dot{V} = 0\} \). Owing to the fact that \( Q_c = Q_c^T > 0 \), \( \dot{V} = 0 \) only if \( x_c = 0 \), which in turn implies \( \dot{x}_c = 0 \) and \( u = 0 \) (as \( -\frac{1}{2} \Gamma^T y_c \)). With \( x_c = \dot{x}_c = 0 \), it follows that \( B_c \dot{e} = 0 \), and because \( B_c \) has full column rank this implies \( \dot{e} = 0 \). If \( \dot{e} = 0 \) then \( \dot{E} = 0 \), and from (6) it follows that
$\omega \times E = 0$. This in turn implies $\omega = 0$. From $\omega = 0$ we have $\dot{\omega} = 0$. Additionally, from (2), (9), and (10), $E = 1$. Therefore, the largest invariant set in $\mathcal{E}$ is the set $\mathcal{M} = \{ (E, \omega, x_c) \in \mathcal{E} \mid E = 1, \omega = 0, x_c = 0 \}$. From LaSalle’s invariance principle it follows that trajectories of the closed-loop system asymptotically approach $\mathcal{M}$, that is, $(E, \omega, x_c) \to (1, 0, 0)$ as $t \to \infty$. \hfill \square

From [8], a derivative free implementation of the control law can be identified. Note that $y_c = \mathcal{G}_c \dot{e}$ can be written

$$y_c(s) = sG_c(s)e(s).$$

As such, rather than using (4) with $u_c = \dot{e} = p(\omega)$, the controller

$$\begin{align*}
x_{c'} &= A_c x_{c'} + B_c e, \\
y_c &= C_c x_{c'},
\end{align*}$$

$$\begin{align*}
&= B_c^T P_c (A_c x_{c'} + B_c e) \\
&= B_c^T P_c A_c x_{c'} + B_c^T P_c B_c e,
\end{align*}$$

(12)

with input $e$ can be implemented without knowledge of $\omega$ or $\dot{e}$.

IV. Numerical Example

The control law shown in (9) and (12) will now be demonstrated in the context of spacecraft attitude control. Two simulations are presented and the following parameters are common to both. Consider a rigid-body spacecraft in a circular orbit around the Earth at an altitude of 450 $(km)$ and an inclination of $87^\circ$ with moment of inertia matrix

$$I = \text{diag} \{17, 10, 15\} \ (kg \cdot m^2).$$

The argument of perigee, time of perigee passage, and the angle of the ascending node are zero. The initial attitude of the spacecraft is described by Euler axis/angle parameters where

$$\begin{align*}
a(0) &= 1/\sqrt{14} \ [2 \ 1 \ -3]^T, \\
\phi(0) &= \pi/4 \ (rad),
\end{align*}$$

and the initial angular velocity is

$$\omega(0) = [0.02 \ 0.01 \ -0.0075]^T. \ A \ residual \ magnetic \ disturbance \ torque \ as \ well \ as \ a \ gravity \ gradient \ torque \ act \ as \ disturbances. \ \ The \ residual \ magnetic \ disturbance \ torque \ is \ \tau_m = m_s b, \ where \ \ m_s = [0.1 \ 0.1 \ 0.1]^T \ (A \cdot m^2) \ is \ the \ residual \ magnetic \ dipole \ and \ b \in \mathbb{R}^3 \ is \ the \ Earth’s \ magnetic \ field \ vector \ expressed \ in \ the \ spacecraft \ body \ frame \ [9]. \ \ The \ gravity \ gradient \ torque \ is \ given \ by \ \tau_g = (3\mu/r^5) r^2 r, \ where \ \mu \ is \ the \ Earth’s \ gravitational \ constant, \ r \ is \ the \ spacecraft \ position \ expressed \ in \ the \ spacecraft \ body \ frame, \ and \ r = \sqrt{r^2}. \ \ The \ desired \ attitude \ is \ set \ as \ C_{da} = 1.$$

The proportional gain is set to $k_p = 0.03 \ (N \cdot m)$ and the SPR controller state-space matrices, $(A_c, B_c, C_c)$, are chosen as in [8]. Briefly, $A_c, B_c$, and $Q_c$ are selected as

$$\begin{align*}
-A_c &= B_c = 0.11, \\
Q_c &= -k_d (B_c^{-1} A_c + B_c^{-1} A_c)^T,
\end{align*}$$

where $k_d$ is a constant such that $0 < k_d < \infty$. Matrices $P_c$ and $C_c$ are found from (5) as

$$P_c = k_d B_c^{-1}$$

and

$$C_c = k_d I.$$

In the following simulations, the derivative gain $k_d$ is set to $k_d = 0.2 \ (N \cdot m)$. Note that the above parameters along with the initial conditions satisfy (10).

![Fig. 1: $G_c(s)$ frequency response.](image)

The frequency response of the controller, $G_c(j\omega)$, is shown in Fig. 1. The maximum singular value of $G_c(j\omega)$ is $\sqrt{\lambda\{G_c^T (−j\omega) G_c(j\omega)\}}$, where $\lambda(\cdot)$ is the maximum eigenvalue. The minimum Hermitian part of $G_c(j\omega)$ is $\frac{1}{2} \lambda\{G_c^T (−j\omega) + G_c(j\omega)\}$ where $\lambda(\cdot)$ is the minimum eigenvalue. The maximum singular value is representative of the gain of the controller, while the minimum Hermitian part must be strictly positive for an SPR controller. Thus, the plots of the maximum singular value and the minimum Hermitian part provide a graphical measure of the gain and the SPR nature of $G_c(j\omega)$ [11]. If $\dot{e}$ is available, then $G_c(s)$ can be used directly. In this case, noise with frequency higher than 0.1 $(rad/s)$ will be rejected. In both simulations that follow $\dot{e}$ is assumed to be unavailable and the derivative free implementation of the controller, $s G_c(s)$, is utilized. The frequency response of $s G_c(s)$ is shown in Fig. 2. Although $s G_c(s)$ does not roll-off, the high frequency gain has been limited thus preventing significant amplification of high frequency noise. Thus, although noise is not completely rejected at high frequency, noise is mitigated. The high frequency gain of the controller can be modified by changing the parameters of the SPR controller.
In the first simulation the error rotation matrix is assumed to be measured exactly. The attitude error, the norm of angular velocity, and the norm of the control torques are shown in Fig. 3. The attitude error is shown in terms of the axis-angle parameter $\phi$, extracted from $E$. Clearly, the spacecraft is successfully controlled and the attitude error goes to zero.

The second simulation considers a more realistic scenario in that the error rotation matrix must be constructed from noisy vector measurements. Here, the TRIAD algorithm [12] is used to estimate $E$. Let $y_0^1 = [1 \ 0 \ 0]^T$ and $y_0^2 = [0 \ 1 \ 0]^T$ be reference vectors in $F_a$. These vectors are measured in the body frame according to, $y_b^j = C_{\nu_b} C_{\nu_a} y_a^j$, $j = 1, 2$, where $C_{\nu_b}$ are rotation matrices associated with the measurement noise. Written explicitly, $C_{\nu_b} = C_3(\nu_{3,3}\sin(\nu_{p,3}))/C_2(\nu_{2,3}\cos(\nu_{p,3}))C_1(\nu_{1,3}\sin(\nu_{p,3}))$, where $C(i), i = 1, 2, 3$ are principal rotations [9]. For $C_{\nu_b}$, $\nu_{p,1} = 1 (s^{-1})$, $\nu_{1,1} = 1/4^\circ$, $\nu_{2,1} = 1/3^\circ$, and $\nu_{3,1} = 4/5^\circ$. Similarly, for $C_{\nu_b}$, $\nu_{p,2} = 5 (s^{-1})$, $\nu_{1,2} = 1/5^\circ$, $\nu_{2,2} = 1/7^\circ$, and $\nu_{3,2} = 1/9^\circ$. The inertial vectors expressed in the desired frame are $y_d^j = C_{\nu_d} y_a^j$, $j = 1, 2$. Employing the TRIAD algorithm, the error rotation matrix is $E = [y_b^1 \ y_b^2 \ y_b^3 \ y_d^1 \ y_d^2]^T$, where $y_b^1 = y_b^1$.

![Fig. 2: $sG_c(s)$ frequency response.](image)

![Fig. 3: Perfect measurements; attitude error, norm of $\omega$, and norm of $\tau_c$ versus orbit.](image)

The frequency response of $G_c(s)$ is shown in Fig. 4. The controller $G_{c,2}(s)$ is close to pure constant gain derivative control for frequencies less than 100 ($rad/s$). It should be noted that $G_{c,2}(s)$ can be realized by a transfer matrix, $G_{c,2}(s) = C_{c,2}(sI - A_{c,2})^{-1}B_{c,2}$, where $A_{c,2} = -B_{c,2} = -100I$, and $C_{c,2} = k_dI$. It is simple to show that $G_{c,2}(s)$ is SPR by using (5) with $Q_{c,2} = 2k_d I$. 

$$
\begin{align*}
\nu_b^2 &= \nu_b^1 x_b^2/|\nu_b^1 x_b^2|, \\
\nu_d^2 &= \nu_d^1 x_d^2/|\nu_d^1 x_d^2|, \\
\nu_d^3 &= \nu_d^1 x_d^2.
\end{align*}
$$

Referring to Fig. 2 and the values of $\nu_{p,1}$ and $\nu_{p,2}$, the noise is not fully rejected by the controller, but because the gain is limited to about $-30$ (dB) the noise is significantly mitigated.

Ideally, to emphasize the design freedom and noise mitigation properties the SPR controller affords, (9) should be compared to a controller with constant gain derivative control, such as $\tau_c = k_p E|\dot{E}^T - k_d \omega$ where $\omega$ is corrupted by noise. However, this is not a valid comparison because (12) acts to mitigate the noise in the measurement of $E$ and not $\omega$. Therefore, in place of constant gain derivative control, the controller shown in (9) is used for comparison where $y_c$ is calculated as

$$
\begin{align*}
y_c(s) &= k_d \frac{100}{s + 100} \dot{e}(s) \\
&= G_{c,2}(s) \dot{e}(s) \\
&= s G_{c,2}(s) e(s).
\end{align*}
$$

The frequency response of $G_{c,2}(s)$ is shown in Fig. 4. The controller $G_{c,2}(s)$ is close to pure constant gain derivative control for frequencies less than 100 ($rad/s$). It should be noted that $G_{c,2}(s)$ can be realized by a transfer matrix, $G_{c,2}(s) = C_{c,2}(sI - A_{c,2})^{-1}B_{c,2}$, where $A_{c,2} = -B_{c,2} = -100I$, and $C_{c,2} = k_dI$. It is simple to show that $G_{c,2}(s)$ is SPR by using (5) with $Q_{c,2} = 2k_d I$. 

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and solving for $P_{c,2} = k_d/1001$. Thus, the stability proof shown in Sec. III holds and a derivative free implementation, $sG_{c,2}(s)$, can be constructed as in (12). The frequency response of $sG_{c,2}(s)$ is shown in Fig. 5. Note that the high frequency gain for $sG_{c,2}(s)$ is much higher than for $sG_{c}(s)$. Thus, the results presented here represent a comparison of the derivative free SPR controller when the controller parameters have been chosen well in terms of noise mitigation, as in $sG_{c}(s)$, and when they have been chosen poorly, as in $sG_{c,2}(s)$.

The closed-loop responses using both $sG_{c}(s)$ and $sG_{c,2}(s)$ are shown in Fig. 6. All control gains and initial conditions remain identical for both cases. The overall attitude error and angular velocity response of both controllers are very similar. The norm of $\tau_c$ when $sG_{c}(s)$ is used is significantly less than when $sG_{c,2}(s)$ is used. This is because $sG_{c}(s)$ is more successful in mitigating the noise in the measurement of $E$ than $sG_{c,2}$ due to its lower high frequency gain. The main advantage of the SPR controller is that the effective angular velocity control can be tuned over a frequency range to allow the mitigation of noise and disturbances, as was demonstrated in this example.

V. CLOSING REMARKS

This paper has investigated rigid-body attitude control. A derivative free controller based on the error rotation matrix was developed. Constant gain feedback using the error rotation matrix serves as proportional control, while derivative control is realized by an SPR system. The
The equilibrium point associated with the closed-loop system is shown to be asymptotically stable provided initial conditions are bounded by a constant related to the proportional gain. Two numerical simulations are presented. The first demonstrates successful set-point regulation to a desired attitude when using perfect measurements of the error rotation matrix. The second considers a realistic scenario where the error rotation matrix must first be estimated from two noisy vector measurements. It was shown that the controller successfully mitigates noise. As mentioned previously, the main contribution of this work is the form of the controller which permits rigid-body attitude control using only the rotation matrix and without angular velocity measurements.

REFERENCES