A Continuous Robust Control Design for a Class of Non-Affine Nonlinear Dynamics with Non-vanishing Disturbance

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Abstract—In this paper, the problem for the control of a class of single-input-single-output (SISO) non-affine nonlinear system with non-vanishing disturbance is investigated. A continuous nonlinear feedback structure is utilized to tackle with the uncertain dynamics in the system. By taking the time derivative of the origin system, a transformed affine-like form is derived. The first order derivative of control input appears linearly in the augmented dynamics with unknown control direction. Nussbaum-type function is incorporated to estimate the unknown control direction. A revised Lyapunov based analysis is carried out to prove that under some moderate assumptions, the Semi-Global Uniformly Ultimately Bounded (SGUUB) tracking result is achieved and all the closed loop signals are bounded. Numerical simulation results are presented to illustrate the performance of the proposed control law.

I. INTRODUCTION

The robust of the integral of the sign of the error (RISE) feedback control design has been widely employed in many practical areas [1]-[5] after being proposed in [6]. It provides a convenient method to achieve asymptotic tracking result due to its simple structure and less dependence on the knowledge of the system’s dynamics. Combined with different control strategies (i.e. adaptive methodology, neural network, optimal control, predictor-based control) [7]-[9], RISE method is extended to the control of more general nonlinear systems. In [8], RISE method is proved to be robust with unknown time delay in the system. In [9], RISE method combined with online estimating strategy is incorporated on a class of system with 2-dimension non-symmetric control matrix. To our best knowledge, RISE method has never been used in the control for the non-affine-in-control system.

Non-affine system is a class of system whose control input appears in the system dynamics as one variable of a nonlinear function or is coupled with the system states in a nonlinear fashion. A lot of practical systems are non-affine in control inputs, i.e., pH neutralization, magnetic levitation systems etc. The non-affine property brings great challenges for researchers to develop control laws due to the following reasons. First of all, the mathematical correlation between the control input and the system states can not be solved explicitly. Secondly, the direction of the control input is often unknown. Numerous work have been devoted to this challenge area in recent years [10]-[13]. In [12], a fast dynamic updating law for the control input is presented to search a dynamic inversion stabilized solution under the assumption that the system dynamics is exactly known. In [13], three control strategies are considered for a second-order non-affine system and global asymptotic stable is obtained. However, all the aforementioned papers accept the fact that the control direction is known a prior.

If the knowledge about the sign of the control direction is absent, it makes the control design much more difficult. A high frequency switching function (i.e. direction), called Nussbaum-type function, is developed first to solving the problem of unknown direction in the adaptive control area [14]. Later, this method is extended to nonlinear systems [15] and non-affine systems [16][17]. In recent ten years, a lot of work has been carried on the expansion for the application of Nussbaum-type function to different non-affine nonlinear systems. In [18], a class of generalized Nussbaum-type function is defined and broaden the key lemma with a wide range of the control coefficient (from \([I^-1, I^+]\) to \([-\infty, -I])\).

In this paper, we combine the Nussbaum-type function into the RISE feedback method for the control problem of the SISO nonlinear non-affine system suffering non-vanishing disturbance. After taking the time derivative of the origin system, the control input appears in the dynamics linearly in its first-order derivative form. Unlike using the Taylor expansion method [17] or the mean-value theorem [19] to transfer the non-affine system into the affine-like form, the augmented system doesn’t need the trim value for the control input or to design an additional term to compensate the unexpected high-order term. Motivated by the previous work [16][18], we combine the RISE method with a Nussbaum-type function to estimate the unknown control direction. Besides, we develop a new Lyapunov function candidate different from the one in [6] to accomplish the stability analysis. Unfortunately, the direct integration of Nussbaum-type function and RISE is impossible to be implemented due to the shortness for the measurable system signals. Thus a novel second-order filter is designed which is employed to provide feedback signals for the control implementation.

The paper is organized as follows. Section 2 describes the dynamic model and some assumptions required for the control development. Section 3 presents the preliminaries and main idea of the proposed control law. Section 4 gives the main result of this paper. Simulation results that demonstrate the performance of the proposed controller are presented in Section 5. Conclusion remarks are described in Section 6.

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II. PROBLEM STATEMENT

Consider the following SISO dynamics with unknown external disturbance

\[ \dot{x}(t) = F(x, u) + d(t) \]

where \( x(t) \) denotes the system state, \( u(t) \in \mathbb{R} \) is the control input, \( F(x, u) \) represents a smooth unknown non-affine nonlinear function, \( d(t) \) denotes the unknown non-vanishing disturbance, and \( y(t) \in \mathbb{R} \) is the output to be controlled. The objective is to design a nonlinear robust controller for the non-affine uncertain system in (1) such that the output \( y(t) \) follows a pre-defined trajectory \( y_d(t) \).

Before the main content is presented, we first state several assumptions as follows

**Assumption 1:** The external disturbances \( d(t) \) are second order differentiable, in the sense that

\[ d^{(i)}(t) \in \mathcal{L}_\infty \quad i = 0, 1, 2. \]  

**Assumption 2:** The pre-defined trajectory is designed such that \( y_d^{(i)}(t) \in \mathcal{L}_\infty \), \( i = 0, 1, 2, 3 \).

**Assumption 3:** There exists a positive constant \( g \) such that

\[ 0 < g \leq \left| \frac{\partial F(x, u)}{\partial u} \right| \]

for all \( x \) and \( u \), and the sign of \( \frac{\partial F(x, u)}{\partial u} \) is unknown.

**Remark 1:** It is not difficult to defer from Assumption 3 that the function \( \frac{\partial F(x, u)}{\partial u} \) is either positive or negative and won’t change the sign. And then, by invoking the implicit function theorem, the equation \( \dot{x} = F(x, u) - d = 0 \) can be solved for \( u \) in terms of \( x, \dot{x} - d \), in the sense that \( u = \varphi(x, \dot{x} - d) \). Sometimes the expression of function \( \varphi(\cdot) \) is hard to be explicitly determined. For example, considering the equation \( \dot{x} - x - \cos(u) - u - d = 0 \), it will be very difficult to express \( u \) in terms of \( x \) and \( \dot{x} - d \) explicitly.

**Assumption 4:** If \( x(t), \dot{x}(t), \ddot{x}(t) \in L_\infty \), then \( F(x, \varphi(x, \dot{x} - d)) \) is bounded, the first partial derivative of \( F(x, u) \) with respect to \( x \) exists and is bounded as \( u = \varphi(x, \dot{x} - d) \), and the derivatives of \( \frac{\partial F(x, \varphi(x, \dot{x} - d))}{\partial x} \) exists and is bounded.

III. PRELIMINARIES FOR THE MAIN RESULT

A. Open loop system development

To meet the control objective, the following tracking error, denoted by \( e(t) \in \mathbb{R} \), is defined:

\[ e(t) = y_d(t) - x(t). \]  

An auxiliary filtered error signal \( r(t) \in \mathbb{R} \) is also defined in the following form

\[ r(t) = \dot{e}(t) + \alpha e(t) \]  

where \( \alpha \in \mathbb{R} \) is a positive constant. Since the filtered error vector \( r(t) \) contains the unmeasurable signal \( \dot{x}(t) \), it can not be used in the feedback control design directly. After taking the time derivative of \( r(t) \) and invoking (4), the following equation can be obtained

\[ \dot{r} = \dot{\dot{e}} + \alpha \dot{e} \]

\[ = \ddot{x}_d + \alpha \dot{e} - \left[ \frac{\partial F(x, u)}{\partial x} \right]^T \dot{x} - \frac{\partial F(x, u)}{\partial u} \dot{u} - \dot{d}(t). \]  

From (6), it is easy to see that the derivative of \( u(t) \) appears linearly in the resulting equation, thus \( \dot{u}(t) \) can be chosen as the new auxiliary control input. Unlike using the Taylor series expansion method or the Mean value theorem to deal with the non-affine problem, the augmented dynamics in (6) will not bring the unexpected high-order term and the trim value for \( u(t) \) is not to be set in our control design.

By defining an auxiliary function \( N(\cdot) \), the open loop dynamics of \( r(t) \) in (6) is re-written into a simple form as:

\[ \dot{r} = N - \frac{\partial F(x, u)}{\partial u} \bigg|_{u=\varphi(x, \dot{x} - d)} \dot{u} - e \]  

where the auxiliary function vector \( N(x, \dot{x}, t) \in \mathbb{R} \) represents the uncertain term of the dynamic system, and is defined as follows

\[ N = \ddot{x}_d + \alpha \dot{e} - \left[ \frac{\partial F(x, u)}{\partial x} \bigg|_{u=\varphi(x, \dot{x} - d)} \right]^T \dot{x} - \frac{\partial F(x, u)}{\partial u} \dot{u} - e. \]  

To facilitate the subsequent control design, we perform the following manipulation on (7). Let the auxiliary function \( N_d(t) \) be defined as

\[ N_d = \ddot{x}_d - \left[ \frac{\partial F(x, u)}{\partial x} \bigg|_{u=\varphi(x, \dot{x} - d)} \right]_{x=x_d, \dot{x}=\dot{x}_d}^T \dot{x}_d - \dot{d}(t). \]  

Due to Assumption 1-3, it is not difficult to check that \( N_d(t), \tilde{N}_d(t) \in \mathcal{L}_\infty \). After adding and subtracting \( N_d(t) \) to the right side of (7), the following open loop dynamics of \( r(t) \) can be obtained

\[ \dot{r} = N_d + \tilde{N} - \frac{\partial F(x, u)}{\partial u} \dot{u} - e \]  

where the auxiliary function \( \tilde{N}(t) \) is defined as:

\[ \tilde{N}(x, \dot{x}, t) = N - N_d. \]  

Since \( \tilde{N}(x, \dot{x}, t) \) is continuously differentiable with respect to \( x \) and \( \dot{x} \), we can show that \( \tilde{N}(t) \) can be upper bounded by the following inequality [6]

\[ \left| \tilde{N} \right| \leq \rho(\|z\|) \|z\| \]  

where \( z(t) \in \mathbb{R}^2 \) is defined as

\[ z(t) = [ e \quad r ]^T \]  

and \( \rho(\cdot) \) is a global invertible, non-decreasing function.
B. Nussbaum-type function

By utilizing the recent results of Nussbaum-type function [18], we can cope with the problem for the unknown direction of input coefficient \( \frac{\partial F(x,u)}{\partial u} \) in (10). A function \( N(\cdot) \) is named Nussbaum-type function, if it has the following properties [18]:

**Property 1:**

\[
\lim_{s \to +\infty} \sup_{\xi \in \mathbb{R}} \frac{1}{s} \int_{s}^{\infty} N(\xi) d\xi = +\infty \quad \text{and} \quad \lim_{s \to +\infty} \inf_{\xi \in \mathbb{R}} \frac{1}{s} \int_{s}^{\infty} N(\xi) d\xi = -\infty. \quad (14)
\]

For example, \( e^{2} \cos(\xi) \) and \( \xi^{2} \cos(\xi) \) are even Nussbaum-type functions.

**Property 2:** For any function \( \tau(\cdot) \) satisfying that \( \tau(\cdot) \in [\varepsilon_{0}, +\infty) \) or \( \tau(\cdot) \in (-\infty, -\varepsilon_{0}] \) in which \( \varepsilon_{0} \) is a positive constant, \( \tau(\cdot)N(\xi) \) is also a Nussbaum-type function.

**Property 3:** For any integrable function \( \varpi(\cdot) \) satisfying \( \varpi(\cdot) \in [-\varepsilon_{1}, \varepsilon_{1}] \) in which \( \varepsilon_{1} \) is a positive constant, \( N(\xi) + \varpi(\cdot) \) is a Nussbaum-type function.

To facilitate the stability analysis, the following lemma about Nussbaum-type function will be invoked.

**Lemma 1:** Let \( V(t) \) be a differentiable function, if \( V(t) \) satisfies

\[
\dot{V} \leq -c_{0}V + N(\xi)\xi \quad (15)
\]

then \( V(t) \) must be SGUUB in finite time and \( \int_{t_{0}}^{t} N(\xi)e^{-(t-r)}dr \) is bounded by some constant. The proof of Property 2-3 and Lemma 1 can be seen in [18].

C. A continuous nonlinear feedback design and stability analysis

In this section, the continuous nonlinear robust control in our previous work [6] is combined with the Nussbaum-type function to deal with the uncertain dynamics and the unknown control direction, and a refined Lyapunov based analysis is addressed in Theorem 1.

Similar to the approaches in [16] and [17], the control input is composed of a Nussbaum-type function which is able to estimate the unknown control direction. The derivative of the control input \( \dot{u}(t) \) is designed as follows:

\[
\dot{u} = N(\xi)(-(k+1)r - \beta \text{sgn}(\xi)) \quad (16)
\]

where \( k \) and \( \beta \) are the positive control gains, \( \text{sgn}(\cdot) \) is the standard sign function, and \( N(\xi) = e^{2} \cos(\xi) \) is a Nussbaum-type function where \( \xi(t) \) is generated via

\[
\xi = \gamma[(k+1)r + \beta \text{sgn}(\xi)]r. \quad (17)
\]

After substituting (16) into the open-loop dynamics of \( r(t) \) in (10), we have

\[
\dot{r} = N + N_{d} - (k+1)r - \beta \text{sgn}(\xi) + \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1)\xi. \quad (18)
\]

Next, we will state the stability result for the proposed control design. We now give the following theorem.

**Theorem 1:** Consider the control law in (16) and the closed loop dynamics in (18), all closed loop signals are uniformly bounded and the output tracking error \( e(t) \) is SGUUB stable, provided that

\[
\beta > 2(\|N_{d}\|_{\infty} + \frac{1}{\alpha} \|N_{d}\|_{\infty}). \quad (19)
\]

**Proof:** Let the Lyapunov function candidate \( V(t) \) be defined as follows:

\[
V = \frac{1}{2}e^{2} + \frac{1}{2}r^{2} + \beta \int_{0}^{t} \dot{e}(\tau)\text{sgn}(e(\tau))d\tau + \beta |e(0)| - N_{d}e. \quad (20)
\]

Noting that

\[
\beta \int_{0}^{t} \dot{e}(\tau)\text{sgn}(e(\tau))d\tau + \beta |e(0)| = \beta |e(t)|, \quad (21)
\]

it is not difficult to check that \( V \) is a positive definite function w.r.t \( e \) and \( r \). Moreover, \( V(t) \) can be upperbounded by

\[
V \leq \frac{1}{2}e^{2} + \frac{1}{2}r^{2} + (\beta + \|N_{d}\|_{\infty}) |e|. \quad (22)
\]

By using (5) and (18), the time derivative of \( V(t) \) is given as

\[
\dot{V} = r\dot{r} + e\dot{e} + \beta \text{sgn}(e)e - \dot{N}_{d}e - N_{d}\dot{e} = \dot{N}r + \alpha N_{d}e - (k+1)r^{2} - \alpha e^{2} - \dot{N}_{d}e + \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1)\xi \leq \frac{1}{2}(\alpha^{2} - \alpha e^{2} - \alpha\dot{N}_{d} - N_{d}\dot{e}) + \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1)\xi \leq -c_{1} |\xi|^{2} - (c_{1} - \frac{1}{\gamma} \rho^{2}(\xi)) |\xi|^{2} - \frac{1}{2}\alpha\beta |e|^{2} + \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1)\xi \quad (23)
\]

where \( c_{1} = \frac{\gamma}{2} \min\{1, \alpha\} \) denotes a positive constant. If the following condition is satisfied

\[
\frac{1}{2} |\xi|^{2} \leq \rho^{-1}(2\sqrt{c_{1}}), \quad (24)
\]

then (23) can be re-written as:

\[
\dot{V} \leq -\min\{c_{2}, \frac{\alpha\beta}{\gamma\rho^{2}(\xi)}\} V + \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1)\xi \leq -c_{2} V + N^{\prime}(\xi)\xi \quad (25)
\]

where the inequality in (22) is used and the positive constant \( c_{2} \) is defined as \( c_{2} = \min\{2c_{1}, \frac{\alpha\beta}{\gamma\rho^{2}(\xi)}\} \). Note, the function \( N^{\prime}(\xi) = \frac{1}{\gamma} \frac{\partial F(x,u)}{\partial u} (N(\xi) + 1) \) is still a Nussbaum-type function according to Property 2-3. From Lemma 1, the closed loop system is proven to be UUB stable. Besides, the following inequality can be established

\[
\int_{0}^{t} N^{\prime}(\xi)e^{-(t-r)}\xi dr \leq \eta_{0} \quad (26)
\]

where \( \eta_{0} \) is some positive constant. By taking integration of (25) w.r.t time \( t \), the following inequality is obtained

\[
\frac{1}{2} |\xi|^{2} \leq V(t) \leq V(0) + \eta_{0}. \quad (27)
\]

Specifically, we can use the right most inequality in (24) and (27) to calculate the region of attraction as follows:

\[
D = \left\{ |\xi| \leq \sqrt{2(V(0) + \eta_{0})} \right\} \subset \Omega \quad (28)
\]
So if the control gain $k$ is chosen according to the following sufficient condition
\[ k \geq \rho^2(\sqrt{2(V(0) + \eta_0)})/4c_1, \tag{29} \]
then for any initial condition for $z(t)$ in the region $D$, the trajectory of $z(t)$ remain in $\Omega$ for all $t > 0$. Note that the region in (24) can be made arbitrary large to include any initial condition by increasing the control gain $k$, so the semi-global performance is also proved.

Remark 2: According to (23), the uncertain dynamics in (18) is totally compensated by the robust term $\int_0^t \beta \text{sgn}(e(\tau))d\tau$ which is the key component in the proposed structure, the ultimate boundedness of the closed loop signals is aroused by the unknown control direction exclusively. Moreover, the ultimate boundedness can be regulated rather small by tuning the positive gain $\gamma$ in (17).

IV. REVISED CONTROL DESIGN

Unfortunately, since $\dot{z}$ in (17) contains the unmeasured term $\dot{e}^2$, the actual control input $u(t)$ is impossible to be calculated. To overcome this problem, a second-order filter is proposed in this section to provide available signals in the feedback structure. And a similar stability analysis result is given in Theorem 2.

A. Filter design and control development

An auxiliary error $e_1(t) \in \mathbb{R}$ is introduced first as follows:
\[ e_1(t) = e(t) + e_f(t) \tag{30} \]
where $e(t)$ is defined in (4), and the filter $e_f(t) \in \mathbb{R}$ will be designed later. We define a revised filtered tracking error signal in the following form
\[ r_1(t) = \dot{e}_1(t) + \alpha e_1(t). \tag{31} \]

To facilitate the subsequent analysis, the following second-order filter is designed
\[ \dot{e}_f = -\alpha e_f + r_f \]
\[ \dot{r}_f = -r_f - (k + 1)r_1 + e - e_f \tag{32} \]
where $e_f(t), r_f(t) \in \mathbb{R}$ are the filter outputs. After taking the time derivative of (31) and substituting (1) into the resulting equation, we can obtain the open loop system for $r_1(t)$ as
\[ \dot{r}_1 = \ddot{y}_d - \frac{\partial F(x,u)}{\partial x} \bigg|_{u=\phi(x,\dot{x}-d)} \dot{x} - \ddot{d} + \alpha \dot{e} - r_f + 2e - e_f. \tag{33} \]
By using (32), (33) can be re-written as follows:
\[ \dot{r}_1 = -(k + 1)r_1 + N_1 - e - \frac{\partial F(x,u)}{\partial u} \dot{u}. \tag{34} \]
The auxiliary function $N_1(\cdot)$ in (34) is defined as
\[ N_1 = \ddot{y}_d - \frac{\partial F(x,u)}{\partial x} \bigg|_{u=\phi(x,\dot{x}-d)} \dot{x} - \ddot{d} + \alpha \dot{e} - r_f + 2e - e_f. \tag{35} \]
Similar to section 3.3, we design the first order derivative of the control law as follows
\[ \dot{u} = N_1(\dot{c_1})(k + 1)r_f - \beta \text{sgn}(e_1) \tag{36} \]
where $c_1(t) \in \mathbb{R}$ is generated via
\[ \dot{c}_1 = \gamma(-(k + 1)r_f + \beta \text{sgn}(e_1))r_1 \tag{37} \]
By substituting (36) into (34), the following closed loop system for $r_1(t)$ is obtained as
\[ \dot{r}_1 = -(k + 1)r_1 + N_1 + \dot{N} + (k + 1)r_f - \beta \text{sgn}(e_1) - e + \frac{1}{2} [\frac{\partial F(x,u)}{\partial u} + 1] \dot{c}_1/r_1 \tag{38} \]
where the auxiliary function $N_{1d}(t)$ be defined as:
\[ N_{1d} = N_1(x, \dot{x}, e_f, r_f, t) \big|_{x=x_d, \dot{x} = \dot{x}_d, e_f = 0, r_f = 0} \tag{39} \]
and the auxiliary function $\tilde{N}_1(t)$ be defined as:
\[ \tilde{N}_1 = N_1 - N_{1d}. \tag{40} \]
Similarly, $\tilde{N}_1(t)$ can be upperbounded by
\[ \|\tilde{N}_1\| \leq \rho_1(\|z_1\|) \|z_1\| \tag{41} \]
where $\rho_1(\cdot)$ is a global invertible, non-decreasing function, $z_1 = [e, e_f, r_f, r]^T \in \mathbb{R}^4$ denotes the error vector.

B. Stability analysis

Theorem 2: The control law in (36) ensures that all the closed loop system signals are bounded and the tracking error $e(t)$ achieve SGUUB stable under closed loop operation, provided that the control gain $\beta$ is selected as:
\[ \beta > 2(\|N_{1d}\| \infty + 1/\alpha \|\tilde{N}_{1d}\| \infty). \tag{42} \]

Proof: Let the Lyapunov function candidate $V_1(t)$ be defined as:
\[ V_1 = \frac{1}{2} r^2 + \frac{1}{2} e^2 + \frac{1}{2} e_f^2 + \frac{1}{2} r_f^2 + \int_0^t \beta \text{sgn}(e_1(\tau))\dot{c}_1(\tau)d\tau + \beta |e_1(0)| - N_{1d}e_1 \tag{43} \]
Based on the similar procedures in section 3.3, we can show that $V_1(t)$ is positive definite and satisfy the following inequality
\[ \frac{1}{2} \|z_1\|^2 \leq V_1 \leq \frac{1}{2} \|z_1\|^2 + (\beta + \|N_{1d}\| \infty) |e_1|. \tag{44} \]
Taking the time derivative of $V_1(t)$ yields
\[ \dot{V}_1 = -(r^2 - \dot{r_f}^2 - \alpha e^2 - \alpha \dot{e}^2 - k r^2 + \dot{N}_1 r - \alpha \beta |e_1| - \alpha (2/3 \text{sgn}(e_1) + N_{1d} - 1/\alpha \|N_{1d}\| e_1 + 1/\gamma |\frac{\partial F(x,u)}{\partial u} + 1|) \dot{c}_1 \leq - \min\{1, \alpha\} \|z_1\|^2 + \frac{1}{\alpha} (\|z_1\|) \|z_1\|^2 + \frac{1}{2} \alpha \beta |e_1| + \frac{1}{\gamma} |\frac{\partial F(x,u)}{\partial u} + 1| \dot{c}_1 \leq -c_3 V_1 + \frac{1}{\gamma} |\frac{\partial F(x,u)}{\partial u} + 1| \dot{c}_1 \]
for $\|z_1\| \leq \rho_1^{-1}(\sqrt{2 \min\{1, \alpha\}})$
\[ \frac{1}{\gamma} \int_0^t |\frac{\partial F(x,u)}{\partial u} + 1| \dot{c}_1 e^{-(t-\tau)} \dot{c}_1 d\tau \leq \eta_1 \tag{46} \]
where $c_3 = \min\{\min\{1, \alpha\}, \frac{\alpha \beta}{2(\alpha + \|N_{1d}\| \infty)}\}$. Following the similar steps in section 3.3, we can prove that...
with \( \eta_1 \) being a positive constant. By adjusting the control gain \( k \geq \frac{e^2(\sqrt{2(V_x(0)+\eta_1)})}{4e^3} \), all the closed loop signals are bounded and the tracking error is SGUUB stable.

C. Calculation for \( r_f \) and \( \varsigma \)

By integrating the second differential equation in (32), we have

\[
r_f = - \int_0^t r_f d\tau - (k + 1)e_1(t) + (k + 1)e_1(0) - (k + 1) \int_0^t e_1 d\tau + \int_0^t (e - e_f) d\tau.
\]

By integrating the differential equation in (37), we have

\[
\varsigma = -(k + 1) \gamma \int_0^t (\dot{e} + \alpha e + r_f) r_f d\tau + \gamma \int_0^t \delta_0 sgn(e_1)(\dot{e}_1 + \alpha e_1) d\tau + \beta \gamma \int_0^t \delta_0 sgn(e_1)(\dot{e}_1 + \alpha e_1) d\tau
\]

\[
= -(k + 1) \gamma \int_0^t e_1(d\tau) - (k + 1) \alpha \int_0^t e_1 d\tau - (k + 1) \gamma \int_0^t e_1^2 d\tau + \beta \gamma (|e_1(t)| - |e_1(0)|) + \gamma \beta \alpha \int_0^t e_1(t) |d\tau|
\]

where the first term \(-(k + 1) \gamma \int_0^t e_1 d\tau\) can be calculated

\[
-(k + 1) \gamma \int_0^t e_1 d\tau = -(k + 1) \gamma r_f e_1 \int_0^t + (k + 1) \gamma \int_0^t e_1 f d\tau = -(k + 1) \gamma r_f(t) e_1(t) + (k + 1) \gamma r_f(0) e_1(0) - \int_0^t e_1^2 d\tau + \gamma \beta \alpha \int_0^t e_1(t) |d\tau|
\]

The last term in (49) is obtained as

\[
-\gamma (k + 1)^2 \int_0^t e_1 d\tau = -\gamma (k + 1)^2 \int_0^t e(\dot{e} + \alpha e + r_f) d\tau = -\gamma (k + 1)^2 t^2 e^2(t) - \gamma e^2(0) + \alpha \int_0^t e^2 d\tau + \int_0^t e_1 f d\tau.
\]

From (48), (49), and (50), the calculation of \( \varsigma \) is completed.

V. NUMERICAL SIMULATION

To validate the proposed control algorithm, we consider the following first order system.

\[
\dot{x} = x - x^2 - 2u - u^3 + d
\]

\[
y = x
\]

The initial condition is set as \( x(0) = 1 \), the reference trajectory \( y_d(t) \) is selected as \( y_d = \sin(t) \), and \( d(t) \) represents the external disturbance in the form that \( d(t) = 0.5 \sin(t) \). The control gains employed in the simulation are given as

\[
k = 2.5 \quad \alpha = 1 \quad \beta = 2 \quad \gamma = 0.5 .
\]

The reference trajectory \( y_d(t) \), the output signal \( y(t) \) and the tracking error \( e(t) \) are illustrated in Fig.1. We can see that the error signal reaches 0 after almost 6 seconds and keep in a very small region around 0 afterwards. Fig.2 shows the control input signal \( u(t) \). In the first 5 seconds, the control input switches very fast when the Nussbaum-type function plays a key role in deciding the correct direction. From the above figures, it’s demonstrated that the proposed control law achieve good tracking results.

VI. CONCLUSION

The RISE feedback structure incorporated with Nussbaum-type function is designed for the SISO nonlinear non-affine system. Comparing with the other control algorithms, the main contribution of this paper are listed as follows. By deriving the augmented affine-like form, we avoid the unexpected high-order term and the need to set the trim value for control input. The external disturbance is dealt with by the simple RISE method. Then, the disturbance is indicated to have no effect on the ultimate error boundedness in theoretical sense. A second-order filter is developed for combining the RISE structure and Nussbaum-type function.
REFERENCES


