A Discrete-time MIMO LPV Controller for the Rejection of Nonstationary Harmonically Related Multisine Disturbances

P. Ballesteros, X. Shu and C. Bohn

Abstract— This paper is an extension of the previous work realized by the authors. Here a discrete-time multiple input multiple output (MIMO) linear parameter-varying (LPV) controller is designed, a polynomial approximation for disturbances with harmonically related frequencies to reduce the number of gain-scheduling parameters is used and the controller is validated experimentally for the rejection of the engine-induced vibrations in a vehicle. Excellent results are achieved in a Golf VI Variant for the rejection of disturbances with nine time-varying frequency components with a controller interpolated from three vertices using only two gain-scheduling parameters. The stability of the controller is guaranteed for changes in the gain-scheduling parameters since LPV control design techniques are used.

I. INTRODUCTION

Gain-scheduling controllers have been used over the last few years for the control of periodic disturbances with time-varying frequencies [1-22]. This control problem is commonly found in active noise control (ANC) and active vibration control (AVC) applications where rotating machinery operates at varying angular velocity, e.g., in automotive applications or aircrafts.

Adaptive filtering updating the filter coefficients through the Filtered-x LMS (FxLMS) algorithm is a common approach in ANC/AVC for the rejection of harmonic disturbances [23]. An off-line analysis of the closed-loop behavior is difficult since the controller is the result of an adaptation process. Also, to date, only “approximate stability results” for the FxLMS algorithm seem available [24].

According to the internal model principle, all controllers contain a model of the harmonic disturbance to achieve disturbance rejection [25]. Different control approaches as general output-feedback controllers [1-13] or observer-based state-feedback controllers [14-22] can be used to obtain a controller including the model of the disturbance. In this paper, a multiple input multiple output (MIMO) observer-based controller is designed.

The use of linear parameter-varying (LPV) gain-scheduling techniques [1-18] guarantees the stability for changes in the gain-scheduling parameters. A polynomial approximation to reduce the number of gain-scheduling parameters for disturbances with harmonically related frequencies is in [9, 10] introduced, however the disturbance model for constant frequencies is used.

An LPV disturbance model for time-varying frequencies is obtained in this paper using the polynomial approximation. A further simplification is here introduced where a triangle as polytope reduces the number of vertices to three. The resulting LPV disturbance model can be interpolated with three vertices and two parameters independently of the number of frequencies. In the control design presented here, the disturbance model is combined with the plant to obtain an LPV augmented system and LPV control design techniques are used to obtain an LPV observer-based controller.

This paper is an extension of the previous work realized by the authors [17-20]. Here, a discrete-time MIMO polytopic LPV (pLPV) observer-based controller for the rejection of harmonic disturbances is designed, a polynomial approximation to reduce the number of scheduling parameters is used and the controller is tested on a vehicle (Golf VI Variant) to suppress the engine-induced vibrations. The controller has only two gain-scheduling parameters and it is interpolated from only three controller vertices through a matrix multiplication. The stability of the controller is guaranteed for changes of the gain-scheduling parameters since LPV control design techniques are used. Excellent results were achieved for the rejection of disturbances with nine frequency components for a specified engine speed range. The future work focuses on augmenting the vibration reduction range of the controller.

The remainder of this paper is organized as follows. The control structure used in this paper is explained in Sec. II. The disturbance model for harmonic disturbances with time-varying frequencies and a polynomial approximation to reduce the number of scheduling parameters are discussed in Sec. III. In Sec. IV pLPV control design techniques are used to obtain a controller for the rejection of time-varying harmonic disturbances. The discrete-time MIMO pLPV disturbance-observer state-feedback controller is validated with experiments in Sec. V. Discussion and conclusions are given in Sec. VI.

II. DISTURBANCE-OBSERVER-BASED STATE-FEEDBACK CONTROL

The disturbance-observer-based state-feedback control structure of [17-20] that fulfills the internal model principle [25] is briefly reviewed in this section. The objective is the rejection of the disturbance acting on a plant.

A state-space representation of a plant is given by

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\[
x_{p,k+1} = A_{p} x_{p,k} + B_{p} (u_{p,k} + y_{d,k}),
\]
\[
y_{p,k} = C_{p} x_{p,k},
\]
and a disturbance \( y_{d,k} \) modeled as the output of an unforced linear time-varying exo-system
\[
x_{d,k+1} = A_{d} x_{d,k},
\]
\[
y_{d,k} = C_{d} x_{d,k},
\]
acts at the input of the plant.

For the control design plant and disturbance model can be combined to obtain the augmented system
\[
\begin{bmatrix}
x_{d,k+1} \\
x_{p,k+1}
\end{bmatrix} =
\begin{bmatrix}
A_{d} & 0 \\
B_{p} C_{d} A_{p}
\end{bmatrix}
\begin{bmatrix}
x_{d,k} \\
x_{p,k}
\end{bmatrix} +
\begin{bmatrix}
0 \\
B_{p}
\end{bmatrix} u_{p,k},
\]
\[
y_{p,k} = \begin{bmatrix} 0 & C_{p} \end{bmatrix} \begin{bmatrix} x_{d,k} \\ x_{p,k} \end{bmatrix}
\]
and it can be written in compact form as
\[
x_{k+1} = A_{k} x_{k} + B_{u} u_{p,k},
\]
\[
y_{p,k} = C_{k} x_{k}
\]
with \( x_{k} = \begin{bmatrix} x_{d,k} \\ x_{p,k} \end{bmatrix} \), \( A_{k} = \begin{bmatrix} A_{d} & 0 \\ B_{p} C_{d} A_{p} \end{bmatrix} \), \( B_{u} = \begin{bmatrix} 0 \\ B_{p} \end{bmatrix} \), and
\[
C_{k} = \begin{bmatrix} 0 & C_{p} \end{bmatrix}.
\]

The control structure is shown in Fig. 1 where \( K \) is a constant state-feedback gain
\[
u_{p,k} = -K \hat{x}_{k} = -\begin{bmatrix} K_{d} & K_{p} \end{bmatrix} \begin{bmatrix} \hat{x}_{d,k} \\ \hat{x}_{p,k} \end{bmatrix}
\]
of the augmented system with the estimated states \( \hat{x}_{k} \) calculated through an identity observer
\[
\hat{x}_{k+1} = (A_{k} - L_{k} C) \hat{x}_{k} + B_{u} u_{p,k} + L_{k} y_{p,k}
\]
with a time-varying observer gain \( L_{k} \). From [17] it follows that as long as a stabilizing state-feedback gain \( K_{p} \) for the linear time-invariant plant and a stabilizing observer gain for the augmented system model is chosen, the overall closed-loop system is stable. \( K_{d} \) has no effect on the overall system stability (as long as \( \hat{x}_{d,k} \) remains bounded). Assuming a perfect disturbance model, complete disturbance cancellation will be achieved after some transient if \( K_{d} = C_{d} \).

A state-space representation of the resulting controller can be written as
\[
x_{c,k+1} = A_{c} x_{c,k} + B_{u} u_{p,k},
\]
\[
u_{p} = C_{c} x_{c,k}
\]
with
\[
A_{c,k} = A_{k} - L_{k} C - B K,
\]
\[
B_{c,k} = -L_{k},
\]
and
\[
C_{c} = -K.
\]

![Figure 1. Disturbance-observer-based state-feedback control structure.](image)

### III. POLYNOMIAL DISTURBANCE MODELING

For harmonic disturbance rejection the controller must contain a model of the disturbance. In this section a polynomial approximation [9, 10] for a discrete-time MIMO harmonic disturbance model is used to reduce the number of gain-scheduling parameters for harmonically related frequencies. In [9, 10], the polynomial approximation is used with a single input single output (SISO) model for constant harmonic disturbances. Here, the polynomial approximation is used for an MIMO time-varying harmonic disturbance model.

A state-space representation in discrete time for an \( n \times n \) MIMO harmonic disturbance with \( n_{d} \) time-varying frequency components is given by (2) with
\[
A_{d,k} = \begin{bmatrix} A_{d,k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{d,n} \end{bmatrix},
\]
\[
A_{d,k} = \cdots = A_{d,n} = \begin{bmatrix} \tilde{A}_{d,k} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{A}_{d,n} \end{bmatrix},
\]
\[
C_{d} = \begin{bmatrix} C_{d} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{d} \end{bmatrix},
\]
with \( r < 1 \) and close to one, \( \Omega_{i,k} = 2 \pi f_{i,k} T \) and \( f_{i,k} \in [f_{i,k_{\text{min}}}, f_{i,k_{\text{max}}}] \) for \( i = 1, \ldots, n_{d} \). For a classical LPV representation of this model two gain-scheduling parameters are needed per frequency \( (2n_{d}) \). The polynomial
approximation of [9, 10] is very useful to reduce the number of gain-scheduling parameters if the frequencies are harmonically related.

The cosine function \( r \cos(\Omega_{i,k}) \) and the sine function \( r \sin(\Omega_{i,k}) \) are approximated as

\[
\begin{align*}
\cos(\Omega_{i,k}) & \approx a_{i,0} + a_{i,\Omega_{i,k}} + \ldots + a_{i,\Omega_{i,k}^n} \quad \text{for } n_a \text{ harmonically related time-varying frequencies} \\
\sin(\Omega_{i,k}) & \approx b_{i,0} + b_{i,\Omega_{i,k}} + \ldots + b_{i,\Omega_{i,k}^n}
\end{align*}
\]

(13)

(14)

for \( n_a \) harmonically related time-varying frequencies.

The coefficients \( a_{i,0}, \ldots, a_{i,N_a} \) and \( b_{i,0}, \ldots, b_{i,N_a} \) can be obtained through a least squares fit over the respective frequency ranges. In simulation and experimental results, it has been observed that an excellent approximation is obtained through a least squares fit over the respective frequency ranges.

Therefore here the approximations

\[
\begin{align*}
\cos(\Omega_{i,k}) & \approx a_{i,0} + a_{i,\Omega_{i,k}} + a_{i,\Omega_{i,k}^4} \\
\sin(\Omega_{i,k}) & \approx b_{i,0} + b_{i,\Omega_{i,k}} + b_{i,\Omega_{i,k}^4}
\end{align*}
\]

(15)

(16)

(17)

are used.

The time-varying parameters

\[
\begin{align*}
\theta_{1,k} & = \Omega_{i,k}^2 = (2\pi f_{1,k} T)^2, \\
\theta_{2,k} & = \Omega_{i,k}^4 = (2\pi f_{1,k} T)^4
\end{align*}
\]

(18)

(19)

are used to obtain the polynomial approximation of the disturbance model and the matrix \( A_{d,k} \) of (11) can be rewritten with

\[
\begin{align*}
\tilde{A}_{d,k} & \approx \begin{bmatrix} a_{i,0} & b_{i,0} \\ -b_{i,0} & a_{i,0} \end{bmatrix} \theta_{1,k} + \begin{bmatrix} a_{i,2} & b_{i,2} \\ -b_{i,2} & a_{i,2} \end{bmatrix} \theta_{2,k} + \begin{bmatrix} a_{i,4} & b_{i,4} \\ -b_{i,4} & a_{i,4} \end{bmatrix} \theta_{2,k}
\end{align*}
\]

(20)

for \( i = 1, \ldots, n_a \).

The polynomial approximation leads to an LPV disturbance model with only two gain-scheduling parameters independently of the number of frequencies.

IV. Gain-Scheduling Observer Design for PLPV Systems

This section combines the control structure of Sec. II with the polynomial disturbance model of Sec. III to obtain a PLPV model of the augmented system. This model is then used to obtain an MIMO PLPV controller for the rejection of harmonic disturbances with PLPV control design techniques.

A PLPV representation of the augmented system matrix \( A_i \) from Sec. II can be written using the polynomial approximation for \( A_{d,k} \) from Sec. III. In [9, 10] a rectangle is used as a polytope. In this paper, a further simplification is introduced where a triangle as a polytope suffices since the relationship between the parameters is known.

The two scheduling parameters \( \theta_{1,k} = \Omega_{i,k}^2 \) and \( \theta_{2,k} = \Omega_{i,k}^4 \) vary inside a triangle in \( \mathbb{R}^2 \) (this is shown schematically in Fig. 2) with vertices

\[
\bar{\theta}_1 = \left[ \theta_{1,\min} \quad \theta_{2,\min} \right]^T, \quad \bar{\theta}_2 = \left( \left( \theta_{1,\min} + \theta_{1,\max} \right) / 2 \quad \theta_{1,\max} \theta_{2,\max} \right)^T, \quad \bar{\theta}_3 = \left[ \theta_{2,\min} \quad \theta_{2,\max} \right]^T
\]

(21)

\[
\theta_{1,\min} = (2\pi f_{1,\min} T)^2 \text{ and } \theta_{1,\max} = (2\pi f_{1,\max} T)^2.
\]

The augmented system matrix

\[
A_i = A(\theta_i) = A_0 + \theta_{1,i} A_1 + \theta_{2,i} A_2
\]

(22)

depends affinely on the parameters \( \theta_i = [\theta_{1,i} \quad \theta_{2,i}]^T \) with constant matrices \( A_0, A_1, A_2 \) and the parameter vector \( \theta_i \) varies inside a convex polytope \( \Theta \) with three vertices \( \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3 \in \Theta \subset \mathbb{R}^2 \). An arbitrary point \( \theta_i \) inside the polytope \( \Theta \) can be written as a convex combination of these vertices using the time-varying coordinate vector \( \lambda_{j,k} = \left[ \lambda_{j,k} \lambda_{2,j,k} \lambda_{3,j,k} \right]^T \in \mathbb{R}^3 \) via

\[
\lambda_{j,k} \geq 0, \quad \sum_{j=1}^3 \lambda_{j,k} = 1 \quad \text{and} \quad \theta_i = \sum_{j=1}^3 \lambda_{j,k} \bar{\theta}_j.
\]

(23)

The LTI vertex systems are defined with the vertex matrices \( A(\bar{\theta}_j) = A_{v,j} \), \( j = 1, \ldots, 3 \).

Applying the same procedure as (23), the augmented system matrix is calculated through

\[
A_i = A(\theta_i) = A_{1,i} A(\bar{\theta}_1) + \ldots + A_{3,i} A(\bar{\theta}_3).
\]

(24)

The time-varying coordinates are calculated with

\[
\lambda_{j,k} \geq 0, \quad \sum_{j=1}^3 \lambda_{j,k} = 1 \quad \text{and} \quad \theta_i = \sum_{j=1}^3 \lambda_{j,k} \bar{\theta}_j.
\]

(25)

The inverse matrix can be calculated offline and the coordinate vector at each sampling time is the result of a matrix multiplication.

A representation of the augmented system (4) is obtained in PLPV form with three vertex and two gain-scheduling parameters. The control design used by the authors in [17, 18] to obtain a PLPV disturbance-observer controller is

Figure 2. Triangle as a polytope in \( \mathbb{R}^2 \).
briefly reviewed in the following with the control structure of Sec. II.

Three observer gains are calculated for the three augmented system vertices solving linear matrix inequalities (LMIs) based on the $H_2$ norm guaranteeing quadratic stability.

If solutions for the matrix variables $P$, $W$ and $Y_{n,j}$, $j=1,\ldots,3$ can be found that satisfy the seven LMIs

$$\begin{bmatrix} P & PA_{n,j} - Y_{n,j}^T \, C \\ (PA_{n,j} - Y_{n,j}^T \, C)^T & P \end{bmatrix} > 0, \quad j=1,\ldots,3,$$

(26)

$$\begin{bmatrix} W & \tilde{Q}P - \tilde{R}Y_{n,j} \\ (\tilde{Q}P - \tilde{R}Y_{n,j})^T & P \end{bmatrix} > 0, \quad j=1,\ldots,3,$$

(27)

$$\text{trace}(W) < \gamma^2,$$

(28)

with

$$\tilde{Q} = \begin{bmatrix} Q^T \\ 0 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} 0^T \\ R^T \end{bmatrix}, \quad Y_{n,j} = L_{n,j} \, P, \quad j=1,\ldots,3$$

(29)

then the augmented system has an $H_2$ norm bounded by $\gamma$. From the solutions $P$ and $Y_{n,j}$, the observer gain for each vertex system can be calculated as

$$L_{n,j} = P^{-T}Y_{n,j}^T.$$

(30)

In order to guarantee quadratic stability for the whole parameter space, solutions for the matrix variables $P$ and $W$ have to be the same for all vertex systems.

The controller of Sec. II with the vertex observer gains can be written in pLPV form as

$$x_{c,k+1} = A_c \left( \lambda_{j,k} \right) x_{c,k} + B_c \left( \lambda_{j,k} \right) y_p,$$

$$u_p = C_c x_{c,k}$$

(31)

with

$$A_c \left( \lambda_{j,k} \right) = \sum_{j=1}^{3} \lambda_{j,k} \left( A_{n,j} - L_{n,j} \, C \right) - BK,$$

(32)

$$B_c \left( \lambda_{j,k} \right) = \sum_{j=1}^{3} \lambda_{j,k} L_{n,j}$$

(33)

and

$$C_c = -K.$$

(34)

The coordinate vector $\lambda_k$ for the interpolation of the controller is calculated at each sampling time with (25).

V. EXPERIMENTAL RESULTS

The discrete-time MIMO pLPV controller of the previous section is validated experimentally in a Golf VI Variant to reduce the engine-induced vibrations in the chassis of the vehicle. Two inertia mass actuators (shakers) and two accelerometers are attached to the engine mounts. The controller uses the measurements of the accelerometers to generate the cancelling signals of the shakers. Anti-aliasing filters are applied to the output signals and reconstruction filters to the control inputs. A schema of the experimental setup is shown in Fig. 3 and a photograph in Fig. 4.

Standard black-box identification techniques are used to obtain a state-space representation of the $2 \times 2$ MIMO plant with a sampling frequency of 2 kHz. The MIMO plant has an order of 22. The controller is designed for the rejection of nine frequency components ($f_{i,k} \in [45, \ldots, 12]$ with $f_{i,k} \in [25, 28.75]$ the half motor order) and is implemented on a rapid prototyping unit (MicroAutoBox from dSpace). The two gain-scheduling parameters depend on the engine speed measured by the CAN-Bus and the engine speed is used for the interpolation of the controller from three controller vertices. Excellent results are achieved for test drives with constant engine speeds and for time-varying engine speeds.

Amplitude frequency responses in open loop and closed loop are shown in Fig. 5 for a constant frequency $f_{i,k} = 25$ Hz. Accelerations and spectrum of the accelerations for constant engine speed of 3420 rpm ($f_{i,k} = 28.5$ Hz) in third gear are shown in Fig. 6. The control sequence is off/on/off. The frequency components $f_{i,k} \in [45, \ldots, 12]$ are rejected as shown in the acceleration spectrums of Fig. 6.

![Figure 3. Schema of the experimental setup.](image)

![Figure 4. Experimental setup.](image)
Results for time-varying engine speeds are shown in Figs. 7 and 8. The controller is switched on the first hundred seconds and switched off the next hundred seconds. The time-varying engine speed is shown in Fig. 7 and the spectrograms of the acceleration measured on the driver and rider side in Fig. 8. Excellent results are obtained for time-varying frequencies where the frequency components \( f_1, f_4, 5 \ldots 12 \) are rejected as shown in Fig. 8.

The discrete-time MIMO pLPV controller is calculated at each sampling time with a very simple interpolation (a matrix multiplication) using two gain-scheduling parameters obtained from the engine speed and three controller vertices. The stability of the controller is guaranteed even for fast changes of the engine speed since pLPV control design techniques are used. The experimental results show the effectiveness of this controller for the rejection of time-varying harmonic disturbances.

VI. DISCUSSION AND CONCLUSIONS

This paper is an extension of the previous work realized by the authors. In this paper a discrete-time MIMO pLPV
observer-based state-feedback controller for the rejection of time-varying harmonic disturbances is designed and validated experimentally for the suppression of engine-induced vibrations in a Golf VI Variant.

A polynomial interpolation is used for the disturbance model to reduce the number of gain-scheduling parameters. Only two gain-scheduling parameters are used for the rejection of disturbances with nine frequency components. The polytope is a triangle, therefore the controller is interpolated at each sampling time from three controller vertices with a simple matrix multiplication.

Excellent results are obtained in test drives for the suppression of the engine-induced vibrations for constant and time-varying engine speeds for a specified engine speed range. The stability is guaranteed even for fast changes of the engine speed since LPV techniques are used. The future work focuses on augmenting the vibration reduction range of the controller.

REFERENCES


