UAV Circumnavigating an Unknown Target using Range Measurement and Estimated Range Rate

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Abstract—The objective of the paper is to design a control algorithm such that a UAV can circumnavigate an unknown target using range-only measurement. By assuming the availability of both range and range rate measurements, a control algorithm is proposed to accomplish the circumnavigation mission, where the associated control input is always bounded. To eliminate the requirement of range rate measurements, an estimated range rate, obtained via a sliding-mode estimator using range measurement, is used to replace the range rate measurement. By carefully choosing parameters in the estimator, range rate can be accurately estimated in finite time thanks to the boundedness of control input under the proposed control algorithm. As a consequence, the circumnavigation mission can still be accomplished using the proposed control algorithm when range rate measurement is replaced by its estimated value obtained from the estimator. This estimator-based control strategy using range-only measurement is desired for small UAVs under GPS-denied environment when limited sensing capability is allowed considering their payload restrictions.

Index Terms—UAV, Autonomy, Estimation and control, GPS-denied environment

I. INTRODUCTION

As Unmanned Aerial Vehicles (UAVs) gain more and more favor in both military and civilian applications due to its advantages over manned aircraft, it is now a demanding technology that UAVs can be used to accomplish missions with minimum human supervision. In many cases, complete autonomy is often desired in UAV operations unless human supervision is necessary. Technologies are thus needed to increase autonomy for UAV operations [1].

The need of autonomy for UAV operations comes from two perspectives: performance and cost. From the performance’s point of view, UAV with autonomy capability is more reliable than manned aircraft. Human operators are one of the most common sources of errors in complex systems. In many situations, human operators are more likely to make mistakes. It is also well-known that the efficiency of human operators are expected to decrease after a certain period of time. Nevertheless, all these drawbacks can be addressed if autonomy becomes available to replace human operators. From the cost’s point of view, UAV with autonomy is less expensive than manned aircraft as the cost of training and/or replacing a pilot is high. Another factor of cost is that manned aircraft are generally larger in size and more complex in capabilities than UAVs due to the existence of human operators onboard.

Due to FAA regulations, UAVs are now mainly used in military and security applications, such as border patrol, mapping, and surveillance. For instance, UAVs can be used to gather information of a target by orbiting around it at some desired distance. Such a UAV mission is often called circumnavigation [2]–[5]. In [2], [3], a localization-and-control framework was proposed to solve the circumnavigation problem. The control algorithm is designed based on the knowledge of accurate location information of the UAV under some local coordinate frame and bearing/range measurement. In [4], a sliding-mode control algorithm based on range and range rate measurements was designed to solve the circumnavigation problem. But such a controller can only guarantee local stability. In [5], another control algorithm based on range and range rate measurements was proposed to solve the circumnavigation problem. In contrast to the local stability shown in [4], global stability was shown in [5].

One drawback in [4], [5] is that range rate measurement is needed. For small UAVs, range rate measurement is difficult to obtain. In certain situations, such as GPS-denied environment due to jamming [6] or spoofing [7], controller design based on range-only measurement is desired for small UAVs considering their limited sensing capabilities due to payload restrictions.

The objective of this paper is to design a control algorithm such that the circumnavigation mission is accomplished using range-only measurement. The design of this control algorithm is motivated by the study on sliding-mode observer in [8]–[10], where numerous velocity observers were designed based on position information. Finite-time convergence of the sliding-mode observers was shown via either analyzing the majorant curves [8] or introducing Lyapunov functions [9], [10]. Unfortunately, the results in [8]–[10] cannot be applied here because a joint estimation-and-control problem is considered here, where the estimator i.e., observer referred in [8]–[10], appears in the controller design and vice versa. In other words, the estimator and controller need to be considered simultaneously rather than independently.

The objective of the paper is fulfilled via a two-step analysis. First, a control algorithm based on range and range rate measurements is proposed to accomplish the circumnavigation mission. One promising feature of the control algorithm is that the associated control input is always bounded. Second, a sliding-mode estimator using range measurement is designed to accurately estimate range rate in finite time when applying the proposed control algorithm with range rate measurement being replaced by the estimated value. By combining the two steps, the circumnavigation
mission can be accomplished using the proposed control algorithm when range rate measurement is replaced by its estimated value obtained from the sliding-mode estimator. To our best knowledge, this is the first paper that solves the circumnavigation problem using range-only measurement.

The rest of the paper is structured as follows. Section II revisits the circumnavigation problem described in [5] and motivates the problem studied in this paper. In Section III, a control algorithm based on range and range rate measurements is proposed to solve the circumnavigation problem, where the associated control input is always bounded. Because the control algorithm is motivated by [5], its stability analysis is similar to that in [5]. In Section IV, range rate measurement used in the control algorithm in Section III is replaced by its estimated value obtained via a sliding-mode estimator. Finite-time convergence of the estimator is shown via a Lyapunov-based approach. In Section V, an illustrative simulation example is provided as a proof of concept. Finally, Section VI is given to conclude the paper.

II. PROBLEM STATEMENT

In this section, we first briefly revisit the circumnavigation problem studied in [5] and then introduce the problem to be studied in this paper. The definitions and notations used in this paper follows those used in [5].

Circumnavigation concerns with the behavior that a UAV can orbit around some unknown target at some desired distance. For instance, in Fig. 1, let \( T \) denote the unknown target whose location is \([x_T, y_T]^T\) and the blue triangle denote the UAV. The objective is to have the UAV orbit around \( T \) at some desired distance \( r_d \). Consider the UAV model given by

\[
\dot{x} = V \cos(\psi), \quad \dot{y} = V \sin(\psi), \quad \dot{\psi} = \omega, \tag{1}
\]

where \([x, y]^T\) is the 2D location of the UAV, \( \psi \) is the heading of the UAV, \( \omega \) is the control input to be designed, and \( V \) is the (constant) velocity of the UAV. Let the range measurement be given by \( r = \sqrt{(x-x_T)^2 + (y-y_T)^2} \). The objective is to design control input \( \omega \) such that \( r(t) \rightarrow r_d \) as \( t \rightarrow \infty \).

**Definition 2.1**: [5] The circle centered at \( T \) with a radius \( r_d \) is defined as \( C_d \). The UAV is inside (resp., outside) \( C_d \) if \( r(t) < r_d \) (resp., \( r(t) \geq r_d \)).

**Definition 2.2**: [5] Denote the reference vector as the vector from the current location of the UAV to \( T \). The bearing angle \( \theta(t) \in [0, 2\pi) \) at time \( t \) is defined as the angle from the reference vector to the current heading of the UAV measured counterclockwise.

If \( r \) and \( \theta \) are chosen as the state variables, the dynamics (1) can be rewritten as

\[
\dot{r} = -V \cos(\theta), \quad \dot{\theta} = \omega + \frac{V \sin(\theta)}{r}. \tag{2}
\]

To distinguish (1) and (2), we call (1) “Cartesian dynamics” and (2) “polar dynamics”. One control algorithm in [5] is given by

\[
\begin{cases}
  k_1[2r(t)V \cos(\pi - \sin^{-1}(\frac{r_a}{r(t)}))] - 2r(t)\dot{r}(t), & r(t) \geq r_d, \\
  0, & \text{otherwise},
\end{cases} \tag{3}
\]

where \( k \) is the control gain and \( \sin^{-1}(\cdot) : [0, 1] \rightarrow [0, \frac{\pi}{2}] \).

The main idea of (3) is to drive the UAV towards one of the two tangents on \( C_d \). Here (3) is a feedback controller based on the difference between the desired rate of \( r^2(t) \) when the UAV moves towards one tangent point and the actual rate of \( r^2(t) \) when the UAV moves along its current heading. In [5], the UAV was shown to orbit around the target \( T \) clockwise at a distance \( r_a \), which is greater than \( r_d \), if \( k \) is larger than \( \frac{1}{r^2} \). Fortunately, if one chooses \( r_d \) carefully, the resulting radius \( r_a \) can match the desired value.

Controller (3) requires the availability of both range and range rate measurements. Because range rate measurement is often unavailable or suffers from large uncertainties even if available, it is desired that controllers can be designed based on range-only measurement. Moreover, in GPS-denied environment, the use of range-only measurement in the controller design is desirable for small UAVs due to their limited sensing capabilities considering payload restrictions. This is the main motivation of this paper.

III. A NEW CONTROLLER BASED ON RANGE AND RANGE RATE MEASUREMENTS

In this section, a new control algorithm based on range and range rate measurements is proposed to accomplish the circumnavigation mission. This new control algorithm is motivated by (3). The main difference between (3) and the new control algorithm is that (3) does not admit a bounded control input while the new control algorithm does.

Motivated by the idea behind the control algorithm (3), a new control algorithm is proposed as

\[
\omega = \begin{cases}
  k[V \cos(\pi - \sin^{-1}(\frac{r_a}{r(t)}))] - \dot{r}(t), & r(t) \geq r_d, \\
  0, & \text{otherwise},
\end{cases} \tag{4}
\]

Again, (4) is a feedback controller based on the difference between the desired rate of \( r(t) \), instead of \( r^2(t) \) for (3), when the UAV moves towards one tangent point and the actual rate of \( r(t) \), instead of \( r^2(t) \) for (3), when the UAV moves along its current heading. Before conducting stability...
analysis for (4), the following definition of a stable circular motion from [5] is needed.

Definition 3.1: [5] A stable circular motion refers to the behavior that the UAV, with dynamics (1), moves around a target with a constant speed and a constant radius.

By mimicking the analysis in [5], the following lemma presents the property of the stable circular motion when control algorithm (4) is applied in system dynamics (1).

Lemma 3.1: Consider system dynamics (1) subject to control input (4). If a stable circular motion exists, the radius is given by \( r_a = \sqrt{r_d^2 + \frac{1}{\pi}} \). In addition, the UAV rotates clockwise when \( k > 0 \).

Proof: Let the radius of the stable circular motion be given by \( r_a \). Then the magnitude of the nominal angular velocity is given by \( \frac{\sqrt{}}{r_a} \). Because the angular velocity of the UAV is equal to the control input (4), the magnitude of the nominal angular velocity is also equal to \(|\omega|\). For a stable circular motion, \( \theta = \frac{\pi}{2} \) or \( \theta = \frac{\pi}{2} \), which implies that (1) the UAV cannot be inside \( C_d \) and (2) \( \dot{r} = 0 \) based on (2). Then \(|\omega| \) becomes \( kV \cos(\pi - \sin^{-1}(\frac{r_a}{r})) \) for the stable circular motion. That is, \( \frac{\sqrt{}}{r_a} \) and \( kV \cos(\pi - \sin^{-1}(\frac{r_a}{r})) \) should be identical, which happens if and only if \( r_a = \sqrt{r_d^2 + \frac{1}{\pi}} \).

When \( k > 0 \) and \( r(t) = r_a \), it can be computed that \( \omega < 0 \), indicating that the UAV rotates clockwise.

Notice that the final stable radius \( r_a \) is larger than \( r_d \) used in the controller design. Similar to the technique used in [5], by choosing \( r_d \) to be \( \hat{r}_d = \sqrt{r_d^2 - \frac{1}{\pi}} \), where \( k \geq \frac{1}{r_d} \), the resulting radius becomes \( r_d \).

We now show that a stable circular motion indeed exists. Before stating the main result, several lemmas are needed, whose contents and proofs are similar to those in [5]. Therefore, we only present the lemmas without providing the proofs. We refer the readers to the proofs presented in [5].

Lemma 3.2: [5] Consider the UAV dynamics in (1) subject to the control policy in (4). Let there be \( t_0 \geq 0 \) such that \( r(t_0) \geq r_a \) and \( \theta(t_0) \in (\sin^{-1}(\frac{r_a}{r(t_0)}), 2\pi - \sin^{-1}(\frac{r_a}{r(t_0)}) \), then \( r(t) \geq r_d \), \( \forall t \geq t_0 \).

Lemma 3.3: [5] Consider the UAV dynamics in (1) subject to the control policy in (4). The UAV can only move inside \( C_d \) at most once.

Lemma 3.4: [5] Consider the UAV dynamics in (1) subject to the control policy in (4). For any \( \theta(0) \), there exists \( t^* \geq 0 \) such that \( \theta(t) \in [0, \pi] \) for any \( t \geq t^* \).

The proofs of the three lemmas are quite similar to the ones in [5]. The proof of Lemma 3.4 depends on Lemma 3.2. The proof of Lemma 3.3 depends on Lemma 3.3. With the three lemmas, we now present the main result in this section.

Theorem 3.2: Consider the UAV dynamics in (1) subject to the control policy in (4). If \( k > \frac{1}{r_d} \), then \( r(t) \to r_a \) and \( \theta(t) \to \frac{\pi}{2} \) as \( t \to \infty \), where \( r_a = \sqrt{r_d^2 + \frac{1}{\pi}} \).

Proof: According to Lemmas 3.3 and 3.4, there exists a time instant \( t^* \) such that \( r(t) \geq r_d \) and \( \theta(t) \in [0, \pi] \). When \( t \geq t^* \), the control input can be simplified as \( \omega = k[V \cos(\pi - \sin^{-1}(\frac{r_a}{r(t)})) + V \cos(\theta)] \). For \( t \geq t^* \), consider a Lyapunov function candidate given by \( V = 1 - \sin(\theta) + \varphi \), where \( \varphi = \int_{r_a}^{r_a} (\frac{1}{r} - \frac{1}{r_a} + k \cos \sin^{-1}(\frac{\varphi(t)}{r_a})) \) \( \geq 0 \). Therefore, \( V \geq 0 \). Because \( \frac{1}{r_a} = k \cos \sin^{-1}(\frac{\varphi(t)}{r_a}) \), \( \varphi \) can be simplified as \( \int_{r_a}^{r_a} (-\frac{1}{r_a} + k \cos \sin^{-1}(\frac{\varphi(t)}{r_a})) \). Taking derivative of \( V \) yields that

\[
\dot{V} = -\cos(\theta) \dot{\theta} + \varphi
\]

\[
= -\cos(\theta) \left[ kV \left( -\cos \sin^{-1}(\frac{r_a}{r}) + \cos(\theta) \right) + \frac{V \sin(\theta)}{r} \right]
\]

\[
- \left[ k \cos \sin^{-1}(\frac{r_a}{r}) - \frac{1}{r} \right] V \cos(\theta)
\]

\[
= V \cos(\theta) \left[ -k \cos(\theta) - \frac{\sin(\theta)}{r} + \frac{1}{r} \right],
\]

where (2) was used to derive the second equality. When \( \theta \in [\frac{\pi}{2}, \pi] \), \( V \leq 0 \) because \( \cos(\theta) \leq 0 \) and \(-k \cos(\theta) - \frac{\sin(\theta)}{r} + \frac{1}{r} \leq \frac{\sin(\theta)}{r} + \frac{1}{r} \leq 0 \). Therefore, \( V \leq 0 \). Noting that \( V \) is uniformly continuous when \( r \geq r_d \) and \( \theta \in [0, \pi] \) because \( V \) is differentiable for the domain \([0, \pi] \times [r_d, \infty] \), it follows from Lemma 4.3 in [11] that \( V \to 0 \) as \( t \to \infty \). When \( k > \frac{1}{r_d} \), \( V \) is uniformly continuous. It then follows from (2) that when \( \theta(t) = \frac{\pi}{2} \), \( r(t) \) is constant.

IV. A REVISITED CONTROLLER BASED ON RANGE MEASUREMENT AND ESTIMATED RANGE RATE

This section focuses on removing the range rate measurement in the control algorithm (4). In particular, an estimated range rate, obtained via a sliding-mode estimator, is used to replace the range rate measurement used in (4).

Here control algorithm (4) is revised as

\[
\hat{a} = \begin{cases} 
  k[V \cos(\pi - \sin^{-1}(\frac{r_a}{r(t)})) - \hat{x}_2], & r(t) \geq r_d \text{, otherwise,} \\
  0, & \text{otherwise}
\end{cases}
\]

(5)

where \( \hat{x}_2 \) is an estimate of \( \hat{r} \). In particular, \( \hat{x}_2 \) is obtained via the following sliding-mode estimator as

\[
\dot{\hat{x}}_1 = \hat{x}_2 + k_1 |\hat{x}_1 - \hat{x}_2|^\frac{1}{2} \sign(r - \hat{x}_1),
\]

(6)

\[
\dot{\hat{x}}_2 = k_2 \sign(r - \hat{x}_1) + k_3 (r - \hat{x}_1),
\]

if \( r \geq r_d \) and

\[
\hat{x}_1 = 0, \quad \hat{x}_1(t_x) = 2r_d - \hat{x}_1(t_x)
\]

(7)

\[
\hat{x}_2 = 0, \quad \hat{x}_2(t_x) = -\hat{x}_2(t_x)
\]

if \( r < r_d \), where \( \sign(\cdot) \) is the sign function, \( t_x \) is the time when the UAV moves outside \( C_d \), \( t_e \) is the subsequent time.
when the UAV moves inside $C_d$, and $k_i, i = 1, 2, 3$, are positive constants. Here $\hat{x}_1$ can be considered an estimate of $r$. The main idea behind this estimator is that (i) $\hat{x}_1$ and $\hat{x}_2$ satisfy (6) if the UAV is outside $C_d$; (ii) $\hat{x}_1$ and $\hat{x}_2$ remain unchanged if the UAV is inside $C_d$; and (iii) once the UAV moves outside $C_d$, $\hat{x}_2$ is reset as its negate while $\hat{x}_1$ is reset as $2r_d$ plus its negate. As shown in the proof of the following Theorem 4.3, the reset of $\hat{x}_1$ and $\hat{x}_2$ at the time when the UAV moves from inside $C_d$ to outside $C_d$ is crucial in establishing a finite-time convergence of $(\hat{x}_1, \hat{x}_2)$ to $(r, \dot{r})$.

**Remark 4.1:** Instead of using actual range rate in controller (4), estimated range rate is used in controller (5). Because the designed sliding mode estimator guarantees that the estimation error converges to zero in finite time, the well-known “separation principle” can be applied in the controller design. That is, the design of range rate estimator and the design of a feedback controller based on estimated range rate can be decoupled into two separated problems.

Before presenting the main result in this section, the following lemma is needed.

**Lemma 4.1:** Consider the differential equation given by

$$\begin{align*}
\dot{p} &= q - k_1 |p|^{\frac{3}{2}} \text{sgn}(p), \\
\dot{q} &= -k_2 \text{sgn}(p) - k_3 p + f(t, p, q),
\end{align*}$$

(8)

where $|f(t, p, q)| < \delta_1 + \delta_2 |q|$ with $\delta_1 > 0$, $i = 1, 2, 3$. If $k_1 > 0, k_2 > \max \{1 + \frac{\delta_1^2}{k_1}, \frac{1}{2} \delta_2^2 + 2 \delta_2 \}$, and $k_3 > 0$, $(p, q)$ approaches $(0, 0)$ in finite time.

**Proof:** Let $\xi = \begin{bmatrix} [|p|^{\frac{3}{2}} \text{sgn}(p), p, q]^T \end{bmatrix}$ and consider the following Lyapunov function candidate given by

$$V = \xi^T P \xi,$$

(9)

where $P = \begin{bmatrix} 4k_2 + k_3^2 & 0 & -k_1 \\
0 & 2k_3 & 0 \\
-k_1 & 0 & 2 \end{bmatrix}$. Note that $V(x)$ is positive-definite and is differentiable almost everywhere except at $p = 0$. When $p \neq 0$, the derivative of $V$ is given by $\dot{V} = \xi^T P \xi$. Notice that the derivative of $\xi$ is given by $\begin{bmatrix} \frac{3}{2} |p|^{-\frac{1}{2}} \dot{p}, \dot{p}, \dot{q} \end{bmatrix}^T$. By recalling (8), $\dot{V}$ can be rewritten as

$$\dot{V} = -|p|^{-\frac{3}{2}} \xi^T Q_1 \xi - \xi^T Q_2 \xi - k_1 f(t, p, q) |p|^{\frac{3}{2}} \text{sgn}(p) + 2q f(t, p, q),$$

(10)

where $Q_1 = \begin{bmatrix} 2k_2 + k_3^2 & 0 & -k_1 \\
0 & 2k_3 & 0 \\
-k_1 & 0 & 1 \end{bmatrix}$ and $Q_2 = 2k_2 \begin{bmatrix} k_2 + k_3^2 & 0 & 0 \\
0 & 2k_3 & 0 \\
0 & 0 & 1 \end{bmatrix}$.

Because $|f(t, p, q)| \leq \delta_1 + \delta_2 |q|$ under the assumption of the lemma, it can be further obtained that $|k_1 f(t, p, q) |p|^{\frac{3}{2}} \text{sgn}(p)| \leq \delta_1 k_1 |p|^{\frac{3}{2}} + k_1 \delta_2 |q| |p|^{\frac{1}{2}} \leq \delta_1 k_1 |p|^{-\frac{1}{2}} |p|^{\frac{3}{2}} \text{sgn}(p)|^2 + \frac{1}{2} \delta_2^2 q^2 + k_1^2 |p|^{\frac{3}{2}} \text{sgn}(p)|^2$ and $|q f(t, p, q)| \leq \frac{3}{2} |p|^{\frac{3}{2}} \text{sgn}(p)|^2$.

Because $V$ is differentiable almost everywhere except at $p = 0$, (11) is valid almost everywhere except at $p = 0$. Note that $V = 0$ if and only if $p = 0$ and $q = 0$. Therefore, $\dot{V}(t) \rightarrow 0$ in finite time, which implies that $(p, q)$ approaches $(0, 0)$ in finite time.

**Remark 4.2:** In [9], finite-time convergence of (8) with $f(t, p, q) \leq \delta_1 + \delta_2 |q|$ was studied. Lemma 4.1 further analyzed the case when $f(t, p, q) \leq \delta_1 + \delta_2 |q|$.

With Lemma 4.1, we are now ready to present the main result in this section.

**Theorem 4.3:** Consider the UAV dynamics in (1) subject to the control policy in (5). If $k > \frac{1}{r_d}, k_1 > 0, k_2 > \max \{1 + \frac{\lambda_{\max}(Q_1 + Q_2)}{k_1}, \frac{1}{2} \lambda_{\max}(Q_1 + Q_2), \frac{1}{2} k_2 \lambda_{\min}(Q_1 + Q_2) \}$ and $k_3 > 0$, then $r(t) \rightarrow r_a$ and $\theta(t) \rightarrow \frac{\pi}{2}$ as $t \rightarrow \infty$, where $r_a = \sqrt{r_d^2 + \frac{1}{4}}$.

**Proof:** Notice from Theorem 3.2 that $r(t) \rightarrow r_a$ and $\theta(t) \rightarrow \frac{\pi}{2}$ as $t \rightarrow \infty$ by using (4). Then it is sufficient to prove the theorem if (5) and (4) become identical after a finite period of time because one can simply consider the initial time be the time after which (5) and (4) are always identical. Therefore, our focus next is to show (5) and (4) become identical in finite time under the condition of the theorem.

Define $x_1 = r$ and $x_2 \equiv \dot{r}$. By recalling the polar dynamics in (2) and the control policy (5), the derivatives of $x_1$ and $x_2$ are given by

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= V \sin(\theta) \left[ \dot{\theta} + \frac{V \sin(\theta)}{x_1} \right].
\end{align*}$$

(12)

\[\text{If the UAV is initially inside } C_d, t_e \text{ is the initial time.}\]
Let \( p \triangleq x_1 - \hat{x}_1 \) and \( q \triangleq x_2 - \hat{x}_2 \). When \( r \geq r_d \), i.e., \( x_1 \geq r_d \), it follows from (12) and (6) that
\[
\dot{p} = q - k_1 |p|^{\frac{1}{2}} \text{sgn}(p), \\
\dot{q} = V \sin(\theta) \left[ \overline{\omega} + \frac{V \sin(\theta)}{x_1} \right] - k_2 \text{sgn}(p) - k_3 p \\
= V \sin(\theta) \left[ -k_2 \cos(\frac{r_d}{r(t)}) - k_x - k_q + \frac{V \sin(\theta)}{x_1} \right] \\
- k_2 \text{sgn}(p) - k_3 p.
\]

Because \( x_2 = \dot{r} = -V \cos(\theta) \), one can obtain that \( |f(t, p, q)| \leq V(2kV + k|q| + \frac{V}{r_d}) \). By letting \( \delta_1 = V^2(2k + \frac{1}{r_d}) \) and \( \delta_2 = kV \), \( |f(t, p, q)| \leq \delta_1 + \delta_2 |q| \). The property of \( f(t, p, q) \) in Lemma 4.1 is fulfilled under the condition of the theorem. By considering the Lyapunov function \( \mathcal{V} \) in (9), it follows from the analysis in Lemma 4.1 that
\[
\dot{\mathcal{V}} \leq -\eta V^{\frac{1}{2}} 
\]
holds almost everywhere for some positive \( \eta \) that is determined by \( k, k_1, k_2, \) and \( k_3 \) under the condition of the theorem.

Now consider the case when \( r < r_d \), i.e., \( x_1 < r_d \). Because \( \overline{\omega} = 0 \), the function \( f(t, p, q) \) becomes \( \frac{|V \sin(\theta)|}{x_1} \), which is not necessarily bounded since \( x_1 \) might approach zero. The property \( |f(t, p, q)| \leq \delta_1 \) for some positive \( \delta_1 \) and \( \delta_2 \) is not necessarily satisfied. To overcome this issue, we first drop the time when the UAV is inside \( C_d \). Because zero control input is imposed when the UAV is inside \( C_d \) and \( V > 0 \), it takes a finite period of time before the UAV moves outside \( C_d \). Without loss of generality, let \( t_c \) be the time when the UAV moves inside \( C_d \) and \( t_e \) be the time when the UAV moves outside \( C_d \). Clearly, \( t_e - t_c \) is bounded by \( \frac{2r_d}{r_2} \). By only considering the case when the UAV is outside \( C_d \), the time interval \([t_c, t_x]\) is dropped. One consequence is that the state \( (p, q) \) might be different at time instants \( t_c \) and \( t_x \). The change of \( (p, q) \) at the two time points could result in a jump of \( \mathcal{V} \) from \( t_c \) to \( t_x \). If such a jump exists, it is desired that \( \mathcal{V} \) becomes smaller. Then the Lyapunov function still satisfies (13) almost everywhere and jumps to smaller at some distinct instants if the time interval \([t_e, t_x]\) is dropped.

When the UAV is initially outside \( C_d \), let the initial time be unchanged. When the UAV is initially inside \( C_d \), let the initial time be the time when the UAV moves outside \( C_d \) for the first time. Notice that \( \mathcal{V} \) in (9) can be written as
\[
\mathcal{V} = 2k_2 |p| + k_3 p^2 + \frac{1}{2} q^2 + \frac{1}{2} \left[ k_1 |p|^{\frac{1}{2}} \text{sgn}(p) - q \right]^2.
\]

When \( \dot{x}_1(t_x) = 2r_d - \dot{x}_1(t_c) \) and \( \dot{x}_2(t_x) = -\dot{x}_2(t_c) \), it can be computed that \( p(t_x) = x_1(t_x) - \dot{x}_1(t_x) = r_d - 2r_d - \dot{x}_1(t_c) = \dot{x}_1(t_c) - r_d = -p(t_c) \). Recall from (1) that \( \dot{r} = -V \cos(\theta) \) and from the proof of Lemma 5.2 in [5] that \( \theta(t_c) = \pi - \theta(t_c) \) or \( \theta(t_c) = 3\pi - \theta(t_c) \). One can obtain that \( \dot{r}(t_c) = -\dot{r}(t_c) \). Equivalently, \( x_2(t_c) = -x_2(t_c) \). It thus follows that \( q(t_c) = x_2(t_c) - \dot{x}_2(t_c) = x_2(t_c) - [-\dot{x}_2(t_c)] = q(t_c) \). From (14), it can be observed that \( \mathcal{V} \) remains unchanged when both \( p \) and \( q \) are negated. Thus, \( \mathcal{V} \) remains unchanged when the state \((p(t_e), q(t_x))\) changes to \((p(t_x), q(t_x))\) under the proposed control algorithm (5). If the finite time interval \([t_e, t_x]\) is dropped, \( \mathcal{V} \) is continuous and satisfies (13) almost everywhere. By following the analysis in Lemma 4.1, \( \mathcal{V} \) approaches zero in finite time if \([t_e, t_x]\) is excluded. This implies that \( \dot{x}_1 - r = \dot{x}_2 - \dot{r} \) go to zero in finite time if \([t_e, t_x]\) is excluded. In other words, there exists a time instant \( t^* \) such that \( r(t^*) \geq r_d, \dot{x}_1(t) - r(t) = 0, \) and \( \dot{x}_2(t) - \dot{r}(t) = 0 \) for any \( t \geq t^* \) if \([t_e, t_x]\) is excluded. That is, (5) and (4) are identical for \( t \geq t^* \) when \([t_e, t_x]\) is excluded. For \( t \in [t_e, t_x] \), (5) and (4) are identical because both of them are zero. Therefore, (5) and (4) are always identical for all \( t \geq t^* \) if we consider \( t^* \) as the initial time.

By recalling the analysis in the first paragraph of the proof of the theorem, it can be obtained that \( r(t) \rightarrow r_a \) and \( \theta(t) \rightarrow \frac{\pi}{2} \) as \( t \rightarrow \infty \).

Notice that the control algorithm (5) is based on the control algorithm (4), where \( \dot{r} \) is replaced by an estimated value \( \dot{x}_2 \). One may wonder if such a technique can be applied to the control algorithm (3) proposed in [5]. Unfortunately, the answer is no because \( |f(t, p, q)| < \delta_1 + \delta_2 |q| \) does not necessarily hold.

Remark 4.4: One unique feature of the proposed control algorithm (5) is that only range measurement is needed in its implementation. Considering the limited sensing capabilities for small UAVs due to their payload restrictions, the control algorithm (5) and the concept behind it are more applicable in small UAV operations.

V. SIMULATION

In this section, a simulation example is provided to illustrate the effectiveness of the proposed control technique. The parameters are chosen as: \( r_d = 10, [x_T, y_T] = [0, -10] \), \( k = 0.2, V = 1, k_1 = 2, k_2 = 1.2, k_3 = 1 \). The initial state of the UAV is given by \([10, -2, 5\pi/4]\) and the sliding-mode estimator is initialized as \([10, 0] \). By computation, the conditions in Theorem 4.3 are satisfied.

Fig. 2 shows the trajectory of the UAV under the control algorithm (5), where the dashed line represents the desired trajectory and the blue line represents the actual trajectory. It can be observed that the UAV eventually orbits around the target at a radius larger than the desired value. From the simulation example, the actual radius is around 11.18, which matches the theoretical result given by \( \sqrt{r_d^2 + \frac{1}{k^2}} \). Note that one may simply choose \( r_d = 8.6603 \) and the radius of the stable circular motion is then exactly 10.

Fig. 3 shows the estimated range \( \dot{x}_1 \) and the actual range \( r \) under the control algorithm (5). It can be observed that the estimated range approaches the actual range in finite time if the time interval during which the UAV is inside \( C_e \) is excluded. Fig. 4 shows the estimated range rate \( \dot{x}_2 \) and the actual range rate \( \dot{r} \) under the control algorithm (5). Again, the estimated range rate approaches the actual range rate in finite time if the time interval during which the UAV is inside \( C_e \) is excluded. One can also see from Fig. 4 that the reset mechanism in (7) is necessary to guarantee the accurate estimate of \( \dot{r} \) because otherwise the estimated range rate will
be significantly different from the actual one at the time when the UAV exits $C_d$.

VI. CONCLUSION

In this paper, we proposed a control algorithm based on range-only measurement such that a UAV can circumnavigate an unknown target at some desired distance. The design of such a control algorithm can be divided into two steps. First, a control algorithm based on range and range rate measurements was proposed to solve the circumnavigation problem. Second, a sliding-mode estimator was designed to replace the range rate measurement used in the control algorithm proposed in the first step. By choosing several parameters carefully, the sliding-mode estimator is able to accurately estimate the range rate in finite time using range measurement. Thus, the circumnavigation mission can be accomplished if the control algorithm in the first step is applied with range rate measurement being replaced by its estimated value obtained in the second step. Because the proposed control algorithm based on range-only measurement requires a minimum number of measurement for implementation, it is very suitable for small UAVs considering their limited sensing capabilities due to payload restrictions.

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