Nonlinear $H_2$ Control of Sawyer Motors

Youngwoo Lee, Donghoon Shin, Wonhee Kim, and Chung Choo Chung †

Abstract—In this paper we propose a nonlinear $H_2$ control to improve transient response in the position control and to reduce energy consumption for the position tracking of Sawyer motors. The proposed method consists of a nonlinear torque modulation with a commutation scheme and gain scheduling LPV controller. The position control problem for the Sawyer motor is reformulated into the LPV system optimal problem. The nonlinear models of the Sawyer motors are analyzed as a LPV system. However, in LPV formulation, each vertex to guarantee the controllability should be chosen because of relationship between tracking motion and yaw motion. Hence, stabilizing the inner-loop system with state feedback controller is designed to solve above problem. The control gain scheduling is determined by using $H_2$ control based on linear matrix inequality (LMI) approach. Since the proposed method is designed based on optimal control with gain scheduling based on LPV synthesis, the proposed method obtains both improved transient response and reduced energy consumption in the position control.

I. INTRODUCTION

Sawyer motors are widely used to achieve precise position control such as semiconductor manufacturing systems and other automated assembly systems. A Sawyer motor, dual-axis linear motion motor that consists of 4 forcers symmetrically mounted onto a puck, has a high-order dynamics [1]. Each motor is capable of high position resolution and high speed motion with only open-loop microstepping control. However, step-out, long settling time, and miss compensation for disturbance may be arisen in open-loop microstepping. Especially, yaw and yaw rate are closely related with the above control objectives.

Many feedback control methods have been studied for Sawyer motors [1]–[5]. Adaptive control methods have been developed to improve tracking performance and to regulate yaw and yaw rate [1], [2]. An adaptive variable structure controller was proposed to guarantee global asymptotic tracking of a reference trajectory [3]. Linear control approaches such as a proportional-integral-derivative controller and lead-lag controller were designed to improve the tracking performance of a the Sawyer motor [4], [5]. However, it is not easy to design controller to achieve improved transient response and high energy efficiency because the previous methods use static control gains. No turning guideline for the control gain adjustment was provided in the previous methods. Gain scheduling for the linear parameter-varying (LPV) system was used to improve the position tracking performance through feedback control [6]–[8]. In order to apply linear optimal control method to nonlinear system, approaches to nonlinear $H_{\infty}$ controls were proposed [9]–[11].

In this paper, we propose a new approach to position control, nonlinear $H_2$ optimal control based on LPV synthesis. The proposed method consists of a torque modulation with commutation scheme and a LPV feedback controller with gain scheduling. The position control problem for Sawyer motor is reformulated into the LPV system optimal problem. However, in LPV formulation, each vertex to guarantee the controllability should be chosen because of relationship between tracking motion and yaw motion. Hence, stabilizing the inner-loop system with state feedback controller is designed to solve above problem. Sawyer motor is analyzed as LPV system which depends on online accessible time-varying parameters which provide real-time information. A control gain scheduling is determined using $H_2$ control based on linear matrix inequality (LMI) approach. Since the proposed method is designed based on optimal control with gain scheduling based on LPV synthesis, the proposed method obtains both improved transient response and reduced energy consumption. The performance of the proposed method was demonstrated through performance comparison with conventional PID controller via simulations.

II. MODELING

Given the position and yaw, $(x, y, \theta)$ of the center of motor, let us define the position of four actuators: two per each axis with $x_1$, $x_2$, $y_1$ and $y_2$ where $x_1$, $x_2$, $y_1$ and $y_2$ are X-axis position [m] of center of the forcer $X_1$, $X_2$, $Y_1$ and $Y_2$. Then we have $x_1 = x + r_x \sin(\theta), x_2 = x - r_x \sin(\theta), y_1 = y + r_y \sin(\theta), y_2 = y - r_y \sin(\theta)$. With the variables, the dynamics of Sawyer motor can be represented as [1]–[5]

\[
\begin{align*}
\dot{x} &= x_v, \\
\dot{x}_v &= -B_x x_v + F_x - d_x \\
\dot{y} &= y_v, \\
\dot{y}_v &= -B_y y_v + F_y - d_y \\
\dot{\theta} &= \theta_v, \\
\dot{\theta}_v &= -B_\theta \theta_v + \tau - d_\theta \\
\end{align*}
\]  

(1)
\[
\begin{align*}
\dot{i}_{x1a} &= -Ri_{x1a} + \kappa \sin(\gamma x_1)(x_v + r_x \cos(\theta) \theta_v) + v_{x1a} \\
\dot{i}_{x1b} &= -Ri_{x1b} - \kappa \cos(\gamma x_1)(x_v + r_x \cos(\theta) \theta_v) + v_{x1b} \\
\dot{i}_{y1a} &= -Ri_{y1a} + \kappa \sin(\gamma y_1)(y_v + r_y \cos(\theta) \theta_v) + v_{y1a} \\
\dot{i}_{y1b} &= -Ri_{y1b} - \kappa \cos(\gamma y_1)(y_v + r_y \cos(\theta) \theta_v) + v_{y1b} \\
\dot{i}_{x2a} &= -Ri_{x2a} + \kappa \sin(\gamma x_2)(x_v + r_x \cos(\theta) \theta_v) + v_{x2a} \\
\dot{i}_{x2b} &= -Ri_{x2b} - \kappa \cos(\gamma x_2)(x_v + r_x \cos(\theta) \theta_v) + v_{x2b} \\
\dot{i}_{y2a} &= -Ri_{y2a} + \kappa \sin(\gamma y_2)(y_v + r_y \cos(\theta) \theta_v) + v_{y2a} \\
\dot{i}_{y2b} &= -Ri_{y2b} - \kappa \cos(\gamma y_2)(y_v + r_y \cos(\theta) \theta_v) + v_{y2b}
\end{align*}
\]

where

\[
F_{x1} = -\kappa \sin(\gamma x_1)i_{x1a} + \kappa \cos(\gamma x_1)i_{x1b}, \\
F_{y1} = -\kappa \sin(\gamma y_1)i_{y1a} + \kappa \cos(\gamma y_1)i_{y1b}, \\
F_{x2} = -\kappa \sin(\gamma y_2)i_{x2a} + \kappa \cos(\gamma y_2)i_{x2b},
\]

\[
\tau = r_x(F_{x1} - F_{x2}) + r_y(F_{y1} - F_{y2}), \gamma = \frac{2\pi}{p}
\]

\(x_v, y_v\) are X-axis and Y-axis velocity [m/s] of center of the motor, \(\theta_v\) is yaw rate [rad/s] of the motor, \(i_{x1a}, i_{x1b}, i_{x2a}, i_{x2b}, i_{y1a}, i_{y1b}, i_{y2a}, i_{y2b}\) are phase currents [A] in phase A and B of the forcer \(X_1, X_2, Y_1, Y_2\), \(v_{x1a}, v_{x1b}, v_{y1a}, v_{y1b}, v_{x2a}, v_{x2b}, v_{y2a}, v_{y2b}\) are input voltages [V] in phase A and B of the forcer \(X_1, X_2, Y_1, Y_2\), and \(d_x, d_y, d_\theta\) are unknown constant load forces [N] and torque [N \cdot m], respectively. \(\kappa\) is force constant [N/A] of the forcers, \(M\) is mass [kg] of the motor, \(J\) is inertia [kg \cdot m²] of the motor, \(p\) is tooth pitch of the platen, and \(r_x\) and \(r_y\) are the distance from center of the motor to center of the forcers \(X_1, X_2, Y_1, Y_2\).

### III. Nonlinear H₂ Controller Design

#### A. Error dynamics

In this section, we propose the nonlinear optimal controller with torque modulation to stabilize error dynamics. Therefore, tracking errors are defined as follows:

\[
\begin{align*}
\epsilon_x &= x^d - x, \quad \epsilon_{xv} = x_{xv} - x_v, \\
\epsilon_y &= y^d - y, \quad \epsilon_{yv} = y_{yv} - y_v, \\
\epsilon_{\theta} &= \theta^d - \theta, \quad \epsilon_{\theta v} = \theta_{\theta v} - \theta_v, \\
\epsilon_{x1a} &= i^d_{x1a} - i_{x1a}, \quad \epsilon_{x1b} = i_{x1b}^d - i_{x1b}, \\
\epsilon_{x2a} &= i^d_{x2a} - i_{x2a}, \quad \epsilon_{x2b} = i_{x2b}^d - i_{x2b}, \\
\epsilon_{y1a} &= i^d_{y1a} - i_{y1a}, \quad \epsilon_{y1b} = i_{y1b}^d - i_{y1b}, \\
\epsilon_{y2a} &= i^d_{y2a} - i_{y2a}, \quad \epsilon_{y2b} = i_{y2b}^d - i_{y2b}
\end{align*}
\]

where \(x^d, y^d, \theta^d\), and \(y_v^d\) are desired position and velocity of the axis X and Y, respectively. \(i^d_{x1a}, i^d_{x1b}, i^d_{x2a}, i^d_{x2b}, i^d_{y1a}, i^d_{y1b}, i^d_{y2a}, i^d_{y2b}\) are desired phase currents of forcer \(X_1, X_2, Y_1, Y_2\), and \(\epsilon_{xv}, \epsilon_{xv}, \epsilon_{yv}, \text{ and } \epsilon_{yv}\) are position and velocity errors of the axis X and Y, \(\epsilon_{\theta v}\) are yaw error and yaw rate error, \(\epsilon_{x1a}, \epsilon_{x1b}, \epsilon_{x2a}, \epsilon_{x2b}, \epsilon_{y1a}, \epsilon_{y1b}, \epsilon_{y2a}, \text{ and } \epsilon_{y2b}\) are current errors for the forcer \(X_1, X_2, Y_1, Y_2\). Then the error dynamics is defined by

\[
\begin{align*}
\dot{\epsilon}_x &= \epsilon_{xv}, \\
\dot{\epsilon}_{xv} &= \frac{1}{M}(M\dot{x}^d + B_x x_v + \kappa \sin(\gamma x_1)i_{x1a} - \kappa \cos(\gamma x_1)i_{x1b}) \\
&\quad + \kappa \sin(\gamma y_2)i_{x2a} - \kappa \cos(\gamma y_2)i_{x2b}) \\
\dot{\epsilon}_y &= \epsilon_{yv}, \\
\dot{\epsilon}_{yv} &= \frac{1}{M}(M\dot{y}^d + B_y y_v + \kappa \sin(\gamma y_1)i_{y1a} - \kappa \cos(\gamma y_1)i_{y1b}) \\
&\quad + \kappa \sin(\gamma y_2)i_{y2a} - \kappa \cos(\gamma y_2)i_{y2b}) \\
\dot{\epsilon}_{\theta} &= \epsilon_{\theta v}, \\
\dot{\epsilon}_{\theta v} &= \frac{1}{J}(J\dot{\theta}^d + B_\theta \theta_v + \kappa \sin(\gamma x_1)i_{x1a} - \kappa \cos(\gamma x_1)i_{x1b}) \\
&\quad - \kappa \sin(\gamma y_2)i_{x2a} + \kappa \cos(\gamma y_2)i_{x2b}) + \kappa \sin(\gamma y_1)i_{y1a} - \kappa \cos(\gamma y_1)i_{y1b}) \\
&\quad - \kappa \sin(\gamma y_2)i_{y2a} + \kappa \cos(\gamma y_2)i_{y2b})
\end{align*}
\]

To stabilize the error dynamics (4), both torque modulation for stabilization of mechanical error dynamics and input voltages for convergence of electrical errors should be developed. However, in LPV formulation, the case that system becomes uncontrollable may happen because of relationship between tracking motion and yaw motion. Hence, stabilizing the inner-loop system with state feedback controller is required to solve above problem. Therefore, we propose the nonlinear controller as follows

\[
\begin{align*}
\dot{\epsilon}_{x1a} &= -\left(\frac{M\dot{x}^d + B_x x_v + u_x}{4\kappa} + \frac{J\dot{\theta}^d + B_\theta \theta_v + u_\theta}{8\kappa r}\right) \sin(\gamma x_1), \\
\dot{\epsilon}_{x1b} &= \left(\frac{M\dot{x}^d + B_x x_v + u_x}{4\kappa} + \frac{J\dot{\theta}^d + B_\theta \theta_v + u_\theta}{8\kappa r}\right) \cos(\gamma x_1), \\
\dot{\epsilon}_{x2a} &= -\left(\frac{M\dot{x}^d + B_x x_v + u_x}{4\kappa} - \frac{J\dot{\theta}^d + B_\theta \theta_v + u_\theta}{8\kappa r}\right) \sin(\gamma y_2), \\
\dot{\epsilon}_{x2b} &= \left(\frac{M\dot{x}^d + B_x x_v + u_x}{4\kappa} - \frac{J\dot{\theta}^d + B_\theta \theta_v + u_\theta}{8\kappa r}\right) \cos(\gamma y_2).
\end{align*}
\]
\[ \dot{e}_{x} = e_{xv} \]
\[ \dot{e}_{xv} = \frac{1}{M} \left( -B_{x} \sin(y_{1}) e_{x1a} + \kappa \cos(y_{1}) e_{x1b} - \kappa \sin(y_{2}) e_{x2a} + \kappa \cos(y_{2}) e_{x2b} + u_{x} \right) \]
\[ \dot{e}_{y} = e_{yv} \]
\[ \dot{e}_{yv} = \frac{1}{M} \left( -B_{y} \sin(y_{1}) e_{y1a} + \kappa \cos(y_{1}) e_{y1b} - \kappa \sin(y_{2}) e_{y2a} + \kappa \cos(y_{2}) e_{y2b} \right) \]
\[ \dot{e}_{\theta_v} = e_{\theta_v} \]
\[ \dot{e}_{\theta_v} = \frac{1}{J} \left( -B_{\theta} \sin(y_{1}) e_{\theta1a} + \kappa \cos(y_{1}) e_{\theta1b} + \kappa \sin(y_{2}) e_{\theta2a} + \kappa \cos(y_{2}) e_{\theta2b} + u_{\theta} \right) \]

\[ \delta_{i} = \kappa \sin(y_{1}), \quad \delta_{2}(i) = \kappa \cos(y_{1}) \]
\[ \delta_{3}(i) = \kappa \sin(y_{2}), \quad \delta_{4}(i) = \kappa \cos(y_{2}) \]
\[ \delta_{5}(i) = \kappa \sin(y_{1}), \quad \delta_{6}(i) = \kappa \cos(y_{1}) \]
\[ \delta_{7}(i) = \kappa \sin(y_{2}), \quad \delta_{8}(i) = \kappa \cos(y_{2}) \]

Because trigonometric functions, \( \sin(y_{1}), \cos(y_{1}), \sin(y_{2}), \cos(y_{2}), \sin(y_{1}), \cos(y_{1}), \sin(y_{2}), \cos(y_{2}) \), are bounded, the bounded varying parameters \( \delta \) can be represented as follows

\[ \hat{\delta} = -\kappa \leq \hat{\delta} \leq \delta = \kappa \]

where \( i = 1, 2, 3, 4, 5, 6, 7, 8 \). Let us define a tracking error vector and an input vector

\[ e = [e_{M}, e_{E}]^{T}, \quad u = [u_{M}, u_{E}]^{T} \]

where \( e_{M} = [e_{x}, e_{xv}, e_{y}, e_{yv}, e_{\theta}, e_{\theta v}]^{T}, \quad e_{E} = [e_{x1a}, e_{x1b}, e_{x2a}, e_{x2b}, e_{y1a}, e_{y1b}, e_{y2a}, e_{y2b}]^{T}, \quad u_{M} = [u_{x}, u_{y}]^{T}, \quad u_{E} = [u_{x1a}, u_{x1b}, u_{x2a}, u_{x2b}, u_{y1a}, u_{y1b}, u_{y2a}, u_{y2b}]^{T}. \]

The 5th terms of (7) are designed to compensate yaw angle and for transformation from nonlinear error dynamics to linear error dynamics in Sawyer motors. By considering the nonlinear controller (5), (6), (7), the tracking error dynamics is rearranged as

\[ \dot{y}_{1a} = -\left( \frac{M}{\kappa} e_{x1a} + B_{x} e_{x1b} + \kappa \cos(y_{1}) e_{x1b} - \kappa \sin(y_{2}) e_{x2a} + \kappa \cos(y_{2}) e_{x2b} + u_{x} \right) \sin(y_{1}) \]
\[ \dot{y}_{1b} = -\left( \frac{M}{\kappa} e_{x1b} + B_{x} e_{x1b} + \kappa \cos(y_{1}) e_{x1b} - \kappa \sin(y_{2}) e_{x2a} + \kappa \cos(y_{2}) e_{x2b} + u_{x} \right) \sin(y_{1}) \]
\[ \dot{y}_{2a} = -\left( \frac{M}{\kappa} e_{x2a} + B_{y} e_{x2b} + \kappa \cos(y_{1}) e_{x2a} - \kappa \sin(y_{2}) e_{x2b} + u_{y} \right) \sin(y_{2}) \]
\[ \dot{y}_{2b} = -\left( \frac{M}{\kappa} e_{x2b} + B_{y} e_{x2b} + \kappa \cos(y_{1}) e_{x2b} - \kappa \sin(y_{2}) e_{x2b} + u_{y} \right) \sin(y_{2}) \]

\[ \dot{\theta}_{d} = \frac{1}{L} \left( \kappa \sin(y_{1}) e_{v} + \kappa \sin(y_{2}) e_{v} + \kappa \sin(y_{1}) e_{v} \right) \sin(y_{1}) \]
\[ \dot{\theta}_{d} = \frac{1}{L} \left( -\kappa \cos(y_{1}) e_{v} - \kappa \cos(y_{2}) e_{v} - \kappa \cos(y_{1}) e_{v} \right) \sin(y_{1}) \]

\[ \dot{\theta}_{d} = \frac{1}{L} \left( -\kappa \cos(y_{1}) e_{v} - \kappa \cos(y_{2}) e_{v} - \kappa \cos(y_{1}) e_{v} \right) \sin(y_{1}) \]

\[ \dot{\theta}_{d} = \frac{1}{L} \left( -\kappa \cos(y_{1}) e_{v} - \kappa \cos(y_{2}) e_{v} - \kappa \cos(y_{1}) e_{v} \right) \sin(y_{1}) \]

B. Problem statement

In this paper, we are interested in the gain scheduling controller design under the existence of parameter variations and/or output disturbances. However, there are two design considerations in the viewpoint of LPV synthesis. First, the Sawyer motors must have controllable vertex because it is influenced by the yaw and the yaw rate. Therefore, yaw compensation becomes an important control issue in dual-axis motion control system. Second, Sawyer motor has a high-order and complex nonlinear dynamics. Therefore, the nonlinear dynamics must be represented as a simple linear dynamics so that it should be tractable in real-time control implementation. It is a very challenging work that nonlinear dynamics is transformed to parameter-dependent linear dynamics in dual-axis motion control.
Therefore, the error dynamics (8) becomes an LPV system as
\begin{equation}
\dot{e} = A_e(\delta)e + B_eu \\
y = C_e e 
\end{equation}
\hspace{1cm} (13)

**B. Polytopic Parameter Vector**

Given uncertain parameter vector, \(\delta(\theta)\) can be represented as in a polytopic form
\begin{equation}
\delta(x_1,x_2,y_1,y_2) = V\xi(\delta)
\end{equation}
\hspace{1cm} (14)

where \(\xi(\delta) = [\xi_1(\delta), \xi_2(\delta), \ldots, \xi_{16}(\delta)]^T, V = [V_1 \ V_2], V_1 = \text{diag}(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8), V_2 = \text{diag}(\delta_9, \delta_10, \delta_11, \delta_12, \delta_13, \delta_14, \delta_15, \delta_16).\) Here, \(V \in \mathbb{R}^{8 \times 16}\) are vertices and \(\xi(\delta) \in \mathbb{R}^{16}\) is the convex interpolation parameter vector \((i = 1, 2, 3, 4)\). Since \(A_e(\delta)\) is a continuously differentiable function with respect to bounded \(\delta(\theta)\), error dynamics (8) can be represented as the LPV system (13).

Therefore, the polytopic decomposition is defined by
\begin{equation}
A_e^{(1)} = A_e(0) + \delta_1 A_e^{(1)}, \quad A_e^{(2)} = A_e(0) + \delta_2 A_e^{(2)} \\
A_e^{(3)} = A_e(0) + \delta_3 A_e^{(3)}, \quad A_e^{(4)} = A_e(0) + \delta_4 A_e^{(4)} \\
A_e^{(5)} = A_e(0) + \delta_5 A_e^{(5)}, \quad A_e^{(6)} = A_e(0) + \delta_6 A_e^{(6)} \\
A_e^{(7)} = A_e(0) + \delta_7 A_e^{(7)}, \quad A_e^{(8)} = A_e(0) + \delta_8 A_e^{(8)} \\
A_e^{(9)} = A_e(0) + \delta_9 A_e^{(9)}, \quad A_e^{(10)} = A_e(0) + \delta_{10} A_e^{(10)} \\
A_e^{(11)} = A_e(0) + \delta_{11} A_e^{(11)}, \quad A_e^{(12)} = A_e(0) + \delta_{12} A_e^{(12)} \\
A_e^{(13)} = A_e(0) + \delta_{13} A_e^{(13)}, \quad A_e^{(14)} = A_e(0) + \delta_{14} A_e^{(14)} \\
A_e^{(15)} = A_e(0) + \delta_{15} A_e^{(15)}, \quad A_e^{(16)} = A_e(0) + \delta_{16} A_e^{(16)} \hspace{1cm} (15)
\end{equation}

where \(A_e(0) = \begin{bmatrix} A_{11} & A_{12}(\delta_6) \\ A_{21}(\delta_6) & A_{22} \end{bmatrix}\). \(A_{12}(\delta_6)\) and \(A_{21}(\delta_6)\) are nominal matrix when \(\delta(\theta)\) has a designer defined nominal values. By convex interpolation parameters, \(\xi(\delta)\), parameter-dependent matrix \(A_e(\delta)\) is composed as follows
\begin{equation}
A_e(\delta) = \sum_{i=1}^{16} \xi_i(\delta) A_e^{(i)} \hspace{1cm} (16)
\end{equation}

where \(\sum \xi_i(\delta) = 1\), and \(\xi_i(\delta) \geq 0\) \((i = 1, 2, 3, 4)\). Using convex interpolation, we can design state feedback control law, \(u = K(\delta)e\).

**C. Constrained Polytopic Parameter**

It should be noted that a multiple set of \(\xi_i\) produce \(A_e(\delta)\). Thus we are to make an unique set \(\xi_i\) in (16). In Sawyer motor, independent equality conditions for each \(\alpha_j\) exist because all of the actuators have separated operation conditions. Therefore, below Definition 1 is available.

**Definition 1:** Let \(\delta(x_1,x_2,y_1,y_2) = V\xi(\delta), \Sigma\xi(\delta) = 1, \) and \(\xi_i \geq 0\) \((i = 1, 2, 3, 4)\). We define \(\alpha_j\) \((j = 1, 2, 3, 7, 8)\) as constrained polytopic parameters if for any \(\xi_i \geq 0\) there is \(\xi_1(\delta) + \xi_2(\delta) = \alpha_1, \xi_3(\delta) + \xi_4(\delta) = \alpha_2, \xi_5(\delta) + \xi_6(\delta) = \alpha_3, \xi_7(\delta) + \xi_8(\delta) = \alpha_4, \xi_9(\delta) + \xi_{10}(\delta) = \alpha_5, \xi_{11}(\delta) + \xi_{12}(\delta) = \alpha_6, \xi_{13}(\delta) + \xi_{14}(\delta) = \alpha_7, \xi_{15}(\delta) + \xi_{16}(\delta) = \alpha_8\) such that \(\alpha_1 + \alpha_2 + \cdots + \alpha_8 = 1\)

From Definition 1 and uncertain parameter condition (11), \(\alpha_j\) are 0.125 in Sawyer motor. Therefore, we redefine the expanded vertices matrix, \(\tilde{V}\), and uncertain parameter vector, \(\delta(\theta)\), as follows
\begin{equation}
\tilde{V} = \begin{bmatrix} 2V_1 \\ 2V_2 \end{bmatrix} \hspace{1cm} (17)
\end{equation}
\begin{equation}
\delta(x_1,x_2,y_1,y_2) = \begin{bmatrix} \delta(x_1,x_2,y_1,y_2) \end{bmatrix} = \begin{bmatrix} \delta_1(\delta) \\ \vdots \\ \delta_8(\delta) \end{bmatrix} \hspace{1cm} (17)
\end{equation}

where \(M \in \mathbb{R}^{8 \times 16}\) is expanded matrix for \(\tilde{V}\), \(s \in \mathbb{R}^{8 \times 1}\) is expanded vector to satisfy invertibility for \(\xi(\delta) = \tilde{V}^{-1}\delta(x_1,x_2,y_1,y_2)\). Each vertex is chosen to cover the equal constraints, \(\delta_{13,5,7} + \delta_{5,4,6,8} = k^2\sin^2(\gamma U) + \cos^2(\gamma U) = k^2, \gamma \in \{x_1, x_2, y_1, y_2\}\). By construction, we obtain an unique interpolation parameter vector, \(\xi(\delta)\) for the \(\tilde{V}\) and \(\delta(x_1,x_2,y_1,y_2)\). Therefore, we will formulate the position tracking problem as \(H_2\) LPV problem to find the optimal control gain in the next subsection.

**D. \(H_2\) LPV State Feedback Controller**

We need to design the optimal controller to guarantee the closed-loop stability and to track the desired position. The closed-loop LPV system in the state space form is defined as follows
\begin{equation}
\dot{e} = A_e(\delta)e + B_1w + B_2u \\
z = C_1e + D_{12}u \hspace{1cm} (18)
\end{equation}

where \(w \in \mathbb{R}^{14}\) is the exogenous disturbance signal, \(z \in \mathbb{R}^{14}\) is the objective function signal including state combination, \(B_1^T B_1 \in \mathbb{R}^{14 \times 14}\) is covariance matrix, \(C_1 \in \mathbb{R}^{14 \times 14}\) and \(D_{12} \in \mathbb{R}^{14 \times 11}\) are weighting matrices of tracking errors and control inputs, respectively. In this position tracking problem, it is important to improve energy efficiency. \(H_2\) optimal control algorithms have been used for energy optimization. The parameter-dependent \(H_2\) state feedback optimal control law for the energy minimization is as follows
\begin{equation}
u = K(\delta)e \hspace{1cm} (19)
\end{equation}

where \(K(\delta) \in \mathbb{R}^{11 \times 14}\) is feedback control law. Therefore, the closed-loop system of (18) is represented by state feedback

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>(J)</td>
<td>(4 \times 10^{-3}) kg \cdot m^2</td>
<td>(B_1, B_2, B_0)</td>
<td>(1 \times 10^{-5}) N \cdot m \cdot s/rad</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>17 N/A</td>
<td>(\gamma)</td>
<td>(2 \times \pi \times p)</td>
</tr>
<tr>
<td>(p)</td>
<td>(1.016 \times 10^{-3}) m</td>
<td>(r_x, r_y)</td>
<td>0.0485 m</td>
</tr>
<tr>
<td>(R)</td>
<td>2 (\Omega)</td>
<td>(L)</td>
<td>(7 \times 10^{-4}) H</td>
</tr>
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| TABLE I | SAWYER MOTOR PARAMETERS |

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optimal controller (19) as follows
\[
\dot{e} = (A_e(\delta) + B_eK(\delta))e + B_1w
\]
\[
z = (C_1 + D_12K(\delta))e. \tag{20}
\]
with \(\delta(\theta) = V^\xi(\delta)\). \(T_{zw}\) is transfer function from exogenous disturbance signal, \(w\), to the object function signal, \(z\). Lyapunov equations to define the \(H_2\) norm of closed-loop system (20) are as follows [12]
\[
A^T_eX + XA_e + C^T_eC_e = 0
\]
\[
A_eY + YA^T_e + B_eB^T_e = 0 \tag{21}
\]
where \(C_e\) is output matrix, \(X\) and \(Y\) are positive definite symmetric matrices from Lyapunov equation. The \(H_2\) norm of closed-loop system (20), \(\|T_{zw}\|_2\), is as follows
\[
\|T_{zw}(\delta)\|_2 = \sqrt{trace(YB_1B^T_1) + trace(D_12D^T_12K(\delta)XK(\delta)^T)}. \tag{22}
\]
For the design of state feedback optimal controller to find the optimal condition between state (tracking error) and input, minimization of \(H_2\) norm is needed. However, because \(H_2\) norm is changed by varying parameter, upper bound of \(H_2\) norm has been minimized. Therefore, solution to minimize the energy consumption is remarked below in Theorem 1.

**Theorem 1:** Consider the closed-loop LPV system (20). If there exist two symmetric matrices \(X_2, Z > 0\) and \(F(i)(i = 1, 2, \ldots, 15, 16)\) for given \(\gamma > 0\) such that

\[
\begin{bmatrix}
A_e(i)X_2 + (A_e(i)x_2)^T + B_eF(i)^T + (B_eF(i))T & X_2B_1 \\
X_2B_1^T & -I
\end{bmatrix} < 0
\]
\[
\begin{bmatrix}
Z & C_1X_2 + D_12F(i)^T \\
(C_1x_2 + D_12F(i)^T & X_2
\end{bmatrix} > 0 \tag{23}
\]

then the closed-loop uncertain system, \((A_e(\delta) + B_eK(\delta), B_1)\), is parametrically-dependent quadratically stabilizable by the state feedback \(K(\delta) = F(\delta)X_2^{-1}\) with \(F(\delta) = \sum_{i=1}^{16} \xi_i(\delta)F(i)\). Moreover the \(K(\delta)\) guarantees \(H_2\) performance, \(\|T_{zw}(\delta)\|_2 < \gamma\).

**Proof:** For the closed-loop LPV system (20), if there exists stable parameter-dependent solution, \(X\), then Lyapunov inequality is represented as follows
\[
X(A_e(\delta) + B_eK(\delta))^T + (A_e(\delta) + B_eK(\delta))X + B_1B_1^T < 0. \tag{24}
\]
Pre and post multiplying by \(X^{-1}\) produces
\[
(A_e(\delta) + B_eK(\delta))^T X_2 + X_2(A_e(\delta) + B_eK(\delta)) + X_2B_1B_1^TX_2 < 0, \tag{25}
\]
where \(X_2 = X^{-1}\). To solve the inequality, we replace bilinear term, \(K(\delta)X\), with \(F(\delta)\). The LMI condition for (25) is given by
\[
\begin{bmatrix}
A_e(\delta)X_2 + (A_e(\delta)x_2)^T + B_eF(\delta) + (B_eF(\delta))^T & X_2B_1 \\
(X_2B_1)^T & -I
\end{bmatrix} < 0. \tag{26}
\]

To require the additional condition for \(H_2\) performance, \(H_2\) constraint is given as follows in terms of the solution \(X > 0\).
\[
trace((C_1 + D_12K(\delta))X(C_1 + D_12K(\delta))^T) < \gamma^2. \tag{27}
\]
Applying Schur complement to (27) leads to below LMI conditions as follows
\[
\begin{bmatrix}
Z & (C_1 + D_12K(\delta))X \\
(C_1 + D_12K(\delta))^T & X
\end{bmatrix} > 0 \tag{28}
\]
Pre and post multiplying by
\[
\begin{bmatrix}
I & 0 \\
0 & X^{-1}
\end{bmatrix}
\]
produces
\[
\begin{bmatrix}
Z & C_1X_2 + D_12F(\delta)^T \\
(C_1x_2 + D_12F(\delta))^T & X_2
\end{bmatrix} > 0 \tag{29}
\]
\(trace(Z) < \gamma^2\).

The LMI conditions, (26) and (29), are from parameter-dependent Lyapunov inequality and \(H_2\) constraint. It is easy to show that LMIs for each vertex using (16) in (23) can be transformed into (26) and (29).

**V. SIMULATION RESULTS**

We performed simulations to evaluate the performance of the proposed method. The Sawyer motor parameters
listed in Table I were used. The used desired position profiles (ramp) are shown in Fig. 1. To verify the performance of the proposed controller, simulations for two cases (proposed gain scheduling LPV controller and fixed gain PID controller) were performed. Simulations were done using MATLAB/simulink with algebraic model. To confirm a realization of the LPV synthesis, interpolation parameters and sum of the interpolation parameters shown in Fig. 2 were used. To satisfy uncertain parameter conditions, maximum value of interpolation parameters should become 0.125 and sum of all of the interpolation parameters is 1. From these results, we could confirm that the proposed LPV controller was well implemented. The position tracking errors of two cases are shown in Fig. 3. The conventional PID controller had large overshot during the transient period, i.e. the large fixed controller gains resulted in poor transient response in the position control. On the other hand, the proposed LPV controller improved the transient response since the gain scheduling was used. Oscillation of yaw motion in PID method was considerably reduced in proposed LPV method. Note that the relationship between peak error and error convergence is trade off. We can observe that the proposed method has a efficient energy consumption compared to conventional method in Fig. 4.

VI. CONCLUSIONS

In this paper, we proposed nonlinear $H_2$ optimal control based on LPV synthesis for Sawyer motors. The proposed method consists of a new torque modulation with a commutation scheme, a state feedback LPV controller with gain scheduling. Using the LPV formulation, general framework for performance criterion was provided in Sawyer motor. Both well-regulated yaw and improved reference trajectory tracking performance was validated via simulation results. The proposed LPV controller gave us both the improved transient response and the efficient energy consumption in the position control.

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