Three Dimensional Impact Angle Constrained Guidance Law using Sliding Mode Control

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Abstract—In this paper, three dimensional impact angle control guidance laws are proposed for stationary targets. Unlike the usual approach of decoupling the engagement dynamics into two mutually orthogonal 2-dimensional planes, the guidance laws are derived using the coupled dynamics. These guidance laws are designed using principles of conventional as well as nonsingular terminal sliding mode control theory. The guidance law based on nonsingular terminal sliding mode guarantees finite time convergence of interceptor to the desired impact angle. In order to derive the guidance laws, multi-dimension switching surfaces are used. The stability of the system, with selected switching surfaces, is demonstrated using Lyapunov stability theory. Numerical simulation results are presented to validate the proposed guidance law.

I. INTRODUCTION

All practical interceptor-target engagement scenarios are three dimensional (3D) in nature. While designing the guidance law for these interceptors, the usual approach is to decouple the 3D engagement into two mutually orthogonal 2D engagements and then design the guidance laws for both the engagement planes separately. It would be close to reality if the guidance law can be designed without decoupling the engagement dynamics. In modern warfare, the objectives of the guidance laws are not limited only to interception but terminal constraints like terminal impact angles are also of paramount importance. The terminal impact angle increases the effectiveness of the warhead. In this paper, a 3-dimensional impact angle constraint guidance law, based on sliding mode control (SMC) theory [1], are proposed.

In the literature, there exist many papers that address 2-dimensional impact angle control problem [2]–[11]. In [2] and [3], a two stage proportional navigation guidance (PNG) law based impact angle control guidance law is proposed for stationary and non-maneuvering targets, respectively. In [4] and [5], the guidance laws based on biased PNG law are proposed. In [6] and [7], optimal control theory based guidance laws are presented for stationary targets. Sliding mode control, which is a nonlinear robust control technique, has also been used to design 2-dimensional impact angle guidance laws [8]–[11]. In [8], an impact angle control guidance law, which also allows to change the damping of the guidance loop, is presented for stationary targets. In [9]–[11], the sliding mode control based guidance laws which ensures the convergence of the interceptor to the desired impact angle, within a finite time, are also presented. However, only a few papers are available on achieving specified impact angle along with zero miss-distance for 3-dimensional engagements [12]–[20]. Sliding mode control theory has also been used to design 3D impact angle guidance law [12]–[15]. In [12], a three dimensional guidance law, based on variable structure and Lyapunov stability theory, is proposed for stationary as well as maneuvering targets. In [13] and [14], the 3D impact angle control guidance laws are derived using the linear optimal control theory and the sliding mode control theory. In the derivation of both of these guidance laws, the 3D engagement is first decoupled into two 2D engagements and then the guidance laws for both the planes are designed independently. In [15] also, a 3D impact angle control guidance law is proposed using SMC theory.

In the literature, there also exists a few 3D impact angle guidance law based on techniques other than sliding mode. In [16], a guidance law, based on the Lyapunov stability theory, is proposed for stationary targets. In [17], a 3D impact angle constrained guidance law, based on a geometric approach, is proposed for the non-maneuvering targets. This guidance law makes the interceptor follow a relatively circular trajectory which is determined using the concept of a reference circle on a moving coordinate frame fixed to the target. In [18], a nonlinear suboptimal impact angle control guidance law, based on model predictive static programming technique, is proposed for ground targets. In [19] and [20], a nonlinear backstepping control theory is used to design a 3D guidance law for maneuvering targets. In [20], the guidance law is designed in two steps where, in the first step, it guarantees the line-of-sight (LOS) angle in vertical plane to converge to its desired value and then in the second step, it stabilizes the system and minimizes the LOS rate. In [21], a generic PN (combination of standard PN and Retro-PN) based 3D impact angle guidance law is proposed for targets having speed advantage over the interceptor. Note that in [13], [14], [19], the guidance laws are derived using the approach of decoupled dynamics and hence, the performance of these guidance laws might degrade.

The main contribution of this paper is to develop a guidance law for three dimensional engagement geometry without decoupling its dynamics into two mutually orthogonal 2D planes. In order to ensure robustness of the guidance law, the sliding mode control theory has been used to derive it. Conventional and nonsingular terminal SMC has been used to derive the guidance laws which ensures asymptotic and finite time convergence of the interceptor to the desired
with respect to the inertial frame of reference \(X_IY_IZ_I\). With some mathematical analysis of (1), it can be shown that the above problem can also be transformed into a problem of controlling the LOS angle and their derivatives. Suppose, the LOS angles \( \theta_L \) and \( \psi_L \) and their derivatives \( \dot{\theta}_L \) and \( \dot{\psi}_L \) become \( \theta_{Lf} \), \( \psi_{Lf} \), 0 and 0, respectively. As \( \theta_L = 0 \), from (1b), we can say that \( \sin \theta_M = 0 \) and hence \( \theta_M = 0 \). Similarly, if \( \psi_L = 0 \) and assume that \( \theta_{Lf} \neq \pm (\pi/2) \) then \( \sin \psi_M = 0 \) and hence \( \psi_M = 0 \). Now, the velocity vector \( V_M \) is aligned with the line-of-sight and because of \( \theta_L = \theta_{Lf} \) and \( \psi_L = \psi_{Lf} \), the velocity vector makes the angles \( \theta_{Lf} \) and \( \psi_{Lf} \) with respect to the inertial frame of reference \(X_IY_IZ_I\).

Now, the above problem of designing an impact angle control guidance law can be restated as the problem of deriving a guidance law which guarantees the following

\[
\dot{\theta}_L = 0; \quad \theta_L = \theta_{Lf}; \quad \dot{\psi}_L = 0; \quad \psi_L = \psi_{Lf}. \tag{2}
\]

From (2), it can be seen that the problem of controlling impact angle along with interception of targets reduces to a simplified problem of controlling the LOS angles and their rates, in both the elevation as well as azimuth plane, to their desired values simultaneously.

### III. Design of the Guidance Laws

In this section, the guidance law which satisfies the above mentioned objectives will be discussed. The dynamics of \( \theta_L \) and \( \psi_L \), which has relative degree two with respect to the control inputs \( A_{yM} \) and \( A_{zM} \), can be easily obtained by some mathematical manipulations after differentiating (1b) and (1c) and then substituting (1d) and (1e). These dynamics are given by

\[
\begin{align*}
\dot{\theta}_L &= - \frac{2\dot{r}\dot{\psi}_L}{r} - \dot{\psi}_L^2 \sin \theta_L \cos \theta_L - \frac{\cos \theta_M}{r} A_{zM}, \\
\dot{\psi}_L &= - \frac{2\dot{r}\dot{\psi}_L}{r} + 2\dot{\psi}_L \dot{\theta}_L \tan \theta_L + \frac{\sin \theta_M \sin \psi_M}{r \cos \theta_L} A_{zM} - \frac{\cos \psi_M}{r \cos \theta_L} A_{yM}. 
\end{align*} \tag{3a, 3b}
\]

Note that the second terms in both equations (3a) and (3b) represents the coupling effect between the dynamics in elevation and azimuth directions. From the last term of (3b), it can also be noted that the interceptor lateral acceleration \( A_{zM} \) affects not only in the elevation direction but it also affects the dynamics in the azimuth direction. If we design the guidance laws after decoupling the engagement dynamics then it might degrade the performance of the interceptor.

In notational form, the dynamics, given by (3), can also be rewritten as

\[
\begin{bmatrix}
\dot{\theta}_L \\
\dot{\psi}_L
\end{bmatrix} = F + BU, \tag{4}
\]
where,

\[
F = \begin{bmatrix}
-2\dot{r}\dot{\theta}_L \\
-2\dot{r}\dot{\psi}_L \\
-r\cos\theta_M \\
-2\dot{r}\dot{\psi}_L \\
-r + 2\dot{\psi}_L\dot{\theta}_L\tan\theta_L \\
\end{bmatrix}
\]

(5a)

\[
B = \begin{bmatrix}
-\cos\theta_M \\
\sin\theta_M \\
\sin\theta_M \\
-\cos\psi_M \\
\cos\psi_M \\
\end{bmatrix}
\]

(5b)

Note that the control input \( U \) is multiplied by the matrix \( B \).

In order to derive the guidance laws, it is assumed that the matrix \( B \) is nonsingular during the engagement and hence, \( B^{-1} \) exists. In other words, the angles \( \theta_M \) and \( \psi_M \) are not equal to \( \pm(\pi/2) \).

### A. Guidance Laws based on Conventional Sliding Mode Control

In this subsection, the guidance law based on conventional sliding mode control is discussed. In order to design guidance law based on sliding mode control, the switching surfaces are defined as

\[
S = \begin{bmatrix}
S_1 \\
S_2
\end{bmatrix} = \begin{bmatrix}
\dot{\theta}_L + C_1(\theta_L - \theta_{Lf}) \\
\dot{\psi}_L + C_2(\psi_L - \psi_{Lf})
\end{bmatrix}
\]

(6)

where, \( \theta_{Lf} \) and \( \psi_{Lf} \) are the desired impact angles and \( C_1 \) and \( C_2 \) are positive constants. The sliding mode dynamics, which can be easily obtained by making \( S = 0 \), governs the dynamics of the system after occurrence of sliding mode in the system. From these dynamics, it can be easily seen that the errors \( \theta_L - \theta_{Lf} \) and \( \psi_L - \psi_{Lf} \) and their derivative asymptotically go to zero. To determine the equivalent control component of the lateral accelerations, the equivalent control approach [1] is used. On differentiating (6) with respect to time, we get

\[
\dot{S} = \begin{bmatrix}
\dot{S}_1 \\
\dot{S}_2
\end{bmatrix} = \begin{bmatrix}
\dot{\theta}_L + C_1(\theta_L - \theta_{Lf}) \\
\dot{\psi}_L + C_2(\psi_L - \psi_{Lf})
\end{bmatrix} = F + G + BU,
\]

(7)

where,

\[
G = \begin{bmatrix}
C_1\dot{\theta}_L \\
C_2\dot{\psi}_L
\end{bmatrix}
\]

(8)

The equivalent control component, obtained by making \( \dot{S} = 0 \), is given by

\[
U_{eq} = -B^{-1}(F + G),
\]

(9)

where,

\[
B^{-1} = \begin{bmatrix}
\frac{r}{\cos\theta_M} & 0 \\
-r\tan\theta_M\tan\psi_M & \frac{r\cos\theta_L}{\cos\psi_M}
\end{bmatrix}
\]

(10)

In sliding mode control, the controller consists of an equivalent and a discontinuous control component given by

\[
U = U_{eq} + U_{disc}.
\]

(11)

In order to design the discontinuous control component \( U_{disc} \) of the guidance law, consider the Lyapunov function candidate as

\[
V = \frac{1}{2}S^2 = \frac{1}{2}S^T S
\]

(12)

On differentiating (12) and substituting (7) and (11), we get

\[
\dot{V} = S^T \dot{S} = S^T [F + G + B(U_{eq} + U_{disc})]
\]

(13)

Substituting (9) in (13), we get

\[
\dot{V} = S^T [F + G + B\{ -B^{-1}(F + G) + U_{disc}\}]
\]

(14)

On simplifying (14), we get

\[
\dot{V} = S^T BU_{disc}.
\]

(15)

If we choose \( U_{disc} \) as

\[
U_{disc} = -B^{-1}M\text{sign}(S).
\]

(16)

where, \( \text{sign}(S) = \begin{bmatrix} \text{sign}(S_1) \\ \text{sign}(S_2) \end{bmatrix} \) and \( M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \), a positive definite matrix, that is, \( M_1, M_2 > 0 \), then

\[
\dot{V} = -(M_1|S_1| + M_2|S_2|) < 0
\]

(17)

and hence, according to the Lyapunov stability theory, sliding mode occurs. As a result \( \dot{\theta}_L \to 0 \); \( \dot{\psi}_L \to 0 \); and \( \dot{\theta}_L \to \dot{\theta}_{Lf} \); \( \dot{\psi}_L \to \dot{\psi}_{Lf} \); asymptotically and the objectives of guidance law are satisfied.

### B. Guidance Law based on Non-singular Terminal Sliding Mode Control

In the previous subsection, the guidance law was derived using conventional sliding mode and hence, after occurrence of sliding mode in the system, the errors \( \theta_L - \theta_{Lf} \) and \( \psi_L - \psi_{Lf} \) and their derivatives go to zero asymptotically. In this subsection, a guidance law which ensures finite time convergence of these errors is designed based on nonsingular terminal sliding mode theory [23]. Define the error in LOS angles as \( e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \) and their derivative as

\[
e = \begin{bmatrix}
\theta_L - \theta_{Lf} \\
\psi_L - \psi_{Lf}
\end{bmatrix}, \dot{e} = \begin{bmatrix}
\dot{\theta}_L \\
\dot{\psi}_L
\end{bmatrix}, \ddot{e} = \begin{bmatrix}
\ddot{\theta}_L \\
\ddot{\psi}_L
\end{bmatrix},
\]

(18)

where, \( \theta_{Lf} \) and \( \psi_{Lf} \) are the desired impact angles.

In order to design a sliding mode based guidance law, define the switching surfaces \( \sigma = [\sigma_1 \sigma_2]^T \) as

\[
\sigma = e + K\hat{e}^\alpha; \quad 1 < \alpha = (p/q) < 2
\]

(19)

where, \( p \) and \( q \) are positive odd integers and \( K \) is a constant coefficient design matrix and can be written as

\[
K = \begin{bmatrix}
K_1 \\
K_2
\end{bmatrix}, \text{ where } K_1, K_2 > 0
\]

(20)

and \( \hat{e}^\alpha = \begin{bmatrix} e_1^{(\alpha)} \\ e_2^{(\alpha)} \end{bmatrix}^T = \begin{bmatrix} \dot{\theta}_L^{(\alpha)} \\ \dot{\psi}_L^{(\alpha)} \end{bmatrix}^T \)

(21)

From the sliding mode dynamics, obtained by making \( \sigma = 0 \), it can be seen that this choice of switching surface provides a finite time convergence of \( e \) and its derivative to zero.
To design the sliding mode controller, the equivalent control approach [1] is used. On differentiating (19), we get
\[ \dot{\sigma} = \dot{\epsilon} + \alpha K \text{diag} [\hat{e}(\alpha - 1)] \hat{e} \]  
(22)
where,
\[ e^{(\alpha - 1)} = \begin{bmatrix} e_1^{(\alpha - 1)} \\ e_2^{(\alpha - 1)} \end{bmatrix} = \begin{bmatrix} \dot{\beta}_L^{(\alpha - 1)} \\ \dot{\psi}_L^{(\alpha - 1)} \end{bmatrix}^T. \]
(23)
Substituting (18) and (4) in (22), we get
\[ \dot{\sigma} = \dot{\epsilon} + \alpha K \text{diag} [\hat{e}(\alpha - 1)] (F + BU) \]  
(24)
In order to obtain the equivalent control component, using equivalent control approach [1], make \( \dot{\sigma} = 0 \) and hence, the equivalent control component of the sliding mode controller is given by
\[ U_{eq} = -B^{-1} \left[ F + \frac{1}{\alpha} K^{-1} \text{diag} \left[ e^{(1 - \alpha)} \right] \dot{e} \right] \]  
(25a)
\[ = -B^{-1} F - \frac{1}{\alpha} B^{-1} K^{-1} e^{(2 - \alpha)}, \]  
(25b)
where,
\[ e^{(2 - \alpha)} = \begin{bmatrix} e_1^{(2 - \alpha)} \\ e_2^{(2 - \alpha)} \end{bmatrix} = \begin{bmatrix} \dot{\beta}_L^{(2 - \alpha)} \\ \dot{\psi}_L^{(2 - \alpha)} \end{bmatrix}^T. \]  
(26)
The discontinuous control component \( U_{disc} \) is selected in the form given by
\[ U_{disc} = -B^{-1} M \text{sign} (\sigma) \]  
(27)
where,
\[ M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad M_1, M_2 > 0; \{ \text{sign} (\sigma) \}_i = \text{sign} (\sigma_i) \]  
(28)
Here, \( (.)_i \) denotes the \( i \)th element of \( (.) \).
In order to determine the bound on the gain of \( U_{disc} \), consider the Lyapunov function candidate as,
\[ V_1 = \frac{1}{2} \sigma^2 = \frac{1}{2} \sigma^T \sigma \]  
(29)
On differentiating and substituting (24) in (29), we get
\[ \dot{V}_1 = \sigma^T \dot{\sigma} = \sigma^T \left[ \dot{\epsilon} + \alpha K \text{diag} \left[ e^{(\alpha - 1)} \right] (F + BU) \right] \]  
(30)
Substituting (11), (25) and (27) in (30) and then, after some mathematical manipulations, we get
\[ \dot{V}_1 = -\sigma^T \left[ \alpha K \text{diag} \left[ e^{(\alpha - 1)} \right] M \text{sign} (\sigma) \right] \]  
(31)
Since, \( 1 < \alpha = (p/q) < 2 \) where, \( p \) and \( q \) are the positive odd integers, \( \alpha - 1 = (p - q)/q \) is a ratio of even integer and odd integer and hence, for all \( z \neq 0, z^{(\alpha - 1)} > 0 \). From (20) and (28), we know that \( K \) and \( M \) are positive definite matrices and hence, the RHS of (31) is always negative definite for \( \dot{\epsilon} \neq 0 \). According to the Lyapunov stability theory, sliding mode will occur in the system in a finite time and as a result, the objectives of interception of the target with the desired impact angle will be achieved. Note that it can be easily shown that without occurrence of sliding mode in the system, \( \dot{\epsilon} = 0 \) is not an attractor.

### IV. Simulation Results

In this section, the numerical simulation results are presented to evaluate the performance of the three dimensional impact angle control guidance laws. These results validate the claim of achieving the desired impact angle criterion under different engagement situations. To reduce chattering, the signum function in \( a_M^{disc} \) is replaced by sigmoid function with a boundary layer of 0.2. The maximum lateral accelerations, that the interceptor can provide in both the directions, are assumed to be limited as follows,
\[ a_M = \begin{cases} a_{M \text{max}} \text{sign} (a_M) & \text{if } a_M \geq a_{M \text{max}} \\ a_M & \text{if } a_M < a_{M \text{max}} \end{cases} \]  
(32)
where, \( a_{M \text{max}} = 40g \) is the maximum allowable interceptor lateral acceleration. The interceptor’s initial position is assumed to be the origin of the inertial frame of reference. The values of the initial conditions of the interceptor as well as its speed are listed in Table I. The diamond and circle markers in the subsequent figures denote the starting point of the interceptor and the target, respectively. The miss distance is defined as the distance of closest approach between the interceptor and the target. The impact angles are considered as the angle made by the velocity vector of the interceptor with respect to inertial frame of reference at the point of closest approach.

#### A. Conventional SMC based Guidance Law

In this subsection, the simulation results for the guidance law, derived using conventional sliding mode, is presented. In this case, based on several simulation trials, the constants \( C_1 \) and \( C_2 \) used to define the switching surfaces are taken as 1 and the gain \( M \) of the discontinuous control component \( a_M^{disc} \) is taken as \( M = 0.05I_2 \) where, \( I_2 \) is the identity matrix of dimension 2. The desired impact angles \( \theta_{LF} \) and \( \psi_{LF} \) are selected as \( \theta_{LF} = 65^\circ \) and \( \psi_{LF} = 45^\circ \). The simulation results for this case, as shown in Fig. 2, shows the trajectory of interceptor, deviations of switching surfaces, lateral accelerations of interceptor, heading angles of interceptor, and LOS angles. From Figs. 2(a) and (c), it can be observed that the guidance law enables the interceptor to intercept that target at desired impact angles. It can also be noted that after occurrence of sliding mode, as shown in Fig. 2(c), the LOS angles track their desired values and the LOS rates became zero. As discussed in Section II, due to the LOS rates converging to zero, the heading angles with respect to LOS frame, \( \theta_M \) and \( \psi_M \), also converges to zero as shown in Fig. 2(c). As a consequence, the velocity vector...
of the interceptor is aligned with LOS and as LOS angles have already converged to their desired value, the interceptor intercepts the target with the desired impact angles. As shown in Fig. 2(b), the interceptor initially requires high lateral accelerations in both the directions to steer it on desired collision course and then reduces to zero towards interception. The miss distance and impact angle errors in \( \theta_L \) and \( \psi_L \) are 0.76 m, 0° and 0°, respectively.

B. Nonsingular Terminal SMC based Guidance Law

In this subsection, the simulation results for the guidance law, derived using nonsingular terminal SMC theory, is presented for the engagement geometry, given by Table I. In
this case, based on several simulation trials, the design matrix $K$ and gain $M$ of the disc
are selected as $0.8I_2$ and $0.25I_2$, respectively, where $I_2$ is the identity matrix of dimension 2.

The parameters $p$ and $q$ used to define the switching surface, given by (19), are chosen as 17 and 13, respectively. Here also, the desired impact angles of $\theta_{L,J} = 65^\circ$ and $\psi_{L,J} = 45^\circ$ are selected and the results are shown in Fig. 3. It can be observed from Fig. 3 that the interceptor intercepts the target successfully and the corresponding miss distance and impact angle errors in $\theta_{L}$ and $\psi_{L}$ are 0.46 m, 0.02$^\circ$ and 0.02$^\circ$, respectively. In this case, conclusions similar to the previous case can be drawn. In addition, from Figs. 3(b) and (c), it can be observed that after occurrence of sliding mode, the LOS angles converges to zero faster than the previous case and hence also the heading angles. The faster convergence of these variables were expected due to the property of terminal SMC theory and their time of convergence can also be controlled using the design matrix $K$. But, faster is the convergence required, more is the initial interceptor lateral acceleration demanded. In this case, it can be observed from Fig 3(b) that the interceptor initially requires higher lateral accelerations than that for the previous case. So, there is a trade-off between time of convergence and lateral acceleration demand.

V. CONCLUSIONS AND FUTURE WORK

In this paper, three dimensional impact angle control guidance laws, based on conventional and nonsingular terminal SMC theory, are presented for stationary targets. The guidance law based on conventional SMC provides asymptotic convergence while the latter approach, based on terminal SMC, ensures finite time convergence of the interceptor to the desired impact angle. On the other hand, guidance laws based on the former approach requires control effort which is less than the latter one. During the design of these guidance laws, the decoupling of the 3D engagement dynamics into two mutually orthogonal planes, which might degrade its performance, is avoided. Simulations are carried out to evaluate the performances of the proposed algorithm and is shown to work well. The extension of these guidance laws to the maneuvering targets and designing guidance strategies that can avoid the singularity of matrix $B$ are the possible future direction of research. Inclusion of impact time, along with impact angle constraint, in the proposed guidance law is another important area for the future extension of this work.

REFERENCES