Application of the Fornasini-Marchesini First Model to Data Collected on a Complex Target Model*

J.E. Piou, Senior Member, IEEE, and A.J. Dumanian, Member, IEEE

Abstract—This work describes the computation of scatterers that lay on the body of a real target which are depicted in radar images. A novelty of the approach is the target echoes collected by the radar are formulated into the first Fornasini-Marchesini (F-M) state space model [1] to compute poles that give rise to the scatterer locations in the two-dimensional (2-D) space. Singular value decomposition carried out on the data provides state matrices that capture the dynamics of the target. Furthermore, eigenvalues computed from the state transition matrices provide range and cross-range locations of the scatterers that exhibit the target silhouette in 2-D space. The maximum likelihood function is formulated with the state matrices to obtain an iterative expression for the Fisher information matrix (FIM) from which posterior Cramer-Rao bounds associated with the various scatterers are derived. Effectiveness of the 2-D state-space technique is tested on static range data collected on a complex conical target model; its accuracy to extract target length is judged and compared with the physical measurements. Validity of the proposed 2-D state-space technique and the Cramer-Rao bounds are demonstrated through data collected on the target model.

I. INTRODUCTION

Two-dimensional (2-D) feature extraction from radar data collected on manmade targets is emerging as a key enabling technique for target classification. It provides coupled range, range-rate, and damping estimates of scattering centers in the row and column directions of the data set. Consequently, the scatterers can be tracked to generate very high resolution images of the target in 2-D space; these images can be mapped in 3-D by using a few snapshots of the data [2]. This method demonstrates the advantage of super-resolution techniques over conventional imaging methods such as fast Fourier transform (FFT) and back projection. A 2-D system realization method that uses enhanced matrices computed from a 2-D data set collected on a target to estimate scatterer locations was developed in [3]. The technique provides an efficient way to estimate target features and enhance the radar echoes within the observed frequency band and time interval. However, the state transition matrices that give the eigenvalues associated with the 2-D scatterer locations are computed from enhanced data matrices carried out on the rows and columns of the data set. Thus, the computational procedure to obtain the state transition matrices makes [3] costly when it is applied to large data sets.

The techniques presented in [2] and [3] demonstrate the practicality of 2-D signal processing methods to extract scatterer dynamics and target features, enhance target echoes within a frequency band and time interval, and generate high resolution images; however, no performance study of either one of the 2-D extraction techniques was considered. A performance study of space-time adaptive processing (STAP) for a single scatterer in angle and Doppler was developed [4]. However, the bounds presented in [4] are not directly applicable to the 2-D feature extraction technique, where the scattering parameters include range, range-rate, amplitude, and damping. Recently, Cramer-Rao bounds and resolution limit for scatterers from signals with constant amplitudes were presented [5]. This method appears to be suited for point-like targets but it is not directly applicable to targets such as missiles, aircrafts, ships, etc. that exhibit complex structures.

In this paper, target echoes collected by a radar are cast as the first Fornasini-Marchesini (F-M) 2-D state space model [1] from which state variables and matrices are computed without effort to obtain dynamics and locations of the scatterers embedded in the 2-D data set. The advantages of this approach over the method [3] are 1) the state matrices that provide the locations of the scatterers in the 2-D space are computed from only one enhanced data matrix and 2) the computational complexity is reduced by half. The proposed technique is applied to a set of measurements collected on a complex conical target model to extract locations of its scatterers that give rise to its complex structure and characterize its body. Posterior Cramer-Rao bounds proposed in [6] are extended to capture the state dynamics of the complex conical target and provide, for the first time, statistical performances of its key features. In addition, statistical error on the length of the target model is computed for a selected range of aspect angles.

The paper is organized in five sections. Section II describes the 2-D state space technique. Section III formulates the Cramer-Rao bounds from the 2-D state space technique. Results from static range data collected on the target model are described in Section IV, and conclusions are presented in Section V.

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J.E. Piou, and A. J. Dumanian are with Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, MA 02420 USA.
II. 2-D DATA FORMULATION

It is assumed here that the 2-D data samples \( y(m,n) \) corrupted with white Gaussian noise \( w(m,n) \) have the following form:

\[
y(l,k) = \sum_{m,n} a(m,n) p(l,k) + w(l,k); \quad l = 1, \ldots, M; k = 1, \ldots, N
\]

in matrix notation, (1) becomes

\[
Y = \begin{bmatrix}
y(1,1) & \cdots & y(1,N) \\
y(2,1) & \cdots & y(2,N) \\
\vdots & \ddots & \vdots \\
y(M-1,1) & \cdots & y(M-1,N) \\
y(M,1) & \cdots & y(M,N)
\end{bmatrix}
\]

(2)

The parameter \( G \) that appears in (1) denotes the number of scatterers embedded in the data; \( a_i \) refers to the \( i \)-th complex amplitude associated with the \( i \)-th scattering center with pole pair \( (s_i, p_i) \) and is defined by

\[
a_i = |a_i| e^{i\phi_i},
\]

(3)

where \( \phi_i \) denotes the \( i \)-th phase. The poles \( s_i \) and \( p_i \) are computed from the eigenvalues of the state transition matrices of the first F-M state space model carried on an enhanced data matrix derived from (1).

This paper presents a technique that provides direct computation of the F-M state matrices from a single enhanced data matrix, and the coupling between \( s_i \) and \( p_i \) is based on a modal decomposition carried on the state transition matrices. The F-M derived from the 2-D data samples formulated in (2) is presented in the next section.

III. FORNASINI-MARCHESINI STATE-SPACE MODEL

Let the first F-M state space model for a 2-D linear filter be defined by

\[
x_{i+1,k+1} = A_1 x_i, x_k + A_2 x_{i, k+1} + A_3 x_{i, k} + Bu_{i, k}
\]

(4)

and

\[
y_{i, k} = Cx_{i, k},
\]

(5)

where \( x_{i, k} \in \mathbb{C}^{G \times 1} \) is the state, and \( u_{i, k} \in \mathbb{C} \) denotes the input, \( y_{i, k} \in \mathbb{C}^{G \times 1} \) is the system output, \( A_1, A_2 \) and \( A_3 \in \mathbb{C}^{G \times G} \), \( B \in \mathbb{C}^{G \times 1} \) and \( C \in \mathbb{C}^{1 \times G} \) refer to constant matrices.

Based on the 2-D data matrix described by (2), the F-M state space equations defined by (4) and (5) can be used to model the data defined by (1) according to the conditions stated in the following theorem.

**Theorem 1:** The 2-D data samples represented by (1) can be modeled by the impulse response of the F-M state space system described by (4) and (5) if

\[
A_3 = -A_1 A_2; \quad A_1 A_2 = A_2 A_1,
\]

and

\[
A_1 = M_{11} A_1 M_{12}^{-1}; \quad A_2 = M_{12} A_2 M_{11}^{-1}
\]

(7)

where \( A_1, A_2 \in \mathbb{C}^{G \times G} \), \( B \in \mathbb{C}^{G \times 1} \), \( C \in \mathbb{C}^{1 \times G} \), \( \Lambda_1 \) and \( \Lambda_2 \) are diagonal matrices with the \( s_i \) and \( p_i \) as entries on their main diagonals, respectively; \( M_{12} \) is a modal matrix associated with \( A_1 \) and \( A_2 \).

**Proof:** The 2-D z-transform of (4) and (5) yield

\[
X(z_1, z_2) = (z_1 I - A_1)^{-1} (z_1 I - A_2)^{-1} BU(z_1, z_2)
\]

(8)

and

\[
Y(z_1, z_2) = C(z_1 I - A_1)^{-1} (z_1 I - A_2)^{-1} BU(z_1, z_2),
\]

(9)

respectively, while the conditions stated in (6) are met. Therefore, the impulse response of (9) allows

\[
y(l, k) = CA_1^{-1} A_2^{-1} B, \quad l > 0, k > 0.
\]

(10)

Next, use the conditions defined by (7) together with

\[
CM_{12} = [v_1, v_2, \ldots, V_G],
\]

(11)

\[
M_{12} B = \begin{bmatrix}
w_1^* \\
w_2^* \\
\vdots \\
w_G^*
\end{bmatrix}
\]

(12)

to obtain

\[
y(l, k) = \sum_{i=1}^{G} (v_i w_i^*) \phi_i^p; l = 1, \ldots, M; k = 1, \ldots, N
\]

(13)

Comparing (13) with (1), it is not difficult to see in a noise free case that

\[
a_i = v_i w_i^*.
\]

(14)

**Corollary 1:** The F-M state space model described by (4) and (5) that are associated with the 2-D data samples defined by (1) is asymptotically stable if and only if

\[
|\hat{\lambda}_i(A_1)| < 1 \quad \text{and} \quad |\hat{\lambda}_i(A_2)| < 1
\]

(15)

**Proof:** From the steps described in the proof of Theorem 1, the transfer function of the F-M state space model associated with the 2-D samples may be written as

\[
T(z_1, z_2) = C(z_1 I - A_1)^{-1} (z_1 I - A_2)^{-1} B.
\]

(16)

Thus, it is not difficult to see that the stability of the system is guaranteed while (15) is satisfied.

To compute the scatterer locations from the eigenvalues of the state space matrices \( A_1 \) and \( A_2 \), an F-M state space model derived from the enhancement of the 2-D output measurements presented in [3] is considered. The entries of the enhanced matrix follow the approach described in [7] and [8], and may be defined according to

\[
H_{l, k} = Y(l; M - J - 1 + l, k; N - L - 1 + k)
\]

(17)

where the parameters \( J \) and \( L \), respectively denote the correlation windows in row and column directions. They are heuristically chosen to be \( L = [N/2] \) and \( J = [M/2] \), where
the brackets denote the smallest integer less than or equal to the inserted quantity. For example
\[
H_{1,1} = \begin{bmatrix}
y(1,1) & \cdots & y(1,N-L) \\ 
y(2,1) & \cdots & y(2,N-L) \\
\vdots & \ddots & \vdots \\ 
y(M-J,1) & \cdots & y(M-J,N-L)
\end{bmatrix}
\] (18)

The primary interest in this section is computing the state space matrices from the 2-D data defined by the time series represented by (1). First, a meta-matrix with the entry as defined according to
\[
H_e = \begin{bmatrix}
H_{1,1} & H_{1,2} & \cdots & H_{1,J} \\ 
H_{2,1} & H_{2,2} & \cdots & H_{2,J} \\
\vdots & \ddots & \vdots & \vdots \\ 
H_{L,1} & H_{L,2} & \cdots & H_{L,J}
\end{bmatrix}
\] (19)

The first F-M state space model associated with the meta-matrix may be written as
\[
X_{t+1,k+1} = A_1X_{t,k+1} + A_2X_{t+1,k} + A_3X_{t,k} + B_1W_{t,k}
\] (20)

and
\[
H_{1,k} = C_1X_{1,k},
\] (21)

where \( X \in \mathbb{C}^{G(n-N,L)} \) is the state, \( A_1, A_2, A_3 \in \mathbb{C}^{G \times G} \), \( B_1 \in \mathbb{C}^{G(M-J)} \), and \( C_1 \in \mathbb{C}^{G(n-N-L)} \) are constant matrices with appropriate dimensions, and \( W_{t,k} \in \mathbb{C}^{(M-J) \times (N-L)} \) denotes the system input. Using the results of Theorem 1, it is not difficult to see that (10) leads to
\[
H_{1,k} = C_1A_1^{l-1}A_2^{k-1}B_1; \quad l = 1, \ldots, L; k = 1, \ldots, J.
\] (22)

The equation above indicates that the meta-matrix \( H_e \) defined by (19) can be factored. The decomposition of \( H_e \) into a product of two matrices is given by
\[
H_e = \Omega \Gamma
\] (23)

where
\[
\Omega = \begin{bmatrix}
C_1 \\
C_1A_1 \\
\vdots \\
C_1A_1^{L-1}
\end{bmatrix}
\] (24)

and
\[
\Gamma = \begin{bmatrix}
B_1 & A_2B_1 & \cdots & A_2^{L-1}B_1
\end{bmatrix}
\] (25)

\( \Omega \) and \( \Gamma \) have, respectively, the structure of the observability and controllability matrices. By computing the singular value decomposition of the meta-matrix \( H_e \) and its low rank truncation [9], the following \( G \) rank reduction of \( H_e \) is obtained according to
\[
\tilde{H}_e = U \Sigma V^*.
\] (26)

The matrix \( U_{sn} \) denotes the signal components of the left-unitary matrix \( U \), and \( \Sigma_{sn} \) is a diagonal matrix with the signal singular values of \( \tilde{H}_e \) arranged in decreasing order as entries on its main diagonal. Furthermore, \( V_{sn} \) is the signal component of the right-unitary matrix \( V \) and \( * \) denotes conjugate and transpose. Therefore, as described in [3] the observability and controllability matrices are given by
\[
\Omega = U_{sn} \Sigma_{sn}^{1/2}
\] (27)

and
\[
\Gamma = \Sigma_{sn}^{1/2} V_{sn}^*.
\] (28)

respectively. Based on the results presented in [3] and [9], the set of complex matrices \( \{A_1, A_2, B_1, C_1\} \) may be derived from \( \Omega \) and \( \Gamma \). First, the matrices computed from \( \Omega \) are given by
\[
A_1 = (\Omega_{-bf}^{*} \Omega_{-bl})^{-1} \Omega_{-bf}^{*} \Omega_{-bl}
\] (29)

and
\[
C_1 = \Omega(1: M-J,::)
\] (30)

where
\[
\Omega_{-bf} = \Omega(M-J + 1: L(M-J),::)
\] (31)

and
\[
\Omega_{-bl} = \Omega(1:(L-1)(M-J),::)
\] (32)

The matrix \( A_2 \) is computed from \( \Gamma \) according to
\[
A_2 = \Gamma_{-bf}^{-1} \Gamma_{-bl}^{*} \left( \Gamma_{-bf}^{-1} \Gamma_{-bf}^{*} \right)^{-1}
\] (33)

and where
\[
\Gamma_{-bf} = \Gamma(:, N-L+1:J(N-L)),
\] (34)

\[
\Gamma_{-bl} = \Gamma(:, 1:(J-1)(N-L)).
\] (35)

The state matrix \( B_1 \) derived from \( \Gamma \) may be written as
\[
B_1 = \Gamma(:, 1: N-L)
\] (36)

The F-M state space model derived from the enhanced data \( \tilde{H}_e \) defined by (19) to compute the signal parameters that appears in (1) is summarized below.

**Algorithm:**
1. Form the enhanced matrix from the data by using (19) then compute the state transition matrices \( A_1 \) and \( A_2 \) from (29) and (33), respectively, and \( B_1 \) and \( C_1 \) from (36) and (30), respectively.
2. Compute an eigenvalue decomposition of \( -A_3 = A_1A_2 \) to obtain the modal matrix \( M_{12} \) that can be used to “diagonalize” \( A_1 \) and \( A_2 \) according to (7).
3. Compute the eigenvalues of \( A_1 \) and \( A_2 \) then reorder them according to the angular strength of the diagonal entries of \( \Lambda_1 = M_{12}^{-1}A_1M_{12} \) and \( \Lambda_2 = M_{12}^{-1}A_2M_{12} \) (37)
4) Use a least squares of the poles onto the data to compute the amplitudes $a_i$.

It is worth mentioning that $B_1$ and $C_1$ computed above have higher dimensions then, respectively, $B$ that appears in (4) and $C$ that is defined by (5). However, the state transition matrices $A_1$ and $A_2$ from (29) and (33) can be used to derive $B$ and $C$.

Suppose that $M$ and $N$ that appear in (2) satisfy the condition $M\leq N$, by using (10), it is not difficult to see that the following equation holds

$$
\begin{bmatrix}
y(1,1) \\
y(2,1) \\
\vdots \\
y(M-1,1) \\
y(M,1)
\end{bmatrix} = \begin{bmatrix}
CB & CA_B^{M-2}B \\
CA_B & CA_B^{M-2}B \\
\vdots & \vdots \\
CA_B^{M-2} & CA_B^{M-2}B \\
CA_B^{M-1} & CA_B^{M-1}B
\end{bmatrix}
\begin{bmatrix}
y(1,M) \\
y(2,M) \\
\vdots \\
y(M-1,M) \\
y(M,M)
\end{bmatrix}
$$

(38)

The right hand side of (38) can be written as a product of three matrices according to

$$
\begin{bmatrix}
C \\
CA_B \\
\vdots \\
CA_B^{M-2} \\
CA_B^{M-1}
\end{bmatrix}
\begin{bmatrix}
O \\
I \\
\vdots \\
A_1 \\
A_1^{M-1}
\end{bmatrix}
\begin{bmatrix}
B \\
O \\
\vdots \\
A_1 \\
A_1^{M-1}
\end{bmatrix}
\begin{bmatrix}
O \\
O \\
\vdots \\
O \\
O
\end{bmatrix}
$$

(39)

The first and third meta-matrices on the right-hand side of (39) are block-diagonal matrices with $C$ and $B$ as their block diagonal entries, respectively; the entries $O$ and $I$ are, respectively, null and identity matrices with appropriate dimensions. By taking the singular value decomposition (SVD) of the data that appears on the left-hand side of (38) and performing low rank truncation, it is easy to see that

$$
\tilde{Y}_{1M} = U_{1Mon} \Sigma_{1Mon} V_{1Mon}'
$$

(40)

where $\tilde{Y}_{1M}$ represents the low rank truncation of the data matrix

$$
U_{1Mon} = \Sigma_{1Mon} V_{1Mon}
$$

(41)

where $D_c$ is the first block-diagonal matrix of the right hand side of (39) with the matrices $C$ as block diagonal elements, $W_A$ is the second matrix with the identity and products of state transition matrices as entries, and $D_B$ defines the second block-diagonal matrix with $B$ as block diagonal elements. Therefore, the following equations hold

$$
D_B = W_A^{-1} \sum_{i=1}^{M} V_{1Mon}'^* V_{1Mon}
$$

(42)

and

$$
D_c = U_{1Mon} \sum_{i=1}^{M} W_A^{-1}.
$$

(43)

The matrices $B$ and $C$ can be obtained by taking the first $G$ elements of the first column of $D_B$, and the first row of $D_c$, respectively. Because computations of $B$ and $C$ require a squared data matrix, i.e., $N=M$, many rows/columns of (2) will be discarded from the available data set; thus, $D_B$ and $D_c$ may not be full diagonal matrices. Therefore, $B$ and $C$ may not be optimum; the following state space equations are considered to formulate, in the next section, the posterior Cramer-Rao bounds.

Suppose that a sequence of data matrices similar to (2) may be represented by a set of state space equations carried in range or Doppler/cross-range direction according to

$$
z_{k+1} = A_k z_k + B_k v_k; \quad j=1,2
$$

(44)

and

$$
Y_k = C_j z_k + w_k,
$$

(45)

where $z_k \in \mathbb{C}^{Gr \times N}$ is the state vector, $A_j \in \mathbb{C}^{Gr \times Gr}$, $j=1,2$ are the state matrices of the F-M first model that satisfy (6). For $j=1$, the eigenvalues of $A_{k1}$ give the range locations of a target features, and the Doppler or cross-range locations of the target for $j=2$. The matrices $B_{jk} \in \mathbb{C}^{Gr \times N}$ and $C_{jk} \in \mathbb{C}^{M \times Gr}$ are constant and can be obtained from the data; $v_k \in \mathbb{C}^{Gr \times N}$ is a diagonal matrix with Gaussian noise samples on its main diagonal, and $w_k \in \mathbb{C}^{M \times K}$ is a disturbance input. Computing the SVD of each element of the data sequence and taking its $G$-rank truncation allows

$$
\tilde{Y}_k = U_{ks} \Sigma_{ks} V_{ks}'
$$

(46)

where $U_{ks}, \Sigma_{ks}$ and $V_{ks}$ are similarly defined as in (40). Therefore, the constant matrices $B_{jk}$ and $C_{jk}$ can be computed from (46) according to

$$
\tilde{Y}_k = C_{jk} A_{jk} B_{jk}
$$

(47)

By using (42) and (43), it is not difficult to see that

$$
B_{jk} = A_{jk}^{-1} \sum_{i=1}^{G} V_{1Mon}'^* V_{1Mon}
$$

(48)

and

$$
C_{jk} = U_{ks} \sum_{i=1}^{G} A_{jk}^{-1}
$$

(49)

It is worth noting that the data sequence $Y_k$ can be modeled by using the scatterer locations either in range, i.e, $(A_j = A_{k1})$ or range-rate/cross-range $(A_{jk} = A_{k2})$. Once the
state matrices are computed, the range extent of the target can be obtained, variances of the pole locations can be computed from the FIM derived from the Cramer-Rao bounds, and the root means square error on the target length can be estimated. The following section formulates the Cramer-Rao bounds from the state space representation defined by (44) and (45).

IV. POSTERIOR CRAMER-RAO BOUNDS

Suppose that \( v(k) \) and \( w(k) \) defined by (44) and (45), respectively, are independent white noise sequences with covariance matrices

\[
Q_k = E\{v(k)v^T(k)\}
\]

and

\[
R_k = E\{w(k)w^T(k)\},
\]

respectively; if the data matrix \( Y_k \), and \( Z_k \) the matrix formed from the state vectors are defined by

\[
Y_k = [y_0 \ldots y_k]
\]

and

\[
Z_k = [z_0 \ldots z_k]
\]

where \( y_i \in \mathbb{C}^{M \times 1} \) is the \( k \)-th column of the data matrix defined by (2) and \( z_i \in \mathbb{C}^{M \times 1} \), then the joint probability distribution function is given by

\[
p(Y_k, Z_k) = p(z_0) \prod_{j=0}^{k} p(y_j | z_j) \prod_{j=1}^{k} p(z_j | z_{j-1})
\]

where \( p(y_j | z_j) \) and \( p(z_j | z_{j-1}) \) denote conditional probability density functions. Therefore, the Fisher information matrix (FIM) \( F_k \) derived from the state vector and its associated covariance \( \text{Cov}(Z_k) = F_k^{-1} \) may be computed. Assume that the probability density function \( p(z_0) \) is known a priori, and following the steps described in [6], the sequence of posterior FIMs obey the recursion

\[
F_{k+1} = D_k^{22} - D_k^{21} (F_k + D_k^{11})^{-1} D_k^{12}
\]

where

\[
F_0 = E\left(-\frac{\partial^2}{\partial z_0^2} \log p(z_0)\right),
\]

\[
D_k^{22} = E\left(-\frac{\partial^2}{\partial z_k^2} \log p(z_k | z_{k-1})\right),
\]

\[
D_k^{21} = E\left[-\frac{\partial^2}{\partial z_0 \partial z_k} \log p(z_k | z_{k-1})\right],
\]

\[
D_k^{12} = \left[D_k^{21}\right]^T
\]

By using (54) together with the Gaussian nature of \( v_i \) and \( w_i \), it can be shown, by following the steps described in [6], that (56)-(58) can be related to the state matrices according to

\[
D_k^{11} = A_k^t (B_k Q_k B_k^t)^{-1} A_k, \quad D_k^{22} = -A_k^t (B_k Q_k B_k^t)^{-1}
\]

and

\[
D_k^{22} = (B_k Q_k B_k^t)^{-1} + C_k R_k^{-1} C_k^t.
\]

In case that only one set of data is considered, the index \( k \) that appears in (55)-(60) can be dropped off. Thus, it is not difficult to see that the state vector \( Z \) defined by

\[
Z = A_j B_j
\]

allows the eigenvalue decomposition

\[
Z = M \Lambda_j M^{-1} B_j
\]

where \( \Lambda_j \) is a diagonal matrix with the eigenvalues of \( A_j \) on its main diagonal and \( M \) is the modal matrix with the eigenvectors of \( A_j \) as entries. The variance of the states \( Z \) can be computed from the FIM defined by (55) and may be given by

\[
\text{Cov}(Z) = F_1^{-1}
\]

Let \( [\bullet]_{i,j} \) be defined as the \( i \)-th entry on the main diagonal of the inserted quantity, \( [MF^{-1}M^t]_{i,j} \) gives the variance of the right hand-side of (55). Therefore, it can be shown that

\[
\text{Var}(\lambda_i(A_j)) = \frac{[MF^{-1}M]_{i,j}}{([MB]_{i,j})^2 ([MB^t]_{i,j})^2}
\]

where \( \lambda_i(A_j) \) denotes the \( i \)-th eigenvalue of the state transition matrix \( A_j \). \( [\bullet]_{i,j} \) refers to the \( i \)-th entry on the main diagonal of the inserted quantity, \( [\bullet]_{i,j} \) represents the \( i \)-th column of the inserted matrix and \( [\bullet]_{i,j} \) denotes the \( i \)-th row of the inserted matrix. If \( A_j \) is substituted by \( A_i \) that gives the range locations of the scatterers, therefore the variance of the range locations of the scatterers can be obtained.

V. EXPERIMENTAL RESULTS

Static-range measurements taken on the 1.6 m long monocoic target model described in [2] is used to demonstrate the effectiveness of the F-M state-space technique and judge the quality of the posterior Cramer-Rao bounds.

For this experiment, focus was on a segment of 1 GHz data collected from 12–13 GHz in 40 MHZ increments and a target viewing angle ranging from −5 to 5 deg in 0.25 degree step size. Due to the presence of a slip-on ring that provides strong returns to the radar, a model order of 12 (i.e., \( G = 12 \)) is selected a priori to compute the locations of the dominant scatterers that appear in the image plane. Figure (1) shows the pole pairs \( (s_j, p_i) \) that depict the scatterer locations. Because the aspect angles for the considered data segment are known (−5° to 5°) an estimate of the target physical length that is
computed as the ratio between the projection of the target onto the range axis to the cosine of the aspects is 1.61 m. A 2-D FFT on the measurement data provides images; Figure 2(a) (truth), and the model Figure 2(b); excellent corroboration between the two images is depicted. Figures 3 (a) and (b) give, respectively, the slant range/cross-range variances of the states and locations associated with the twelve scatterers. The variances of the states and pole locations were respectively computed from (63) and (64), and were ordered from least to the greatest. It is worth noting that the in-range variances of the states and pole locations start rolling off at the eleventh dominant scatterer. The trend conveys that the model order G=12 is an appropriate choice to compute the state matrices and model the radar echoes from the target.

**Figure 1.** Locations of the 12 dominant peaks of the linear image generated from 1 GHz band data of the target model described in [1].

**Figure 2.** Linear images generated from a 2-D FFT using the 1 GHz band data collected on the monoconic target [2] a) measurements and b) model using 12 dominant scatterers.

**VI. CONCLUSIONS**

Cramer-Rao bounds for real targets that contain multiple complex scatterers have been formulated. A 2-D state space technique has been developed to obtain state transition matrices that allow computation of the bounds. Static range data collected on a complex conical target model over a 1 GHz bandwidth has been used to test the feasibility of the 2-D state-space method and judge the practicality of the Cramer-Rao bounds. It has been demonstrated that when an enhanced data matrix is used, the 2-D state-space technique provides accurate locations of the multiple complex scattering centers that are translated into small errors in the range and cross-range standard deviations derived from the Cramer-Rao bounds. Thus 2-D locations of features such as groves, rings and antenna cavities that lay on a conical target model and their associated statistical errors can be easily estimated from the proposed state-space technique and Cramer-Rao bounds.

**REFERENCES**


