Abstract—In this work, we address the connectivity maintenance problem for a team of mobile robots which move according to a given collective control objective. In our framework, the interaction among the robots is limited by a given visibility radius both in terms of sensing and communication. For this scenario, we propose a bounded control law which can provably preserve the connectivity of the multi-robot system over time even in the presence of any desired bounded control objective. Furthermore, we characterize the effects of the connectivity control term on the collective control objective, in terms of robustness of the desired control objective to the disturbance of the connectivity, by resorting to the set Input-to-State Stability framework (set-ISS). For the validation of the proposed bounded connectivity control law we consider the encirclement problem as an example of collective control objective. Simulations are provided to corroborate the theoretical results.

I. INTRODUCTION

Multi-Robot Systems (MRSs) have been a very active research field over the last three decades. Motivated by the wide range of applications which can be carried out by a team of robots, such as environmental exploration [1], search and rescue operations [2], coverage tasks [3]. Interestingly, the majority of these approaches rely on the capability to exchange information in order to perform a collaborative task. Therefore, the capability to preserve the connectivity of the network over time is of great importance.

Recently, the connectivity maintenance problem has been widely investigated by the scientific community. Generally speaking, decentralized connectivity maintenance strategies exploit locally available data to implement a control law that guarantees that, if the communication graph is initially connected, then it will stay connected as the system evolves. A measure of the connectivity of a graph is the value of the second-smallest eigenvalue of the Laplacian matrix of the graph. Consequently, several connectivity maintenance techniques have been developed based on the algebraic connectivity, as described in the recent survey paper [4].

Decentralized estimation procedures for the computation of the second-smallest eigenvalue of the Laplacian matrix were introduced in [5]–[7]. A gradient based control strategy was then introduced in [8], [9] that provably ensures connectivity maintenance. The main drawback in this control strategy is in the fact that it requires an unbounded control action: then, connectivity cannot be guaranteed in a real scenario where an upper bound on the actuation is given. Connectivity maintenance with bounded control input has been recently investigated in several works. However, to the best of the authors’ knowledge, bounded input techniques available at the state of the art address the local connectivity maintenance problem that aims at preserving over time the original set of links that define the connectivity graph [10]–[15]. Clearly, the preservation of each link of the communication graph is a very restrictive requirement which significantly limits the capability of the multi-robot system itself.

The contribution of this paper is a novel bounded control law which provably preserves the connectivity of a multi-robot system from a global point of view, that is without imposing network topology preservation. Furthermore, we consider the case where the multi-robot system is driven by a given bounded collective control objective: we consider the encirclement problem (i.e. uniformly deploying a set of robots along a circle) as a motivating example for the framework validation.

II. BOUNDED CONNECTIVITY MAINTENANCE LAW

Considering \( n \) mobile robots, we describe the communication architecture among them as an undirected graph \( G = \{V, E\} \). Each robot corresponds to a node of the graph, \( V = \{v_i : i = 1, \ldots, n\} \), and each link between two robots corresponds to an edge of the graph, \( E = \{e_{ij}\} \). Let \( N_i \) be the neighborhood of the \( i \)-th robot, i.e. the set of robots that can exchange information with the \( i \)-th one.

The communication graph can be described by means of the adjacency matrix \( A(G) \in \mathbb{R}^{n \times n} \). Each element \( a_{ij} \) is defined as the weight of the edge between the \( i \)-th and the \( j \)-th robot, and is a positive number if \( j \in N_i \), zero otherwise. Since we are considering undirected graphs, we assume \( a_{ij} = a_{ji} \). The degree matrix of the graph is defined as \( \Delta G = \text{diag}(\{\Delta_i(G)\}) \), where \( \Delta_i(G) \) is the degree of the \( i \)-th node of the graph, i.e. \( \Delta_i(G) = \sum_{j=1}^{n} a_{ij} \).

The (weighted) Laplacian matrix of the graph is defined as \( L(G) = \Delta(G) - A(G) \). Interestingly, the second smallest eigenvalue \( \lambda_2 \) of the Laplacian matrix defines the algebraic connectivity of the graph and in particular \( \lambda_2 > 0 \) if and only if the graph is connected [16]. Therefore, \( \lambda_2 \) provides...
a natural metric to measure the connectivity of the network topology.

Assume each robot $i$ is characterized by single integrator kinematics in a $d$-dimensional space as follows:

$$\dot{p}_i(t) = u_i(t), \quad p_i \in \mathbb{R}^d$$

and denote with $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{nd}$ the stacked vector of robots positions.

Consider an adjacency matrix $A(G)$ whose elements are defined as:

$$a_{ij} = \begin{cases} e^{-\frac{||p_i-p_j||}{\sigma^2}} & \text{if } ||p_i-p_j|| \leq R \\ 0 & \text{otherwise} \end{cases}$$

where $R > 0$ is the visibility range of the robots, and $\sigma > 0$ is a design parameter. Let $v_2$ be the eigenvector corresponding to the eigenvalue $\lambda_2$. From [6], we know that the $\frac{\partial \lambda_2}{\partial p_i}$ can be computed as:

$$\frac{\partial \lambda_2}{\partial p_i} = \sum_{j \in N_i} \frac{\partial a_{ij}}{\partial p_i} \left( v_2^i - v_2^j \right)^2$$

where $v_2^k$ is the $k$-th component of $v_2$.

The following connectivity maintenance control law is proposed for each robot $i$ characterized by the kinematics given in (1):

$$u_i^e = k e^{\frac{-(\lambda_2(p)+\epsilon)}{\epsilon}} \frac{\frac{\partial \lambda_2(p)}{\partial p_i}}{\left\| \frac{\partial \lambda_2(p)}{\partial p_i} \right\|}, \quad u_i^e \in \mathbb{R}^d,$$

where $\epsilon > 0$ is the desired lower-bound for $\lambda_2$, and $c, k \in \mathbb{R}^+$ are tunable positive control gains\(^1\). Details concerning the tuning of the gain $k$ will be provided in the rest of the paper.

Note that, a decentralized implementation of the control law defined in (4) requires a distributed estimation of the global quantities $\lambda_2$ and $v_2$. This can be carried out by exploiting the procedure given in [8].

**A. Connectivity Preservation with an Additional Control Term**

Consider the following dynamics for each robot $i$:

$$\dot{p}_i(t) = u_i^e(t) + u_i^c(t)$$

where the term $u_i^c(t) \in \mathbb{R}^d$ is an additional control term used to model any desired control objective for which the following holds:

$$\|u_i^c(t)\| \leq U_i, \quad \forall i \in \{1, \ldots, n\}.$$

where $U_i \in \mathbb{R}^+$.\(^2\)

We are now interested in proving that, under opportune assumptions, the connectivity can be preserved even in the presence of this additional bounded control term. To this end, define $U_m \in \mathbb{R}^+$ as:

$$U_m = \max_{i=1,\ldots,N} \{U_i\}$$

\(^1\)The dependence of $\lambda_2$ from the stacked vector of robot positions $p$ will be dropped in the sequel when not strictly required for the sake of readability.

\(^2\)Note that we replaced the notation $\dot{p}_i(t) = u_i^c(t)$ with $\dot{p}_i(t) = f_i^c(p)$ to emphasize the fact we are now considering the control objective as the dynamics of the system and the connectivity control as a disturbance on it.

**Lemma 1** Consider the dynamics of the system given in (5) and the condition given in (7). Assume the initial value of the algebraic connectivity to be $\lambda_2 > \epsilon$. Then, if

$$k > U_m$$

the control law $u_i^c$ ensures that $\lambda_2$ never goes below $\epsilon$ as the system evolves.

**Proof:** Consider a continuously differentiable function $V(p) : \mathbb{R}^{nd} \rightarrow \mathbb{R}^+$ defined as follows:

$$V(p) = e^{\frac{-(\lambda_2(p)+\epsilon)}{\epsilon}}.$$

for which the time derivative is:

$$\dot{V}(p) = \nabla_p V(p)^T \dot{p} = \sum_{i=1}^{N} \frac{\partial V^T}{\partial p_i} \dot{p}_i$$

$$= \sum_{i=1}^{N} \left[ -\frac{1}{c} e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} \right]_{T} + k e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} + u_i^o$$

$$\leq 1 \frac{(\frac{\lambda_2}{\epsilon})}{c} \sum_{i=1}^{N} \left[ \frac{1}{c} e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} \right] + k e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} + U_1$$

$$\leq 1 \frac{(\frac{\lambda_2}{\epsilon})}{c} \sum_{i=1}^{N} \left[ \frac{1}{c} e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} \right] + k e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} + U_m$$

It follows that the time derivative $\dot{V}(p)$ is negative, and thus the algebraic connectivity increases, if $\sum_{i=1}^{N} \left[ \frac{1}{c} e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} \frac{\partial \lambda_2}{\partial p_i} \right] \neq 0$ and the following holds:

$$-k e^{\frac{-(\lambda_2+\epsilon)}{\epsilon}} + U_m < 0$$

Which is:

$$\lambda_2 < \epsilon + c \cdot \log \left( \frac{k}{U_m} \right)$$

At this point since $\log \left( \frac{k}{U_m} \right) > 0$ the thesis follows.

**B. Control Objective Performance**

So far we have shown that the connectivity can be preserved with a bounded control action by assuming the additional control term to be bounded itself. By switching perspective, we now focus on the effects of the connectivity control term on the actual control objective of the multi-robot system. To this end, consider again the integrated dynamics for each robot $i$:

$$\dot{p}_i(t) = f_i^o(p) + u_i^c(t)$$

with $f_i^o(p) : \mathbb{R}^{nd} \rightarrow \mathbb{R}^d$ the desired control objective. More precisely, we can consider the term $\dot{p}_i(t) = f_i^o(p)$ as the actual system dynamics\(^2\) and the connectivity control term $u_i^c(t)$ as a bounded disturbance injected into the system.
Therefore, the set Input-to-State Stability (set-ISS) seems to be a very natural framework to investigate the robustness of the system against this “disturbance”. The reader is referred to [17] from an overview of the set-ISS framework. In particular, we seek for a relationship between some desired overall performance parameters of the system (e.g. cohesiveness of the swarm, shape of the formation, encirclement of a target) and the tunable control gain of the connectivity control term.

Consider the multi-robot system described as:

\[ \dot{p} = f(p, u) = f^o(p) + u^c(t), \]  

(\Sigma)

with \( f^o(p) = [f_1^o(p)^T \ldots f_n^o(p)^T]^T \) the stacked vector of the robots control objective and \( u^c(t) = [u^c_1(t)^T \ldots u^c_n(t)^T]^T \) the stacked vector of robots connectivity control term. Assume the system (\Sigma) to be set-ISS with respect to a non-empty set \( \mathcal{A} \subseteq \mathbb{R}^n \). This implies that:

\[ \|p(t, p(0), u^c)\|_\mathcal{A} \leq \beta(\|p(0)\|_\mathcal{A}, t) + \gamma(\|u^c\|_\infty), \]  

(14)

where \( p(0) \) is the initial value of \( p(t) \), \( \beta \) is a class \( \mathcal{K} \) function, \( \gamma \) is the \( \mathcal{K} \) asymptotic gain with respect to \( \mathcal{A} \) and \( \|u^c\|_\infty = \sup \{\|u^c_i(t)\|, t \geq 0\} \). The concept of ISS Lyapunov function can be introduced to study the set ISS property of a system with respect to a set \( \mathcal{A} \subseteq \mathbb{R}^n \). The original result introduced in [19] exists in generally equivalent forms where there are characterizations ranging from \( V \) being smooth [20], to Lipschitz [21], or even locally Lipschitz [22]. The following variant is instrumental for the analysis provided in this work.

**Definition 1** Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function such that:

\[ \alpha_1(\|p\|_\mathcal{A}) \leq V(p) \leq \alpha_2(\|p\|_\mathcal{A}), \]  

(15)

\[ \frac{\partial V}{\partial p} f(p, u) \leq -\alpha_3(\|p\|_\mathcal{A}), \quad \forall \|p\|_\mathcal{A} \geq \chi(\|u^c\|_\infty) \]  

(16)

\( \forall (t, p, u^c) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \), where \( \alpha_1, \alpha_2, \chi \) are class \( \mathcal{K} \subseteq \mathcal{K} \) functions, and \( \alpha_3 \) is a continuous positive definite function. Then the system (\Sigma) is input-to-state stable with respect to the set \( \mathcal{A} \) with asymptotic gain \( \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \chi \), with \( \circ \) the function composition operator.

It should be noticed that, if we consider the autonomous system obtained by zeroing the input:

\[ \dot{p} = f(p, 0) = f^o(p), \]  

(\Sigma_0)

then eq. (14) reduces to

\[ \|p(t, p(0), 0)\|_\mathcal{A} \leq \beta(\|p(0)\|_\mathcal{A}, t). \]  

(17)

which implies the asymptotic stability of the autonomous system (\Sigma_0) with respect to \( \mathcal{A} \) in the sense of [23]. It follows that a measure of the disturbance of the connectivity control term with respect to the desired control objective can be given in terms of magnitude of the asymptotic gain \( \gamma \), which in turns depends upon the magnitude of the connectivity control term. In the following, we provide a general analysis by which an upper bound on the disturbance introduced by the connectivity control term is derived.

In particular, the magnitude of the connectivity maintenance control term depends upon the algebraic connectivity \( \lambda_2 \) and the control gain \( k \) as follows:

\[ \|u^c_i\| = k e^{(-\lambda_2(p+i))} \left\| \frac{\partial \lambda_2(p)}{\partial p_i} \right\| \]  

(18)

We now characterize the lowest algebraic connectivity for which the robots fit a given non-empty set \( \mathcal{A} \in \mathbb{R}^n \). Note that the value of the algebraic connectivity \( \lambda_2 \) depends upon the number of edges and the weights of the Laplacian matrix. Therefore, to derive a lower bound on its value with respect to \( \mathcal{A} \), we should consider the connected graph with the minimum number of edges, i.e. \( n - 1 \), and the smallest value of the weights which is inversely proportional to the inter-robot distance. By assuming a \( R \)-disk graph to model the inter-robot visibility, it follows that the configuration with the smallest algebraic connectivity is the one formed by a line with the maximum inter-robot distance which fits a \( \mathcal{A} \). Denote with \( \lambda_2 \) the value of the algebraic connectivity corresponding to this configuration, and assume \( \lambda_2 > \epsilon \).

Since \( \lambda_2 \geq \lambda_2 \) within the set \( \mathcal{A} \), according to eq. (18) it is possible to conclude that the maximum value of the connectivity maintenance control action inside the set \( \mathcal{A} \) can be expressed as a function of the control gain \( k \) as follows:

\[ \|u^c_i\| \leq k e^{(-\lambda_2(p+i))} \]  

(19)

Therefore, an upper bound on the maximum magnitude of the connectivity control term is given by coupling the inequality in (8) with the inequality in (19), that is:

\[ \max_{i \in V} \{\|u^c_i\|\} < k e^{(-\lambda_2(\epsilon+i))} \]  

(20)

It follows that for any given objective control term with related \( \mathcal{K} \) asymptotic gain \( \gamma \), an upper bound on the disturbance \( \xi \) due to the connectivity control term is:

\[ \xi < \gamma k e^{(-\lambda_2(\epsilon+i))} \]  

(21)

Note that the bound given in (21) is insightful as it relates the disturbance due to the connectivity control term to the algebraic connectivity within the set \( \mathcal{A} \). In particular, it suggests that the control scheme given in (\Sigma) is particularly suitable for all those applications for which a high value of the algebraic connectivity is expected at the equilibrium.

**III. ENCIRCLEMENT: A CASE STUDY**

In this section we consider the encirclement problem in a 2-dimensional space as a case study to investigate the effects of the connectivity control term on the collective control objective. Briefly speaking, the encirclement problem is the problem of uniformly deploying a set of robots along a circumference of radius \( r_e \).

1 For a given non-empty set \( \mathcal{A} \) the configuration with \( n - 1 \) edges and maximum inter-robot distance \( d_{ij} \) can be computed by solving an optimum optimization problem.

2 According to Lemma 1 if the initial configuration is such that \( \lambda_2 > \epsilon \) this condition is enforced over time by the connectivity control law.
Let \( p_i = [p_{xi}, p_{yi}]^T \) be the cartesian coordinate of the robot \( i \) which are related to the polar coordinates \( (r_i, \theta_i) \) by the following relationship:

\[
p_i = r_i \mathbf{R}(\theta_i) \mathbf{1}_2
\]

with

\[
\mathbf{R}(\theta_i) = \text{diag}([\cos(\theta_i), \sin(\theta_i)]).
\]

The encirclement problem can be solved exploiting the following control law:

\[
\dot{p}_i^x = r_i \cos(\theta_i) - \sin(\theta_i) \dot{\theta}_i
\]

\[
\dot{p}_i^y = r_i \sin(\theta_i) + \cos(\theta_i) \dot{\theta}_i
\]

where the dynamics \( \dot{r}_i \) and \( \dot{\theta}_i \) are defined as:

\[
\dot{r}_i = -k_r \text{ sat}(r_i - r_e)
\]

\[
\dot{\theta}_i = -k_\theta \text{ sat}(\theta_i + 1 - \theta_t)
\]

where \( k_r, k_\theta \in \mathbb{R}^+ \) are positive control gains and \( \text{sat}_x(\cdot) \) and \( \text{sat}_\theta(\cdot) \) are smooth saturation functions, appropriately defined based upon the characteristics of the actuators of the robots. Note that, for the sake of simplicity the relative ordering of the robots along the circumference is defined a priori as denoted by the control law on the angular quantities. Clearly, the connectivity of the network topology must be preserved over time in order to implement such a control law.

Furthermore, the control law given in (24) can be expressed in vector form as:

\[
\dot{p}_i = \dot{r}_i \mathbf{R}(\theta_i) \mathbf{1}_2 + \dot{\theta}_i \dot{\mathbf{R}}(\theta_i) \mathbf{1}_2
\]

with

\[
\dot{\mathbf{R}}(\theta_i) = \text{diag}([-\sin(\theta_i), \cos(\theta_i)]).
\]

Consider now the following set:

\[
\bar{A} = \{ p_i \in \mathbb{R}^2 : \| x_i \| = r_e \}
\]

and the related metrics \( \| p_i \|_{\bar{A}} \) defined as:

\[
\| p_i \|_{\bar{A}} = \begin{cases} \| p_i \| - r_e & \text{if } \| p_i \| \geq r_e \\ r_e - \| p_i \| & \text{if } \| p_i \| < r_e \end{cases}
\]

Consider now the presence of the connectivity maintenance control action. In this case, the control law in eq. (26) is modified as follows:

\[
\dot{p}_i = \dot{r}_i R(\theta_i) \mathbf{1}_2 + \dot{\theta}_i \dot{R}(\theta_i) \mathbf{1}_2 + u_i(t)
\]

This control will now be shown to be set-ISS, with respect to the set \( \mathcal{A} \) defined in eq. (30), according to Definition 1. For this purpose, we will consider the dynamics of the system in a small neighborhood of \( \mathcal{A} \), where the algebraic connectivity is sufficiently big and, consequently, the connectivity maintenance control law is dominated by the encirclement control law.

**Lemma 2** Consider the saturated encirclement control law defined in eq. (32), and the set \( \mathcal{A} \) defined in eq. (28). Then, the system is set-ISS according to Definition 1.

**Proof:** Consider now the following Lyapunov function:

\[
V(p) = \sum_{i=1}^n V_i = \frac{1}{2} \sum_{i=1}^n \| p_i \|_{\mathcal{A}}^2
\]

We will first consider the case \( \| p_i \| \geq r_e \). In this case, considering the control law in eq. (32), it follows that the time derivative of the Lyapunov function can be computed as follows:

\[
\dot{V}(p) = \sum_{i=1}^n \frac{\| p_i \|_{\mathcal{A}}}{\| p_i \|} p_i^T \dot{p}_i
\]

\[
= \sum_{i=1}^n \frac{\| p_i \|_{\mathcal{A}}}{\| p_i \|} \left[ (r_i \mathbf{R}(\theta_i) \mathbf{1}_2 + \dot{\theta}_i \dot{\mathbf{R}}(\theta_i) \mathbf{1}_2) + (p_i^T u_i) \right]
\]

\[
= \sum_{i=1}^n \frac{\| p_i \|_{\mathcal{A}}}{\| p_i \|} \left[ r_i \mathbf{R}(\theta_i) \mathbf{1}_2 + \dot{\theta}_i \dot{\mathbf{R}}(\theta_i) \mathbf{1}_2 + p_i^T u_i \right]
\]

\[
= \sum_{i=1}^n [-k_r \text{ sat}_x(\| p_i \|_{\mathcal{A}}) + \sum_{i=1}^n \| p_i \|_{\mathcal{A}} \text{ sat}_x(\| p_i \|_{\mathcal{A}}) + n \| u_i \|_{\infty} \| p_i \|_{\mathcal{A}}
\]

where the fact \( \mathbf{R}(\theta_i) \mathbf{1}_2 \perp \dot{\mathbf{R}}(\theta_i) \mathbf{1}_2 \) has been used.

The Lyapunov derivative can then be upper bounded as follows:

\[
\dot{V}(p) \leq -k_r \text{ sat}_x \left( \sum_{i=1}^n \| p_i \|_{\mathcal{A}}^2 \right) + \sum_{i=1}^n \| p_i \|_{\mathcal{A}} \| p_i \|_{\mathcal{A}} \| u_i \|_{\infty}
\]

\[
\leq -k_r \text{ sat}_x \left( \| p \|_{\mathcal{A}}^2 \right) + \sum_{i=1}^n \| p_i \|_{\mathcal{A}} \| u_i \|_{\infty}
\]

\[
\leq -k_r \text{ sat}_x \left( \| p \|_{\mathcal{A}}^2 \right) + n \| u \|_{\infty} \| p \|_{\mathcal{A}}
\]

Let us now find a continuous positive function \( \tilde{\alpha}_3 \) as required by Definition 1 for the case \( \| p_i \| \geq r_e \). In particular, the following upper bound holds:

\[
\dot{V}(p) \leq -\frac{1}{2} k_r \text{ sat}_x \left( \| p \|_{\mathcal{A}}^2 \right)
\]

if the following condition is satisfied:

\[
-\frac{1}{2} k_r \text{ sat}_x \left( \| p \|_{\mathcal{A}}^2 \right) + n \| u \|_{\infty} \| p \|_{\mathcal{A}} < 0
\]
This condition is satisfied if the following holds:

\[
\frac{1}{2} k_r \text{sat}_r \left( \frac{\|p\|^2_\mathcal{A}}{\|p\|_\mathcal{A}} \right) > n \frac{\|u^c\|_\infty \|p\|_\mathcal{A}}{k_r} \quad \text{sat}_r \left( \frac{\|p\|^2_\mathcal{A}}{\|p\|_\mathcal{A}} \right) = 2 \frac{n \|u^c\|_\infty}{k_r}
\]  

(38)

As we are considering the dynamics of the system in a neighborhood of \( \mathcal{A} \), we assume to be in the range of linearity of the saturation function. Hence, the condition in eq. (38) can be rewritten as follows:

\[
\|p\|_\mathcal{A} > 2 \frac{n \|u^c\|_\infty}{k_r}
\]

(39)

Consider now the case \( \|p_i\| < r_e \). In this case, considering the control law in eq. (32), it follows that the time derivative of the Lyapunov function can be computed again as in eq. (34). This implies that upper bounds can be derived following the same arguments as in the previous case.

Therefore, Definition 1 can be applied by considering:

\[
\begin{align*}
\alpha_1(\|p\|_\mathcal{A}) &= \alpha_2(\|p\|_\mathcal{A}) = \sum_{i=1}^{n} \frac{1}{2} \|p_i\|^2_\mathcal{A} \\
\alpha_3(\|p\|_\mathcal{A}) &= \frac{1}{2} k_r \text{sat}_r \left( \frac{\|p\|^2_\mathcal{A}}{\|p\|_\mathcal{A}} \right) \\
\chi(\|u^c\|_\infty) &= 2 \frac{n \|u^c\|_\infty}{k_r}
\end{align*}
\]

(40)

from which it follows that the asymptotic gain is \( \gamma = \chi \).

It is worth remarking that, from Lemma 2, the asymptotic stability of the set \( \mathcal{A} \) follows, when the connectivity control term is zero, that is considering only the encirclement control strategy in eq. (24).

The characteristics of the set \( \mathcal{A} \) defined in eq. (30) will now be derived, computing the smallest value \( \lambda_2 \) for the algebraic connectivity such that all the robots fit \( \mathcal{A} \) while creating a connected graph. To this end, let us recall that the set \( \mathcal{A} \) represents the circle of radius \( r_e \). Therefore, the lowest algebraic connectivity is achieved when the robots are evenly distributed, that is when the inter-robot distance is maximized, and the number of edges is minimized. Interestingly, this represents the steady state of the proposed control law: therefore it follows that the smallest value of the algebraic connectivity is achieved when the robots achieve the steady state configuration. For the sake of simplicity, let us assume the radius \( r_e \) of the encirclement for a system with \( n \) robots with visibility radius \( R \) to be:

\[
r_e < \frac{R}{2 \sin \left( \frac{\pi}{n} \right)}
\]

(41)

It follows that the network topology at the steady state can be described by a ring topology and the related Laplacian matrix is a tridiagonal matrix for which the eigenvalues can be written in a closed form [25]. In particular, the algebraic connectivity is:

\[
\lambda_2 = 2e^{-\left( \frac{2 r_e \sin \left( \frac{\pi}{n} \right)}{\pi} \right)^2} \left( 1 - \cos \left( \frac{2 \pi}{n} \right) \right)
\]

(42)

It follows that according to (21) an upper bound on the disturbance \( \xi \) due to the connectivity control term for the considered encirclement problem is:

\[
\xi < \gamma \left( k e^{\left( \frac{-\lambda_2 \gamma^2}{\pi} \right)} \right)
\]

(43)

Simulations have been carried out to corroborate the theoretical results. We considered a system composed of 8 robots for which the visibility \( R \) was fixed to 2.6m and \( r_e \) was fixed to 2.7m to satisfy (41). The gains \( k_r \) and \( k_o \) in (25) were chosen unitary, while the \( \text{sat}_r \) and \( \text{sat}_\theta \) functions were chosen respectively to 2 and \( \pi \). Moreover, the following parameters were used: \( \epsilon = 0.02 \), \( c = 0.011 \), \( k_r = 1 \). According to eq. (42), the value of the algebraic connectivity obtained with this parameter set is \( \lambda_2 = 0.0693 \).

According to eqs. (24) and (25), the upper bound on the encirclement control law is \( U_m = 2 + \pi \). Hence, since the gain \( k \) of the connectivity maintenance control action is to be chosen such that \( k > U_m \), as demonstrated in Lemma 1, then we impose \( k = 5.5 \).

To better understand the disturbance introduced by the connectivity control term on the collective control objective, we consider an initial configuration for which the objective could be achieved while remaining connected even without introducing the connectivity control term. In particular,
Encirclement without Connectivity Control
Encirclement with Connectivity Control

Fig. 2. Convergence Results: Encirclement without connectivity (solid line), and Encirclement with connectivity (dashed line).

Fig. 1(a) shows the initial configuration, while Figs. 1(b) and 1(c) depict the final configuration without an with the connectivity control.

Fig. 2 shows the distance of the stacked state vector $p$ as defined in (31) from the (cartesian product) set $A = \mathcal{A} \times \ldots \mathcal{A}$ in the case without (solid line) and with (dashed line) the connectivity control term, respectively. It can be noticed how in the second case this distance does not go to zero asymptotically due to the disturbance introduced by the connectivity control term: in particular, the final value of the error is $\|p\|_A = 0.1265$. It is possible to show that this value is consistent with the upper bound on the disturbance derived in eq. (43). In fact, considering the asymptotic gain $\gamma(\cdot) = \frac{2n(\cdot)}{k}$ then the upper bound on the disturbance can be computed as: $\xi < 2 \cdot 8 \cdot 5.5 \cdot e^{-0.0493} = 0.9955$. Then, the final value of the error obtained in the simulation, that is $\|p\|_A = 0.1265$, satisfies the condition $\xi <= 0.9955$ given above.

IV. CONCLUSION

In this work we addressed the connectivity maintenance problem for a team of mobile robots which move accordingly to a given collective control objective. We provided a bounded connectivity control law and we proved that the connectivity is preserved over time even in the presence of any desired bounded control objective. In addition, we provided a theoretical analysis of the effect of the connectivity control term on the collective control objective, in terms of robustness of the desired control objective to the disturbance of the connectivity, by exploiting the ISS framework. We considered the encirclement control law as an example of collective control objective to corroborate the theoretical findings. Future work will be mainly focused on an experimental validation of the proposed bounded connectivity control law.

REFERENCES


