Stability Analysis of Piecewise Affine Systems with Sliding Modes

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Abstract—This paper proposes new sufficient conditions for stability analysis of Piecewise Affine (PWA) systems. The conditions are based on a convex combination of Piecewise Quadratic (PWQ) Lyapunov functions and are given in terms of Linear Matrix Inequalities (LMIs), which can be solved efficiently using available software packages. There are three contributions of the new conditions presented in this paper. First, the conditions guarantee exponential stability of the state dynamics even in the presence of non-destabilizing sliding modes of all possible dimensions smaller than the dimension of the state space. Second, the conditions can handle the important case where the equilibrium point is located at a boundary between affine subsystems. Third, sufficient conditions for stability of systems independently of the parametrization of the boundary surfaces are derived as a corollary. The new method presented in this paper leads to a unified methodology for stability analysis of switched affine systems and piecewise affine systems with sliding modes.

I. INTRODUCTION

The problem of stability analysis for hybrid and switched affine systems has received considerable attention over the past two decades. Several approaches to construct Lyapunov functions and provide sufficient conditions for stability are now available in the literature (see for instance the surveys [1], [2]). Considering the case of switched affine systems, the use of PWQ Lyapunov functions is an interesting approach to reduce conservativeness compared to a quadratic Lyapunov function. However, it is a common misunderstanding in the literature to believe that if there is a continuous PWQ function that is positive definite and decreasing with time along each vector field of a PWA system then the system is stable (see [3] for details). Even Piecewise Linear (PWL) systems composed exclusively by stable subsystems can become unstable in the presence of a sliding mode [3], [4].

In continuous-time systems, sliding modes are a well understood phenomenon [5] that plays an important theoretical role as a mathematical model of complex dynamics found in many practical applications [6]. The analysis of sliding modes can be quite complex, and for this reason, it is rare to find methodologies considering the cases where sliding modes exist. Importantly for stability analysis of PWA systems with attractive sliding modes are found in [7], [4], [3]. It is proposed in [7] to add the sliding dynamics to the modes of the system. However, this needs a-priori information about the sliding modes, which is typically hard to get. In [4, p.64], an extra condition is introduced to extend the analysis to systems with attractive sliding modes. However, the conditions are never satisfied for the case where the origin belongs to a boundary between affine subsystems. In [3], stability is verified without the need of a-priori information about the sliding modes. However, systems containing sliding modes are treated only by using common Lyapunov functions and the conservativeness introduced by not using PWQ Lyapunov functions requires the use of common Lyapunov functions of higher degree. Furthermore, the case where the equilibrium point is located at a boundary between affine subsystems has not been considered for PWA systems before in the literature, excluding a wide range of important cases from the analysis. For instance, applications where state-dependent surfaces are designed for tracking references that are not the equilibrium point of any of the subsystems [9], such as power electronic converters, cannot be analyzed by current existing methods.

This paper presents novel sufficient conditions for stability of PWA systems considering the presence of sliding modes. The results guarantee global exponential stability of the state dynamics even if not destabilizing sliding modes occur along any switching surface of the system and even if the origin is located in a boundary between affine subsystems. The new method combines ideas from two approaches, stability analysis of PWA systems in [3] and stabilization of switched affine systems with sliding modes [9], providing a unified theory for both systems. PWA systems are a particular class of state-based switched affine systems where the boundaries are fixed. The conditions are based on a convex combination of PWQ Lyapunov function and are formulated as LMIs. The method can handle PWA systems with discontinuous vector fields, which may lead to the existence of sliding modes involving any number of subsystems. However, there is no need for a-priori information about the sliding modes. If the conditions are satisfied for a system that contains sliding modes, then the sliding modes are guaranteed to be stable. As a by-product, sufficient conditions for stability for any possible switching surfaces are derived as a corollary, allowing to check stability independently of the complexity of the boundary.

The paper is organized as follows. Section II is devoted to some preliminaries and definitions. The main results are presented in Section III. Section IV is devoted to illustrate the paper contributions through some numerical examples. Finally, some concluding remarks end the paper.

Notation. \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space.
\( \mathbb{R}^{n \times m} \) is the set of \( n \times m \) real matrices. \( \mathbb{I}_q \) denotes the set of integers \{1, \ldots, q\}. \( \| \cdot \| \) stands for the euclidian norm of vectors and its induced spectral norm of matrices. Block matrix terms that can be deduced from symmetry are represented by \( \ast \). The \( i \)-th row of a matrix \( M \) is represented by \( \text{row}_i(M) \). \( \theta_{p \times q} \) denotes a \( p \times q \) matrix of zeros. \( I_r \) is the \( r \times r \) identity matrix. For a real matrix \( M, M' \) denotes its transpose and \( M > 0 \) \((M < 0)\) means that \( M \) is symmetric and positive-definite (negative-definite). The symbol \( \triangleright \) \((\preceq)\) is used for strict (non-strict) element-wise inequalities. \( \nabla f(.) \) denotes the gradient of a function \( f(.) \). The symbol \( \otimes \) denotes the Kronecker product. \( \varrho(\Pi) \) represents the set of all vertices of a given polytope (or simplex) \( \Pi \). For a set \( \mathcal{R}, \mathcal{R}' \) denotes its closure. An empty set is denoted by \( \emptyset \). For two sets \( \mathcal{U}, \mathcal{V} \) the notation \( \mathcal{U} \subset \mathcal{V} \) denotes \( \mathcal{U} \) is a subset of \( \mathcal{V} \). The symbols \( \cup \) and \( \cap \) denote the operators for union and intersection of sets, respectively.

II. Preliminaries

This section presents the background mathematical notation used in the rest of the paper. First, the dynamics of a PWA system can be written as

\[
\dot{x} = f_i(x) = A_i x + a_i, \quad x \in \mathcal{R}_i \tag{1}
\]

where \( x(t) \in \mathcal{R}_i \subset \mathbb{R}^n \) is the state vector with initial condition \( x(0) = x_0 \), \( A_i \in \mathbb{R}^{n \times n}, a_i \in \mathbb{R}^n \). The state space is partitioned into \( m \) open regions \( \mathcal{R}_i, i \in \mathbb{I}_m := \{1, \ldots, m\} \), such that

\[
\bigcup_{i=1}^m \mathcal{R}_i = \mathbb{R}^n, \quad \mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \quad i \neq j \tag{2}
\]

The Filippov definition of trajectories \[6\] is considered for solutions of (1).

**Definition 1** (Filippov solution \[6\]): A continuous function \( x(t) \) is regarded to be a Filippov solution of (1) if it is a solution of the differential inclusion

\[
\dot{x} \in \mathcal{F}(x) \tag{3}
\]

for all \( t \geq 0 \) where

\[
\mathcal{F}(x) := \sum_{i \in \mathcal{T}(x)} \theta_i(x) (A_i x + a_i), \quad \mathcal{T}(x) := \{ i \in \mathbb{I}_m | x \in \mathcal{R}_i \} \tag{4}
\]

where \( \theta(x) = [\theta_1(x) \cdots \theta_m(x)]' \) satisfies

\[
\theta \in \Theta := \left\{ \theta : \theta_i \geq 0, \sum_{i=1}^m \theta_i = 1 \right\} \tag{5}
\]

If \( x \in \mathcal{R}_i \), then \( \mathcal{F}(x) = \{f_i(x)\} \).

Based on Definition 1, consider a more general system representation of (1), which includes any possible sliding mode dynamics:

\[
\dot{x} = A_\theta x + a_\theta, \quad \theta \in \Theta, \quad x \in \mathbb{R}^n \tag{6}
\]

where

\[
A_\theta = \sum_{i=1}^m \theta_i(x) A_i, \quad a_\theta = \sum_{i=1}^m \theta_i(x) a_i \tag{7}
\]

The description (6) is general enough to represent the system dynamics at a boundary that is the intersection between any number of regions.

For stability of the origin of (6) we must have \( \dot{x} = 0 \). Therefore the following assumption is necessary.

\[
\exists \theta(0) \in \Theta \text{ such that } \sum_{i \in \mathcal{I}(0)} \theta_i(0) a_i = 0 \tag{8}
\]

**Remark 1:** When \( a_i \neq 0 \) for some \( i \in \mathcal{I}(0) \), assumption (8) implies that if the PWA system is stable, the equilibrium is maintained by an intermittent switching. The assumption (8) is more general than the assumption available in the literature, where it is assumed that \( a_i = 0 \) if \( i \in \mathcal{I}(0) \).

The subsystem \( i \) is active when \( x \in \mathcal{R}_i \), then \( \theta_i(x) = 1, \theta_j(x) = 0, \forall j \neq i \). When \( x \in \mathcal{R}_i \cap \mathcal{R}_j \neq \emptyset \), then \( \theta(x) \) assumes a specific value in \( \Theta \) for that point \( x \). The dependence of \( \theta \) with respect to \( x \) can be nonlinear and difficult to take into account to formulate convex problems. For this reason, the dependence will be omitted and we will use a (possibly) more conservative approach where \( \theta \) is treated as a free parameter that can assume any value inside the simplex \( \Theta \).

Note that system (6) can be rewritten as

\[
\dot{x} = \sum_{i=1}^m \theta_i A_i x, \quad A_i = \begin{bmatrix} A_i & a_i \\ 0_{1 \times n} & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{9}
\]

We end this preliminary section with a version of the Finsler’s Lemma and the definition of a linear annihilator.

**Lemma 1** (Finsler’s Lemma): Let \( \mathcal{U} \subset \mathbb{R}^s \) be a given polytopic set, \( S(.) : \mathcal{U} \rightarrow \mathbb{R}^{q \times q}, K(.) : \mathcal{U} \rightarrow \mathbb{R}^{r \times q} \) be given matrix functions, with \( S(.) \) symmetric. Let \( Q(w) \) be a matrix whose columns are basis vectors for the null space of \( K(w) \). Then the following are equivalent:

(i) \( \forall w \in \mathcal{U} \) the condition \( \sum \! S(w)z > 0 \) is satisfied \( \forall z \in \mathbb{R}^q \) such that \( K(w)z = 0 \).

(ii) \( \forall w \in \mathcal{U} \) there exists a matrix function \( L(.) : \mathcal{U} \rightarrow \mathbb{R}^{s \times q} \) such that \( S(w) + L(w)K(w) + K(w)L(w)' > 0 \).

(iii) \( \forall w \in \mathcal{U} \) the condition \( Q(w)'S(w)Q(w) > 0 \) is satisfied.

Two cases have particular interest to this paper. The first case is if \( K(.) \) is an affine function, then we constrain \( L \) to be constant and (ii) becomes a polytopic LMI condition that is sufficient for (i). The second case is if \( K \) is constant, then \( Q \) is constant as well and (iii) is a polytopic LMI that is equivalent to (i). Note that the interest of the above polytopic LMI problems is that they are efficient alternatives to the condition (i) that is difficult to be numerically tested. Finally, Lemma 1 is still valid if the inequalities are replaced by equality conditions. See for instance \[10\], \[11\] for more details on the Finsler’s Lemma.

**Definition 2** (Annihilator): Given a vector function \( f(.) : \mathbb{R}^q \rightarrow \mathbb{R}^s \) and a positive integer \( r \), a matrix function \( Ff(.) : \mathbb{R}^q \rightarrow \mathbb{R}^{q \times q} \) will be called an annihilator of \( f(.) \) if \( Ff(z)z = 0, \forall z \in \mathbb{R}^q \). If \( Ff(.) \) is a linear function it will be referred to as a linear annihilator.

For this paper, we are interested in a general formula for a linear annihilator for the case of \( f(z) = z = [z_1 \ldots z_q]' \in \mathbb{R}^q \).
Taking into account all possible pairs $z_i, z_j$ for $i \neq j$ without repetition, i.e. $\forall i, j \in \mathbb{I}_q$ with $j > i$, we get a linear annihilator given by the formula

$$\mathcal{K}_z(z) = \begin{bmatrix} \phi_1(z) & Y_1(z) \\ \vdots & \vdots \\ \phi_{i(q-1)}(z) & Y_{i(q-1)}(z) \end{bmatrix} \quad (10)$$

$$Y_i(z) = -z_i I_{(q-i)}, \quad i \geq 1, \quad \phi_i(z) = \begin{bmatrix} z_2 & \ldots & z_q \end{bmatrix}, \quad i \geq 2.$$ 

In what follows, annihilators are used jointly with the Finsler’s Lemma to reduce the conservativeness of parameter dependent LMIs. See, for instance [11] where linear annihilators are also used to reduce the conservativeness of state dependent LMIs.

### III. MAIN RESULTS

This section presents the main results for the stability analysis of PWA systems, where the regions can be described by

$$\mathcal{R}_i = \{ x : E_i x + e_i > 0 \} \quad (11)$$

where $E_i \in \mathbb{R}^{p_i \times n}$ and $e_i \in \mathbb{R}^{p_i}$. Stability of Filippov solutions in (6) will be proved using a Lyapunov function. The candidate Lyapunov function considered in this paper is a convex combination of PWQ functions with the following structure:

$$V(x) = \sum_{i=1}^{m} \theta_i x^T P_i x, \quad P_i = \begin{bmatrix} P_i & * \\ q_i' & r_i \end{bmatrix} \quad (12)$$

Before presenting the theorem for stability, consider the following definitions, auxiliary notation and LMI conditions.

Let $\mathcal{S}_k \subset \mathbb{R}^n$ be the set of points $x$ belonging to the $k$-th surface between any number of adjacent regions and let $\mathcal{S}_k$ be the set of all regions $i$ sharing the $k$-th surface, where $k \in \mathbb{I}_q$ and $g$ is the total number of surfaces.

Note that $row_j(E_i)x + row_j(e_i) = 0, \quad j \in \mathbb{I}_q$, represents each of the $p_i$ surfaces surrounding $\mathcal{R}_i$. For all $k \in \mathbb{I}_q$, define $l_k$ as one (it can be any) of the integers in $\mathcal{S}_k$. Therefore, if $i = l_k$, then $i \in \mathcal{S}_k$. Define for all $k \in \mathbb{I}_q$, $E_k := row_j(E_k)$ and $e_k := row_j(e_k)$, where $j$ is such that $row_j(E_k)x + row_j(e_k) = 0 \quad \forall x \in \mathcal{S}_k$. Therefore, $E_k x + e_k = 0, \quad \forall x \in \mathcal{S}_k$

Consider the auxiliary notation

$$C_{a_k} = E_k, \quad \overline{C}_{a_k} = \begin{bmatrix} E_k & e_k \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & \ldots & P_m \end{bmatrix}, \quad q = \begin{bmatrix} q_1 & \ldots & q_m \end{bmatrix}, \quad r = \begin{bmatrix} r_1 & \ldots & r_m \end{bmatrix} \quad (13)$$

$$A = \begin{bmatrix} A_1 & \ldots & A_m \end{bmatrix}, \quad a = \begin{bmatrix} a_1 & \ldots & a_m \end{bmatrix} \quad (14)$$

$$\alpha M = \begin{bmatrix} \alpha_1 & \ldots & \alpha_m \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 & \ldots & \alpha_m \end{bmatrix} \quad (15)$$

$$C_i = \mathcal{K}_\theta \otimes I_n, \quad \overline{C}_i = \begin{bmatrix} \mathcal{K}_\theta \otimes I_n & 0_{dn \times m} \\ 0_{dx \times n} & \mathcal{K}_\theta \otimes I_m \end{bmatrix} \quad (16)$$

$$\mathcal{K}_\theta \in \mathbb{R}^{d \times m}, \quad M = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \in \mathbb{R}^{1 \times m} \quad (17)$$

$$C_{b_k} = C_{a_k} (M \otimes I_n), \quad \overline{C}_{b_k} = \overline{C}_{a_k} \begin{bmatrix} M \otimes I_n & 0_{0 \times m} \\ 0_{1 \times nm} & M \end{bmatrix} \quad (18)$$

$$\overline{C}_{b_k}(P_1 - P_j) \overline{C}_{a_j} = 0, \quad \forall i, j \in \mathcal{S}_k, \quad i \neq j \quad (35)$$

where $\mathcal{K}_\theta$ is the linear annihilator of $\theta$ as defined in (10).

1) LMI conditions for positivity of $V(x)$ with $V(0) = 0$:

$$r_i = 0, \quad \text{if} \quad i \in \mathbb{I}(0) \quad (21)$$

$$q_i = 0, \quad \text{if} \quad a_i = 0 \quad (22)$$

$$q_i' = \sum_{j=1}^{p_i} \gamma_j \text{row}_j(E_i), \quad \text{if} \quad a_i \neq 0 \quad (23)$$

$$P_i \geq \varepsilon I_n, \quad \text{if} \quad a_i = 0 \text{ and } e_i \neq 0 \quad (24)$$

$$P_i \geq \varepsilon I_n, \quad \text{if} \quad e_i = 0 \quad (25)$$

$$Q_{a_k} (P_i - \varepsilon I_n) Q_{a_k} \geq 0, \quad \text{if} \quad e_i = 0 \quad (26)$$

$$Q_{a_k} (P_i - \epsilon I_n) Q_{a_k} \geq 0, \quad \text{if} \quad a_i \neq 0 \text{ and } e_i \neq 0 \quad (27)$$

where $\epsilon \succ 0$, $\gamma_j \succ 0$, $\forall i \in \mathbb{I}_m$, $\forall j \in \mathbb{I}_p$, and $Q_{a_k}, \overline{Q}_{a_k}$ are given matrix basis for the null spaces of $C_{a_k}, \overline{C}_{a_k}$, respectively.

2) LMI conditions for decay of $V(x)$:

$$P_A x + A_P x + \alpha_P = 0, \quad \text{if} \quad a_i = 0 \text{ and } e_i \neq 0 \quad (29)$$

$$P_A x + A_P x + \alpha_P = 0, \quad \text{if} \quad a_i = 0 \text{ and } e_i = 0 \quad (30)$$

$$P_A x + A_P x + \alpha_P = 0, \quad \text{if} \quad a_i \neq 0 \quad (31)$$

$$Q_{Q_a} (P' x + \alpha' x + L C(\theta) + C(\theta)' L_4') Q_{Q_a} \geq 0, \quad \forall \theta \in \vartheta(\Theta), \quad \text{if} \quad a_i = 0 \text{ and } e_i = 0 \quad (32)$$

$$Q_{Q_a} (P' x + \alpha' x + L C(\theta) + C(\theta)' L_4') Q_{Q_a} \geq 0, \quad \forall \theta \in \vartheta(\Theta), \quad \text{if} \quad a_i = 0 \text{ and } e_i = 0 \quad (33)$$

where $L_4$ has the dimensions of $C(\theta)'$ and $Q_{Q_a}, \overline{Q}_{Q_a}$ are given matrix basis for the null spaces of $C_{Q_a}, \overline{C}_{Q_a}$, respectively.

3) LMI conditions for continuity of $V(x)$:

$$\overline{Q}_{Q_a} (P_i - P_j) \overline{Q}_{a_j} = 0, \quad \forall i, j \in \mathbb{S}_k, \quad i \neq j \quad (35)$$

Taking into account all possible pairs $i, j \in \mathbb{S}_k$ for $i \neq j$ without repetition, i.e. $\forall i, j \in \mathbb{S}_k$ with $j > i$, we avoid declaring redundant LMIs in (35).

**Remark 2:** Compared to the current literature, conditions (23), (27), (28), (32)-(34) are new. Condition (23) allows feasibility when the origin is located in a boundary between
affine subsystems and (27), (28), (32)-(34) guarantee stability
of any sliding mode dynamics that may occur. □

The result for global stability analysis is formalized in the following theorem.

**Theorem 1:** Consider the system (6) with assumption (8) and regions described by (11). With the auxiliary notation
(13)-(20), let \( Q_{i_0}, Q_{b_0}, Q_{b_1} \) be given matrix basis for the null space of \( C_{i_0}, C_{i_1}, C_{i_2}, \) respectively, and \( L_{i}, \bar{L}_{i} \) be matrices to be determined with the dimensions of \( C_i(\theta)^T, \bar{C}_i(\theta)^T \).

Suppose \( \exists P_{j}, \bar{P}_{j}, \lambda_{j}, \bar{\Lambda}_{j}, L_{j}, \bar{L}_{j}, \epsilon > 0, \gamma_{j} > 0 \) and given decay rates \( \alpha > 0 \) solving the LMI (21)-(35) for all \( i \in I_n \) and for all \( k \in I_{g} \). Then (12) is a Lyapunov function for the system (6) and is globally exponentially stable. □

**Proof:** The proof is structured as follows. First, continuity of \( V(x) \) is ensured \( \forall x \in \mathbb{R}^n \), followed by positivity of \( V(x) \), \( \forall x \in \mathbb{R}^n \). In the sequence, the proof for decay of \( V(x) \), \( \forall x \in \mathbb{R}^n \) is divided in two parts, \( \forall x \in \mathcal{R}_i, \forall i \in I_m \) and \( \forall x \in S_k, \forall k \in I_g \) (note that the union of \( \mathcal{R}_i \) for all \( i \in I_m \) and \( S_k \) for all \( k \in I_g \) results in \( \mathbb{R}^n \) in the end). The results are summarized and the conclusion about stability is presented.

Consider the Lyapunov function candidate (12) rewritten as \( V(x) = \sum_{i=1}^{m} \theta_i V_i(x) \), where \( V_i(x) = x^T P_i x \). Noticing that \( \overline{C}_{i_0} x = 0 \) and then using the Finsler's Lemma, it follows from (35) that for any \( x \in \mathcal{S}_k, \forall k \in I_g \), \( V(x) = V_i(x), \forall i, j \in I_m \),

Therefore, \( V(x) \) is continuous \( \forall x \in \mathbb{R}^n \). In addition, constraint (21) implies that \( V(0) = 0 \). Note in Definition 1 that if \( x \in \mathcal{R}_i \), then \( \theta_i = 1, \gamma_i = 0, \forall i \neq j \), therefore \( V(x) = V_i(x), \forall x \in \mathcal{R}_i \).

The proof that positivity of \( V(x) \) is ensured \( \forall x \in \mathbb{R}^n \) is divided in three parts, contemplating all cases of \( e_i \) and \( a_i \):

1. If \( e_i = 0 \) and \( a_i = 0 \), we conclude from (21), (22), (24) that for all \( x \neq 0 \in \mathcal{R}_i \) (including \( x \in \mathcal{S}_k \)), \( V_i(x) = x^T P_i x \geq \epsilon \| x \|^2 > 0 \), therefore \( V(x) = \sum_{i=1}^{m} \theta_i V_i(x) \geq \min_{i \in I_m} \{ V_i(x) \} \geq \epsilon \| x \|^2 > 0, \forall \theta \in \Theta \).

2. If \( e_i = 0 \), we have \( \mathcal{R}_i = \{x | E_i x > 0\} \), then for any \( Z_i \) with appropriate dimensions and non-negative entries, for all \( x \in \mathcal{R}_i \), \( x^T E_i^T Z_i E_i x \geq 0 \). Also note that (23) with \( \gamma_i > 0 \) implies \( q_i x = \sum_{j=1}^{m} \gamma_j \text{row}_j(E_i) x \geq 0 \) for all \( x \in \mathcal{R}_i \). In this case, (21), (22), (23), (25) yield

\[
V(x) = V_i(x) = x^T P_i x + 2q_i x \geq x^T P_i x \\
\geq x^T E_i^T Z_i E_i x + \epsilon \| x \|^2 \geq \epsilon \| x \|^2 > 0 \tag{36}
\]

for all \( x \neq 0 \in \mathcal{R}_i \). To ensure positivity of \( V(x) \), \( \forall x \in \mathcal{S}_k, \forall k \in I_g \), the condition is

\[
V(x) = \sum_{i=1}^{m} \theta_i x^T P_i x \geq \sum_{i=1}^{m} \theta_i x^T P_i x \geq \epsilon x^T x = \epsilon \| x \|^2 > 0, \forall \theta \in \Theta \to \mathbb{R}^n, C_{i_0} x = 0 \tag{37}
\]

Evaluating \( \theta_i \) in (37) \( \epsilon x^T P_i x \geq \epsilon x^T x \) for all vertices of \( \Theta \) and noticing that \( C_{i_0} x \) is an annihilator of \( x \) with constant entries only, we get (27) by using the Finsler's Lemma.

3. If \( e_i \neq 0 \) and \( a_i = 0 \), we have \( \mathcal{R}_i = \{x | \bar{E}_i x > 0\} \) and similarly to the previous case, condition (26) yields

\[
V(x) = V_i(x) = x^T P_i x \geq x^T E_i^T \bar{E}_i E_i x + \epsilon \| x \|^2 \geq \epsilon \| x \|^2 > 0 \tag{38}
\]

for all \( x \neq 0 \in \mathcal{R}_i \). To ensure positivity of \( V(x) \), \( \forall x \in \mathcal{S}_k, \forall k \in I_g \), the condition is

\[
V(x) = \sum_{i=1}^{m} \theta_i x^T P_i x \geq \sum_{i=1}^{m} \theta_i x^T P_i x \geq \epsilon x^T x = \epsilon \| x \|^2 > 0, \forall \theta \in \Theta \to \mathbb{R}^n, C_{i_0} x = 0 \tag{37}
\]

Evaluating \( \theta_i \) in (37) \( \epsilon x^T P_i x \geq \epsilon x^T x \) for all vertices of \( \Theta \) and noticing that \( C_{i_0} x \) is an annihilator of \( x \) with constant entries only, we get (27) by using the Finsler's Lemma.

The next steps show how to obtain the conditions for \( V(x) \) decreasing for \( x \in \mathcal{S}_k, \forall k \in I_g \). Consider the compact notation \( P_\theta := \sum_{i=1}^{m} \theta_i P_i \) and \( q_\theta = r_\theta A_\theta a_\theta a_\theta \) defined in a similar way.

1. If \( a_i = 0, \forall i \in I_k \), this condition can be characterized by

\[
\nabla V(x)^T A_\theta x = 2x^T P_\theta A_\theta x = x^T (P_\theta A_\theta + A_\theta^T P_\theta) x \\
\leq -2\alpha_\theta x^T P_\theta x = -2\alpha_\theta V(x) < 0 \tag{41}
\]

2. If \( a_i = 0 \) and \( e_i = 0 \), we have \( \mathcal{R}_i = \{x | E_i x > 0\} \), then for any \( A_i \) with appropriate dimensions and non-negative entries, for all \( x \in \mathcal{R}_i \), \( x^T E_i^T A_i E_i x \geq 0 \). In this case, (30) yields

\[
\nabla V(x)^T A_\theta x = \nabla V_i(x)^T A_i x < -\alpha_\theta x^T P_i x - x^T E_i^T A_i E_i x \\
\leq -\alpha_\theta x^T P_i x = -\alpha_\theta V_i(x) = -\alpha_\theta V(x) < 0 \tag{42}
\]

for all \( x \neq 0 \in \mathcal{R}_i \).

3. If \( a_i = 0 \), \( \forall i \in I_k \), similarly to the previous case, condition (31) yields

\[
\nabla V(x)^T (A_i x + a_i) = \nabla V_i(x)^T (A_i x + a_i) = 2x^T P_i \bar{A}_i x \\
\leq -\alpha_\theta x^T P_\theta x = -\alpha_\theta V(x) < 0 \tag{43}
\]

for all \( x \neq 0 \in \mathcal{R}_i \).

(44) and (45)
where \( x_\theta = \theta \otimes x = [\theta_1 x' \ldots \theta_m x']' \in \mathbb{R}^{mn} \). Noticing that \( C_t(\theta) \) is a linear annihilator for \( x_\theta \) (i.e. \( C_t(\theta)x_\theta = 0 \)), we insert it in condition (45) by using the Finsters’ Lemma and obtain the LMI (32). Moreover, if \( \epsilon_i = 0 \), \( \forall i \in \mathbb{N}_m \), it is possible to use the annihilator \( C_{b_i}x_\theta = 0 \), along with the Finsters’ Lemma, to reduce the conservativeness of (32), obtaining (33).

(2) If \( \exists i \in \mathbb{N}_2 : a_i \neq 0 \), the condition for \( V(x) \) decreasing for \( x \in \mathcal{S}_k \), \( \forall k \in \mathbb{I}_p \), can be characterized by

\[
\nabla V(x)' (A_\theta x + a_\theta) = \dot{x}' \left[ \begin{array}{c} P_0 A_\theta + A_\theta^T P_0 \\ \alpha_0 q_0 A_\theta + \alpha_0 q_0^T A_\theta \end{array} \right] \dot{x} < 0
\]

See [9] for details on the first equality of (46). Using (14)-(20), the condition (46) can be rewritten as

\[
\nabla \theta Y \nabla \theta < 0 \quad , \quad \nabla \theta = \left[ \begin{array}{c} \theta \otimes x \\ \theta \end{array} \right]
\]

As in the previous case, note that \( \mathcal{C}_t(\theta)x_\theta = 0 \) and \( \mathcal{C}_{b_i}x_\theta = 0 \). By using the Finsters’ Lemma to insert these annihilators to relax the condition (47), we get the LMI condition (34).

The last situation that needs to be considered is when \( V(x) \) is not differentiable at a point \( x \). Due to continuity and differentiability of the functions \( V_i(x) \), such points can only occur when a trajectory reaches a surface, because at this instant the value of \( \theta \) may change instantly to any value in \( \Theta \). Continuity of \( V(x) \) implies that \( V(x) \) cannot increase at the points where \( V(x) \) is not differentiable. Furthermore, \( \theta \) may be discontinuous at the boundaries but \( V(x) \) is guaranteed to be decreasing because the conditions (32)-(34) hold \( \forall \theta \in \Theta \).

In summary, \( V(x) \) is continuous, positive definite and satisfies the bounds (40). Moreover, \( V(x) \) is globally strictly decreasing for the dynamics of the system (6), that includes the subsystem dynamics and the sliding mode dynamics that may eventually occur at any switching surface, and global exponential stability follows from [6, p.155].

The following interesting corollary is derived from Theorem 1.

**Corollary 1** (Stability independent of the boundaries): If it is possible to find a solution for Theorem 1 by replacing the variables \( \mathcal{Z}, \mathcal{Z}_i, \mathcal{L}_i, \gamma_j \) by zeros and \( \mathcal{Q}_{0i}, \mathcal{Q}_{0j}, \mathcal{Q}_{b_i}, \mathcal{Q}_{b_j} \) by identity matrices, all with appropriate dimensions, then this system is globally exponentially stable for any boundaries. □

**Proof:** Follows trivially as a particular case of the proof of Theorem 1, noticing that by fixing the decision variables as suggested in Corollary 1, the LMIs are now checked without inserting any information about any specific surface to relax the conditions. □

**Remark 3:** Note that in Corollary 1, the continuity condition (35) is replaced by

\[
\mathcal{P}_i - \mathcal{P}_j = 0 \quad , \quad \forall i, j \in \mathbb{N}_m \quad , \quad i \neq j
\]

which is satisfied only if it is possible to force all Lyapunov functions to be equal, reducing the problem to finding a single quadratic Lyapunov function. This may be more conservative than Theorem 1, but a stronger result is obtained as stability is guaranteed even if the boundaries change. □

**IV. Numerical Examples**

In the examples that follow we have used SeDuMi with Yalmip interface [12] to solve the LMIs and Simulink to obtain the state trajectories. Example 1 shows that the technique does not fail for systems with unstable sliding modes, even that the Lyapunov function is not common for all regions. Example 2 illustrates the case where the origin is located at a boundary between affine subsystems. Example 3 shows the application of Corollary 1.

**Example 1:** Consider the system

\[
\begin{cases}
\dot{x} = A_1 x + a_1, & \text{if } x_2 \geq 0 \\
\dot{x} = A_2 x + a_2, & \text{if } x_2 \leq 0
\end{cases}
\]

with the following matrices \( A_1, A_2, a_1, a_2 \), respectively [3]:

\[
\begin{bmatrix}
1 & -2 \\
2 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
-2 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Although this system has both \( A_1 \) and \( A_2 \) Hurwitz, it presents an unstable sliding mode, as shown in [3].

The regions \( \mathcal{R} \) can be expressed as in (11) with

\[
E_1 = [0 \ 1], \quad E_2 = [0 \ -1], \quad e_1 = e_2 = 0.
\]

The LMIs to be solved in this case are (21), (25), (27), (30), (33), (35). It is not possible to find a feasible solution, which is consistent with the expected result. Without the condition (33) for inclusion of sliding modes dynamics, the LMI problem would be feasible, providing a wrong conclusion about the stability of the system. Reference [3] gets the same infeasible result by using a more conservative approach with a common quadratic Lyapunov function. □

**Example 2:** Consider the system (49), with the regions parameterized by (51), with the matrices \( A_1, A_2, a_1, a_2 \) given by, respectively.

\[
\begin{bmatrix}
-2 & -2 \\
4 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & 2 \\
-4 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
\delta
\end{bmatrix}
\]

where \( \delta \) is a given fixed parameter. This system presents only stable sliding modes for \( \delta \geq 0 \). First, consider the case where \( \delta = 0 \). According to [4, p. 84], it is not possible to find a quadratic or class \( C^1 \) PWQ function for this system. However, Theorem 1 does not require a class \( C^1 \) function to be satisfied. Solving the same LMIs of Example 1, a feasible solution is found. Reference [3] solves this case by using a sixth order \( C^1 \) polynomial Lyapunov function. □

For the case where \( \delta > 0 \), note that the system satisfies assumption (8) with \( \theta_i(0) = 1/2, \forall i \in \mathbb{N}_m \). In this case the LMIs to be solved are (21), (23), (25), (27), (30), (34), (35). The conditions are tested with \( \delta = 2 \), for which some trajectories are shown in Fig. 1, and a feasible solution is found. Note that the origin is located at a boundary between affine subsystems, for which case there is no other stability analysis method available in the current literature. □
Example 3: Consider the system
\[
\begin{aligned}
    \dot{x} &= A_1 x + a_1, & \text{if } x_2 + 1 &\geq 0 \\
    \dot{x} &= A_2 x + a_2, & \text{if } x_2 + 1 &\leq 0 \\
\end{aligned}
\]  
(53)
with the matrices \(A_1, A_2, a_1, a_2\) given by, respectively,
\[
\begin{bmatrix}
    -1 & -2 \\
    2 & -2
\end{bmatrix}, \begin{bmatrix}
    -1 & 2 \\
    -2 & -2
\end{bmatrix}, \begin{bmatrix}
    0 \\
    0
\end{bmatrix}, \begin{bmatrix}
    \delta \\
    0
\end{bmatrix}
\]  
(54)
where \(\delta\) is a given fixed parameter. The regions \(\mathcal{R}_i\) are parameterized as in (11) by
\[
E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad e_1 = 1, \quad e_2 = -1. \quad (55)
\]

For the case of \(\delta = 0\), the LMIs to be solved are the same as in Example 1. A feasible solution is found, showing that the system (53) is stable for the given surface, even with the occurrence of a sliding mode. Moreover, it is possible to prove that the system (53) is stable for any possible surfaces by fixing the matrices \(Z_i, \Lambda_i\) equal to zero and \(Q_{\delta k}, \tilde{Q}_{\delta k}, \tilde{Q}_{\delta}^{k}\) equal to identity matrices, as mentioned in the Corollary 1, then solving the same set of LMIs, which is feasible.

For the case of \(\delta \neq 0\), subsystem 2 is affine and Corollary 1 is not feasible because the equilibrium point of this subsystem, given by
\[
x_{eq2} = -A_2^{-1}a_2 = \frac{1}{3} \begin{bmatrix}
    \delta \\
    -\delta
\end{bmatrix}
\]  
(56)
is not the origin. Recalling that \(A_2\) is Hurwitz, it is easy to realize that for boundaries that let \(x_{eq2} \in \mathcal{R}_2\), the system is not globally stable. Therefore, for the boundary given in (53), the origin of the PWA system is not globally stable for \(\delta \geq 3\). For this case, the LMIs to be solved to analyze stability are (21), (24), (29) for \(i = 1\), (26), (28), (31) for \(i = 2\), and (34), (35). As expected, it is not possible to find a solution when \(\delta \geq 3\), but it is possible when \(\delta < 3\). In the latter case, the closer \(\delta\) gets to 3, the closer \(a_2\) must get to 0. Fig. 2 shows the occurrence of a stable sliding mode for the particular case of \(\delta = 1\). □

V. CONCLUDING REMARKS

Sufficient conditions for stability of PWA systems were formulated as convex problems. The conditions are sufficient for checking the stability of systems even in the occurrence of sliding modes. There is no need to know a-priori in which switching surfaces a sliding mode happens, if it does. As a by-product, sufficient conditions for stability for any switching surfaces are derived at the expense of some additional conservativeness. Three examples are used to illustrate the application and the advantages of the proposed method.

REFERENCES


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