Lagrangian Mechanics and Lie Group Variational Integrators for Spacecraft with Imbalanced Reaction Wheels

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Abstract—This paper presents an analytic dynamic model and a geometric numerical integrator for spacecraft with reaction wheel assemblies. According to Lagrangian mechanics on an abstract Lie group, Euler-Lagrange equations are derived without any restrictive assumptions on the configuration of reaction wheels. This yields the most generalized reaction wheel dynamic model, that can possibly include the effects of arbitrary mass distribution about their spin axes, such as reaction wheel imbalance. The second part is focused on constructing a geometric numerical integrator, referred to as Lie group variational integrator, that provides long-term structural stability in simulating reaction wheel dynamics accurately. These are illustrated by a numerical example.

I. INTRODUCTION

The objective of this paper is to provide both of a comprehensive analytic model and an accurate simulation tool for spacecraft with reaction wheel assembly (RWA), that can be particularly used for analysis of RWA faults. Actuator faults can lead to loss or delay of mission as has been seen in the past with expensive and long-term spacecraft missions, such as the two recent RWA failures for the NASA Kepler Space Telescope [1] and thruster and RWA failures for the JAXA Hayabusa Spacecraft mission [2].

However, analyzing or predicting actuator faults on spacecraft with RWA is challenging. This is mainly due to the following two reasons. First, actuator hardware faults on spacecraft typically occur slowly over many years [3], [4]. Therefore, to predict such hardware malfunction earlier stages, the dynamics of spacecraft should be propagated over a long-time period accurately and efficiently. Conventional numerical integration schemes, such as popular variable stepsize Runge–Kutta methods, have undesirable long-term numerical properties, and they sometimes yield erroneous results [5]. Second, most of the existing dynamic models of RWA rely on several assumptions on the mass distribution of each reaction wheel. For example, it is usually assumed that the spin axis passes through the mass center of each reaction wheel and the mass distribution is axially symmetric about the spin axis. Therefore, they cannot be applied to various RWA hardware fault scenarios such as imbalanced reaction wheel.

To address these two issues, the first part of this paper is focused on deriving an analytic model for spacecraft with RWA. By considering its configuration space as a Lie group, it is shown that the equations of motion can be derived in a unified fashion according to Lagrangian mechanics. This model may incorporate the effects of reaction wheels with arbitrary mass distributions.

The second part of this paper deals with a geometric numerical integrator for spacecraft with RWA. Geometric numerical integration is concerned with developing numerical integrators that preserve geometric features of a system, such as invariants, symmetry, and reversibility [5], [6]. A geometric numerical integrator, referred to as a Lie group variational integrator [7], [8], is developed for spacecraft with RWA. The proposed geometric numerical integrator preserves symplecticity and momentum maps, and exhibits desirable energy conservation properties. It also respects the Lie group structure of the configuration manifold, and avoids the singularities and computational complexities associated with the use of local coordinates. In short, this paper provides a mathematical model and a reliable numerical simulation tool that characterizes the nonlinear coupling between several dynamic modes of spacecraft with RWA, and they can be used to study non-local, large maneuvers of spacecraft with arbitrary configurations of RWA over a long period of time.

II. SPACECRAFT WITH REACTION WHEELS

Consider a spacecraft model with n reaction wheels. Throughout this paper, the base spacecraft denotes the spacecraft without reaction wheels, and the (whole) spacecraft means the complete spacecraft including the base spacecraft and n reaction wheels.
Define an inertial frame, and a body-fixed frame that is located at the mass center of the base spacecraft. The configuration of the base spacecraft is described by \((R, x) \in \text{SO}(3) \times \mathbb{R}^3\), where \(x \in \mathbb{R}^3\) is the location of its mass center with respect to the inertial frame, and \(R \in \text{SO}(3)\) is the rotation matrix representing the linear transformation of the representation of a vector from the body-fixed frame to the inertial frame. Here, the special orthogonal group is denoted by \(\text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I_{3 \times 3}, \det[R] = 1 \}\).

The attitude kinematic equation for the base spacecraft is
\[
\dot{R} = R \dot{\Omega},
\]  
(1)

where \(\Omega \in \mathbb{R}^3\) is the angular velocity of the base spacecraft represented with respect to the body-fixed frame. The hat map \(\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)\) is a linear isomorphism between \(\mathbb{R}^3\) and \(3 \times 3\) skew-symmetric matrices, defined by the condition that \(\hat{x} y = x \times y\) for all \(x, y \in \mathbb{R}^3\). The inverse of the hat map is denoted by the vee map \(\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3\).

Next, we describe the configuration of reaction wheels. Let \(\rho_i \in \mathbb{R}^3\) be the vector from the origin of the body-fixed frame to the base of the spin axis of the \(i\)-th reaction wheel. The spin axis of the \(i\)-th reaction wheel is denoted by \(s_i \in S^2\), where \(S^2 = \{ q \in \mathbb{R}^3 \mid q^T q = 1 \}\). All of \(\rho_i\) and \(s_i\) are represented with respect to the body-fixed frame, they are fixed, i.e., \(\dot{\rho}_i = \dot{s}_i = 0\).

We introduce the reaction wheel frame where its first axis corresponds to \(s_i\), and the remaining two axes are chosen such that they constitute an orthogonal frame. Define \(Q_i \in \text{SO}(3)\) be the rotation matrix representing the linear transformation of a vector from the body-fixed frame to the\(i\)-th reaction wheel frame. The definition of \(Q_i\), we have \(s_i = Q_i e_i\), where \(e_j \in \mathbb{R}^3\) is the \(j\)-th standard basis of \(\mathbb{R}^3\), (e.g., \(e_1 = [1,0,0]^T \in \mathbb{R}^3\)). It is assumed that the \(i\)-th reaction wheel frame is fixed to the wheel, i.e., it rotates about the spin axis.

Let \(\theta_i \in S^1\) be the rotation angle of the \(i\)-th reaction wheel from the reference configuration given by \(Q_i = I_{3 \times 3}\), where \(S^1 = \{ q \in \mathbb{R}^2 \mid q^T q = 1 \}\). The angular velocity of the \(i\)-th reaction wheel is \(\dot{\theta}_i\), with respect to the \(i\)-th reaction wheel frame, or equivalently it is \(\dot{s}_i\), with respect to the body-fixed frame. The corresponding kinematics equation for \(Q_i\) is
\[
\dot{Q}_i = Q_i \dot{e}_i \dot{\theta}_i = \dot{s}_i Q_i \dot{\theta}_i.
\]  
(2)

In short, a configuration of the presented spacecraft with reaction wheels is described by \((R, x, \theta_1, \ldots, \theta_n)\). The configuration space is a Lie group \(G = \text{SO}(3) \times \mathbb{R}^3 \times (S^1)^n\), and the kinematics equations are given by (1) and (2).

### III. Lagrangian Mechanics

#### A. Lagrangian

The dynamics of spacecraft with reaction wheels have been studied widely. However, the dynamic models are often based on several simplifying assumptions. For example, reaction wheels are assumed to be inertially symmetric about its spin axis, and the spin axis corresponds to the principal axis of the spacecraft. The translational dynamics are often ignored. In some cases, the definition of the angular momentum or inertia matrices are unclear and confusing. For example, the point about which an inertia matrix is defined, or the reference frame with which it is represented is not explicitly specified. Here, we present the equations of motion for spacecraft reaction wheels without relying on any simplifying assumptions, according to Lagrangian mechanics.

a) Kinetic Energy: Consider a mass element \(dm\) of the base spacecraft. Let its location with respect to the body-fixed frame be \(\rho \in \mathbb{R}^3\). As the base spacecraft is rigid, we have \(\dot{\rho} = 0\). The location and velocity of the mass element are \(x + R \rho\), and \(\dot{x} + R \dot{\Omega} \rho\), respectively with respect to the inertial frame.

Next, consider a mass element \(dm_i\) of the \(i\)-th reaction wheel that is located at \(\xi_i\) with respect to the \(i\)-th reaction wheel frame. As each reaction wheel is rigid, we have \(\dot{\xi}_i = 0\). Its location with respect to the inertial frame is given by \(x + R(\rho_i + Q_i \xi_i)\), and its velocity with respect to the inertial frame is \(\dot{x} + R \dot{\Omega}(\rho_i + Q_i \xi_i) + R Q_i \dot{e}_i \xi_i \dot{\theta}_i\).

The kinetic energy is composed of the kinetic energy of the base spacecraft and the reaction wheels:
\[
T = \int_{B_0} \frac{1}{2} \|\dot{x} + R \dot{\Omega} \rho\|^2 dm(\rho)
+ \sum_{i=1}^n \int_{B_i} \frac{1}{2} \|\dot{x} + R \dot{\Omega}(\rho_i + Q_i \xi_i) + R Q_i \dot{e}_i \xi_i \dot{\theta}_i\|^2 dm_i(\xi_i),
\]
where \(B_0\) and \(B_i\) denote the volume occupied by the base spacecraft and the \(i\)-th reaction wheel, respectively.

Expanding the right hand side of the above expression and rearranging, we obtain
\[
T = \frac{1}{2} \{ \int_{B_0} dm + \sum_{i=1}^n \int_{B_i} dm_i \} \|\dot{x}\|^2
+ \dot{x}^T R \dot{\Omega} \{ \int_{B_0} \rho dm + \sum_{i=1}^n \int_{B_i} (\rho_i + Q_i \xi_i) dm_i \}
+ \dot{x}^T \sum_{i=1}^n R Q_i \dot{e}_i \xi_i \int_{B_i} \dot{\xi}_i dm_i
+ \frac{1}{2} \Omega^T \left\{ \int_{B_0} -\dot{\rho}^2 dm + \sum_{i=1}^n \int_{B_i} -((\rho_i + Q_i \xi_i)^\wedge)^2 dm_i \right\} \Omega
+ \frac{1}{2} \sum_{i=1}^n \dot{e}_i^T \left\{ \int_{B_i} -\xi_i^2 dm_i \right\} e_i \dot{\theta}_i^2
+ \sum_{i=1}^n \Omega^T \left\{ \int_{B_i} -((\rho_i + Q_i \xi_i)^\wedge Q_i \xi_i dm_i \right\} s_i \dot{\theta}_i.
\]  
(3)

Let \(m_b, m_i \in \mathbb{R}\) be the mass of the base spacecraft, and the mass of the \(i\)-th reaction wheel, respectively. The total mass is given by \(m = m_b + \sum_{i=1}^n m_i \in \mathbb{R}\). Next, we introduce several variables that describe the mass distribution of the base spacecraft and the reaction wheels. Since the origin of the body-fixed frame is located at the mass center of the base spacecraft, we have \(\int_{B_0} \rho dm = 0\). Let \(J_b \in \mathbb{R}^{3 \times 3}\) be
the inertia matrix of the base spacecraft with respect to the body-fixed frame:

\[ J_b = \int_{B_b} -\hat{\rho}^2 dm. \]  

(4)

Note that it is defined about the mass center of the base spacecraft, instead of the mass center of the whole spacecraft. Also, let \( I_i \in \mathbb{R}^3 \) and \( J_i \in \mathbb{R}^{3 \times 3} \) be the first and the second mass moment of inertia for the \( i \)-th reaction wheel:

\[ I_i = \int_{B_i} \xi_i dm_i, \quad J_i = \int_{B_i} -\hat{\xi}_i^2 dm_i. \]  

(5)

Note that they are defined about the base of the spin axis, and they are represented with respect to the \( i \)-th reaction wheel frame. We also introduce

\[ J = J_b - \sum_{i=1}^{n} m_i \hat{\rho}_i^2, \]  

(6)

which corresponds to the inertia of the whole spacecraft when \( I_i = 0 \) and \( J_i = 0 \), i.e., the reaction wheel is replaced by a point mass.

Substituting these into (3), and using \( Q_i \hat{e}_1 = s_i \), the kinetic energy can be rearranged as

\[ T = \frac{1}{2} m \| \dot{x} \|^2 + \dot{x}^T \sum_{i=1}^{n} \{ \dot{\Omega} (m_i \hat{\rho}_i + Q_i I_i) + Q_i \hat{e}_1 I_i \hat{\theta}_1 \} + \frac{1}{2} \Omega^T \left\{ J + \sum_{i=1}^{n} (-\hat{\rho}_i Q_i \hat{I}_i - Q_i \hat{I}_i \hat{\rho}_i + Q_i J_i Q_i^T) \right\} \Omega + \sum_{i=1}^{n} e_1^T J_i e_1 \hat{\theta}_i^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega^T (-\hat{\rho}_i Q_i \hat{I}_i + Q_i J_i) e_1 \hat{\theta}_i. \]  

(7)

b) Lagrangian: It is assumed that there exists a potential \( U(x, R) : \text{SO}(3) \times \mathbb{R}^3 \rightarrow \mathbb{R} \) that is dependent of the attitude and the position of spacecraft. The Lagrangian is given by

\[ L(x, R, \theta, \dot{x}, \Omega, \hat{\theta}) = T(R, \theta, \dot{x}, \Omega, \hat{\theta}) - U(x, R). \]  

(8)

B. Euler-Lagrange Equations

Euler-Lagrange equations for mechanical systems evolving on an abstract Lie group are presented in [8]. As the configuration space of the given spacecraft model is also a Lie group, the equations of motion can be derived from the abstract formulation of [8] as follows.

c) Derivatives of Lagrangian: The derivatives of the Lagrangian with respect to \( x \) and \( \dot{x} \) are given by

\[ D_x L = -\frac{\partial U}{\partial x} \triangleq f_u, \]

\[ D_{\dot{x}} L = m \ddot{x} + R \sum_{i=1}^{n} \{ \dot{\Omega} (m_i \hat{\rho}_i + Q_i I_i) + Q_i \hat{e}_1 I_i \hat{\theta}_1 \}. \]  

(10)

The infinitesimal variation of a rotation matrix can be written as

\[ \delta R = \frac{d}{dc} \bigg|_{c=0} \text{exp}(c \hat{\eta}) = R \hat{\eta}, \]  

(11)

for \( \eta \in \mathbb{R}^3 \) [8]. Using this expression, the left-trivialized derivative of \( L \) with respect to \( R \) is given by

\[ (T^* L_R \cdot D_R L) \cdot \eta = D_R L \cdot \delta R = D_R L \cdot \hat{R} \hat{\eta} = \dot{\delta R} \hat{\eta} \sum_{i=1}^{n} \{ \dot{\Omega} (m_i \hat{\rho}_i + Q_i I_i) + Q_i \hat{e}_1 I_i \hat{\theta}_1 \} + M_u \cdot \eta \]

\[ = -R^T \dot{x} \sum_{i=1}^{n} \{ \dot{\Omega} (m_i \hat{\rho}_i + Q_i I_i) + Q_i \hat{e}_1 I_i \hat{\theta}_1 \} \cdot \eta + M_u \cdot \eta, \]  

(12)

where \( M_u = -T^* L_R \cdot D_R U \in \mathbb{R}^3 \) is the moment due to the potential (see [9] for a formal definition of left trivialization). The derivative of \( L \) with respect to \( \Omega \) can be written as

\[ D_{\Omega} L = -R^T \dot{x} \sum_{i=1}^{n} (m_i \hat{\rho}_i + Q_i I_i) \]

\[ + \{ J + \sum_{i=1}^{n} (-\hat{\rho}_i Q_i \hat{I}_i - Q_i \hat{I}_i \hat{\rho}_i + Q_i J_i Q_i^T) \} \Omega + \sum_{i=1}^{n} (-\hat{\rho}_i Q_i \hat{I}_i + Q_i J_i) e_1 \hat{\theta}_i. \]  

(13)

As the \( i \)-th reaction wheel frame has one degree of freedom corresponding to rotations about its spin axis, the variation of \( Q_i \) can be written as

\[ \delta Q_i = Q_i \hat{e}_1 \delta \hat{\theta}_i = \delta_i Q_i \delta \hat{\theta}_i. \]  

(14)

Using this, the derivative of the Lagrangian with respect to \( \hat{\theta}_i \) is given by

\[ D_{\hat{\theta}_i} L = \dot{x}^T R \{ \dot{\Omega} Q_i \hat{e}_1 I_i + Q_i \hat{e}_1 I_i \hat{\theta}_i \} \]

\[ + \frac{1}{2} \Omega^T (-\hat{\rho}_i Q_i \hat{I}_i - Q_i \hat{I}_i \hat{\rho}_i + Q_i \hat{I}_i \hat{\theta}_i) \Omega + \frac{1}{2} \sum_{i=1}^{n} e_1^T J_i e_1 \hat{\theta}_i. \]  

(15)

The derivative of the Lagrangian with respect to \( \hat{\theta}_i \) is

\[ D_{\hat{\theta}_i} L = \dot{x}^T R Q_i \hat{e}_1 I_i + e_1^T J_i e_1 \hat{\theta}_i + \frac{1}{2} \Omega^T (-\hat{\rho}_i Q_i \hat{I}_i + Q_i J_i) e_1 \hat{\theta}_i. \]  

(16)

d) Virtual Work: Let \( f_e \in \mathbb{R}^3 \) be the external force applied to the mass center of the base spacecraft, represented with respect to the inertial frame, and let \( M_e \in \mathbb{R}^3 \) be the external moment about the mass center of the base spacecraft, represented with respect to the body-fixed frame. The internal control torque at the \( i \)-th reaction wheel about its spin axis \( s_i \) is denoted by \( \tau_i \in \mathbb{R} \). The virtual work done by these force and moments are given by

\[ \delta W = f_e \cdot \delta x + M_e \cdot \eta + \sum_{i=1}^{n} \tau_i \delta \hat{\theta}_i. \]

e) Euler-Lagrange Equations: From [8], the Euler-Lagrange equations on the given configuration space of Lie group can be written as

\[ \frac{d}{dt} D_x L - D_{\dot{x}} L = f_e, \]

\[ \frac{d}{dt} D_{\Omega} L + \Omega \times D_{\Omega} L - T^* L_R \cdot D_R L = M_e, \]  

(18)
\[ \frac{d}{dt} \mathbf{p}_x - m \mathbf{D}_\mathbf{p}_x L = \tau_i. \] (19)

We can substitute (9)–(16) into these, and find the expressions for \( \ddot{x}, \dot{\Omega}, \) and \( \ddot{\theta} \). This will be straightforward but tedious algebraic operations, and it is omitted in this paper.

Alternatively, we introduce the following Legendre transformations to obtain \( p_x, p_\Omega \in \mathbb{R}^3 \) and \( p_{\theta_i} \in \mathbb{R} \) as
\[
p_x = \mathbf{D}_x L, \quad p_\Omega = \mathbf{D}_\Omega L, \quad p_{\theta_i} = \mathbf{D}_{\theta_i} L. \tag{20}\]

From this, the Euler-Lagrange equations are transformed into Hamilton’s equations:
\[ \dot{p}_x - \mathbf{D}_x L = f_x, \tag{21} \]
\[ \dot{p}_\Omega + \Omega \times p_\Omega - \mathbf{T}_R^T \mathbf{D}_R L = M_e, \tag{22} \]
\[ \dot{p}_{\theta_i} - \mathbf{D}_{\theta_i} L = \tau_i. \tag{23} \]

Together with kinematics equations (1) and (2), these can be used to obtain \( (p_x(t), p_\Omega(t), p_{\theta_i}(t), \mathbf{x}(t), R(t), \theta(t)) \), and the velocity variables \( (\dot{\mathbf{x}}(t), \Omega(t), \dot{\theta}(t)) \) can be obtained from the Legendre transformation.

IV. LIE GROUP VARIATIONAL INTEGRATORS

Geometric numerical integration deals with numerical integration techniques that preserve the underlying geometric properties of a dynamical system, such as invariants, symmetries, or the structure of configuration manifolds [5], [6]. Variational integrators provide a systematic method to construct geometric numerical integrators for mechanical systems [10], where a numerical integrator is developed according to a discrete analogue of Hamilton’s variational principle. Numerical flows of variational integrators can have desirable properties such as symplecticity and momentum preservation, and they can exhibit good energy behavior over a long time period.

In particular, Lie group variational integrators are developed for Lagrangian/Hamiltonian systems evolving on a Lie group [8]. They inherit the desirable computational properties of variational integrators, and they also preserve the Lie group structures of a configuration manifold naturally by updating a group element using the group operation. These are in contrast to projection-based methods where projection at each time-step may corrupt conservation properties, or constrained-based methods where a nonlinear constraint needs to be solved at each time-step. They also avoid singularities introduced by local parameterizations.

The unique feature of Lie group variational integrators is that they preserve both the symplecticity of mechanical systems and the nonlinear structure of a Lie group configuration manifold concurrently, and it has been shown that this is critical for accurate and efficient simulations of multibody dynamics [7]. In this section, we construct a Lie group variational integrator for spacecraft with reaction wheels presented in the previous section.

A. Discrete Lagrangian

Let \( h > 0 \) be a fixed time step, and the subscript \( k \) is used to denote the value of any variation at \( t = kh \). Therefore, the configuration of spacecraft with reaction wheel at \( t = t_k \) is described by \( g_k = (R_{k}, x_k, \theta_k) \in G \). We define a discrete-time kinematic equation as follows. Define \( f_k = (F_k, \Delta x_k, \Delta \theta_k) \in G \) such that \( g_{k+1} = g_k f_k \):
\[
(R_{k+1}, x_{k+1}, \theta_{k+1}) = (R_k F_k, x_k + \Delta x_k, \theta_k + \Delta \theta_k). \tag{24}\]

Therefore, \( f_k \) represents the update relation between two integration steps. This ensures that the Lie group structures are preserved since \( g_k \) is updated by the Lie group action. For example, \( R_{k+1} = R_k F_k \in SO(3) \) as the group operation of \( SO(3) \) corresponds to matrix multiplication.

A discrete Lagrangian \( L_d(g_k, f_k) : G \times G \rightarrow \mathbb{R} \) is an approximation of the Jacobi solution of the Hamilton–Jacobi equation, which is given by the integral of the Lagrangian along the exact solution of the Euler-Lagrange equations over a single time step, \( L_d(g_k, f_k) \approx \int_0^h L(\hat{g}(t), \hat{g}^{-1}(t) \hat{g}(t)) \, dt \), where \( \hat{g}(t) : [0, h] \rightarrow G \) satisfies Euler-Lagrange equations with boundary conditions \( \hat{g}(0) = g_k, \hat{g}(h) = g_k f_k \). The resulting discrete-time Lagrangian system approximates the Euler-Lagrange equations to the same order of accuracy as the discrete Lagrangian approximates the Jacobi solution.

The velocities \( \dot{x} \) and \( \dot{\theta} \) at \( t_k \) are approximated by
\[
\dot{x}_k = \frac{1}{h} \Delta x_k, \quad \dot{\theta}_k = \frac{1}{h} \Delta \theta_k.
\]

From the attitude kinematics equation (1), the angular velocity can be approximated by
\[ \dot{\Omega}_k = \frac{1}{h} R_k^T (R_{k+1} - R_k) = \frac{1}{h} (F_k - I_{3 \times 3}). \]

Substituting these into (8), a discrete Lagrangian is given by
\[
L_d(g_k, f_k) = \frac{1}{h} T(R_k, \dot{\theta}_k, \Delta x_k, (F_k - I_{3 \times 3})^\top, \Delta \theta_k) - \frac{h}{2} U(x_k, R_k) - \frac{h}{2} U(x_k + \Delta x_k, R_k F_k). \tag{25}
\]

Using properties of the hat map, it can be rearranged as
\[
L_{d_k} = \frac{1}{2h} m \| \Delta x_k \|^2 + \frac{1}{h} \text{tr}[(I_{3 \times 3} - F_k) A_k] + \frac{1}{h} \sum_{i=1}^n \Delta x_i^T R_k (F_k - I_{3 \times 3})(m_i \rho_i + Q_{ik} I_i) + \frac{1}{h} \sum_{i=1}^n \Delta x_i^T R_k Q_{ik} \dot{e}_i I_i \Delta \theta_{ik} + \frac{1}{2h} \dot{e}_i^T J e_i \Delta \theta_{ik}^2 - \frac{h}{2} U_k - \frac{h}{2} U_{k+1}, \tag{26}
\]

where the matrices \( A_k, J_{d_k}, J_{Q_i} \in \mathbb{R}^{3 \times 3} \) are defined as
\[ A_k = J_{d_k} + \sum_{i=1}^n \frac{1}{2} (-\hat{\rho}_i Q_{ik} \hat{I}_i + Q_{ik} J_i) \dot{e}_i \Delta \theta_{ik}^\top, \]
\[ J_{d_k} = \frac{1}{h} \text{tr}[J_{Q_k} I_{3 \times 3} - J_{Q_k}], \]
\[ J_{Q_k} = J + \sum_{i=1}^n (-\hat{\rho}_i Q_{ik} \hat{I}_i - Q_{ik} \hat{I}_i \hat{\rho}_i + Q_{ik} J_i Q_i^T). \]
B. Discrete Euler-Lagrange Equations

f) Derivatives of Discrete Lagrangian: The derivatives of the discrete Lagrangian with respect to $\Delta x_k$ and $x_k$ are given by

$$
\mathbf{D}_{\Delta x_k} L_{d_k} = \frac{m}{h} \Delta x_k + \frac{1}{h} R_k (F_k - I_{3 \times 3}) \sum_{i=1}^{n} (m_i \rho_i + Q_{ik} I_i) + \frac{1}{h} \sum_{i=1}^{n} R_k Q_{ik} \hat{e}_i I_i \Delta \theta_{ik} + \frac{h}{2} f_{uk+1},
$$
(27)

$$
\mathbf{D}_{x_k} L_{d_k} = \frac{h}{2} f_{uk} + \frac{h}{2} f_{uk+1}.
$$
(28)

Using (14), the derivatives of the discrete Lagrangian with respect to $\theta_{ik}$ and $\hat{\theta}_{ik}$ can be written as

$$
\mathbf{D}_{\theta_{ik}} L_{d_k} = \frac{1}{h} e^T I_{j} e_1 \Delta \theta_{ik} + \frac{1}{h} \Delta x_k^T R_k Q_{ik} \hat{e}_i I_i + \frac{1}{2h} \text{tr}[(I_{3 \times 3} - F_k)(-\hat{\rho}_i Q_{ik} I_i + Q_{ik} e_1 e_1)],
$$
(29)

$$
\mathbf{D}_{\hat{\theta}_{ik}} L_{d_k} = \frac{1}{h} \Delta x_k^T R_k (F_k - I_{3 \times 3}) Q_{ik} \hat{e}_i I_i + \frac{1}{h} \Delta x_k^T R_k Q_{ik} \hat{e}_i^2 I_i \Delta \theta_{ik} + \frac{1}{h} \text{tr}[(I_{3 \times 3} - F_k)(J_{d_k}) + \frac{1}{2} (\hat{\rho}_i Q_{ik} \hat{e}_i I_i + Q_{ik} \hat{e}_i e_1 I_i) e_1 \Delta \theta_{ik}],
$$
(30)

where

$$
\mathcal{J}_{d_k} = \frac{1}{2} \text{tr}[\mathcal{J}_{Q_k} I_{3 \times 3} - \mathcal{J}_{Q_k}],
$$

$$
\mathcal{J}_{Q_k} = I_{j} e_1 (\hat{\rho}_i Q_{ik} I_i) - (Q_{ik} \hat{e}_i I_i)^T \hat{\rho}_i + Q_{ik} (\hat{e}_i I_i + I_i \hat{e}_i) Q_{ik}^T.
$$

Similar to (11), the variation of $F_k$ can be written as $\delta F_k = F_k \xi_k$ for $\xi_k \in \mathbb{R}^3$. Using this, the left-trivialized derivative of $L_{d_k}$ with respect to $F_k$ can be written as

$$
(\mathcal{T}^T F_k \cdot \mathbf{D} F_k L_{d_k}) \cdot \xi_k = \frac{1}{h} (A_k F_k - F_k^T A_k^T) v_k \cdot \xi_k - \frac{1}{h} \sum_{i=1}^{n} \overrightarrow{R_{k+1}} \Delta x_k (m_i \rho_i + Q_{ik} I_i) \cdot \xi_k + \frac{h}{2} M_{uk+1} \cdot \xi_k.
$$
(31)

Similarly, the derivative with respect to $R_k$ is

$$
(\mathcal{T}^T R_k \cdot \mathbf{D} R_k L_{d_k}) \cdot \eta_k =
\frac{1}{h} \sum_{i=1}^{n} \overrightarrow{R_{k}} \Delta x_k (m_i \rho_i + Q_{ik} I_i) \cdot \eta_k - \frac{1}{h} \sum_{i=1}^{n} \overrightarrow{R_{k}} \Delta x_k Q_{ik} \hat{e}_i I_i \Delta \theta_{ik} \cdot \eta_k + \frac{h}{2} (M_{uk} + F_k M_{uk+1}) \cdot \eta_k.
$$
(32)

g) Virtual Work: The integral of the virtual work over a single time step is approximated by

$$
\delta W_d = \frac{h}{2} (f_{ek} \cdot \delta x_k + f_{ek+1} \cdot \delta x_{k+1}) + \frac{h}{2} (M_{ek} \cdot \eta_k + M_{ek+1} \cdot \eta_{k+1}) + \frac{h}{2} \sum_{i=1}^{n} (\tau_{ik} \delta \theta_{ik} + \tau_{ik+1} \delta \theta_{ik+1}).
$$

h) Lie group variational integrator: Substituting (27)–(32) into the discrete Euler-Lagrange equations on an abstract Lie group [8], a Lie group variational integrator for the presented spacecraft with reaction wheels is given by

$$
\mathbf{D}_{\Delta x_k} L_{d_k} - \mathbf{D}_{\Delta x_{k+1}} L_{d_{k+1}} + \mathbf{D}_{x_k} L_{d_{k+1}} = -h f_{ek+1},
$$
(33)

$$
\mathcal{T}^T_k L_{F_k} \cdot \mathbf{D} F_k L_{d_k} - \mathcal{T}^T_k L_{F_{k+1}} \cdot \mathbf{D} F_{k+1} L_{d_{k+1}} + \mathcal{T}^T_k L_{R_k} \cdot \mathbf{D} R_k L_{d_k} - \mathcal{T}^T_k L_{R_{k+1}} \cdot \mathbf{D} R_{k+1} L_{d_{k+1}} = -h M_{ek+1},
$$
(34)

$$
\mathbf{D}_{\theta_k} L_{d_k} - \mathbf{D}_{\theta_{k+1}} L_{d_{k+1}} + \mathbf{D}_{\hat{\theta}_k} L_{d_{k+1}} = -h \tau_{k+1},
$$
(35)

For a given $(g_k, f_k) = ((R_k, x_k, \theta_k), (F_k, \Delta x_k, \Delta \theta_k))$, the configuration at the next step $g_{k+1}$ is obtained from the kinematics equation (24). Then, we solve (33)–(35) to obtain $f_{k+1}$. These yield a discrete Lagrangian flow map $(g_k, f_k) \mapsto (g_{k+1}, f_{k+1})$ and it is repeated.

Alternatively, discrete Legendre transformation is given by

$$
p_{x_k} = \mathbf{D}_{\Delta x_k} L_{d_k} - \mathbf{D}_{x_k} L_{d_k} = \frac{h}{2} f_{ek},
$$
(36)

$$
p_{\Omega_k} = \mathbf{D}_{\theta_k} L_{d_{k+1}} - \mathbf{D}_{\hat{\theta}_k} L_{d_{k+1}} = \frac{h}{2} \tau_k.
$$
(37)

The corresponding discrete Hamilton’s equations are

$$
p_{x_{k+1}} = p_{x_k} + \mathbf{D}_{\Delta x_k} L_{d_k} = \frac{h}{2} (f_{ek} + f_{ek+1}),
$$
(39)

$$
p_{\Omega_{k+1}} = \mathbf{D}_{\theta_k} L_{d_k} + \mathbf{D}_{\hat{\theta}_k} L_{d_{k+1}} = \frac{h}{2} (M_{ek} + M_{ek+1}),
$$
(40)

$$
p_{\rho_{k+1}} = p_{\rho_k} + \mathbf{D}_{\hat{\theta}_k} L_{d_{k+1}} = \frac{h}{2} (\tau_k + \tau_{k+1}).
$$
(41)

For a given $(g_k, p_k) = ((R_k, x_k, \theta_k), (P_{\Omega_k}, p_{x_k}, p_{\rho_k}))$, we solve (36)–(38) for $f_k$. Then, $g_{k+1}$ is obtained from (24), and $p_{k+1}$ is obtained from (39)–(41). These yield a discrete Hamiltonian flow map $(g_k, p_k) \mapsto (g_{k+1}, p_{k+1})$.

V. NUMERICAL EXAMPLE

We consider a spacecraft model with $n = 3$ reaction wheels. Mass properties of the base spacecraft are given by

$$
m = 500 \text{ kg}, \quad J = \text{diag}[120, 120, 60] \text{ km}^2/\text{s}.
$$

The initial position and velocity are chosen as zero: $x(0) = v(0) = 0_{3 \times 1}$, and the initial attitude is $R = I$ and the initial angular velocity is $\Omega(0) = [0.5, 0.1, -0.2] \text{ rad/s}$. 

3126
The configurations of reaction wheels are selected as
\[ s_i = e_i, \quad \rho_i = 0.6s_i, \quad \text{for} \quad 1 \leq i \leq 3. \]

And, reaction wheel frames are defined as
\[ Q_1 = \exp(\theta_1 e_1), \quad Q_2 = \exp(\theta_2 e_2)Q_0, \quad Q_3 = \exp(\theta_3 e_3)Q_0^T, \]
where \( Q_0 = [e_2, e_3, e_1] \in SO(3). \) These are chosen such that \( Q_i e_1 = s_i \) for \( 1 \leq i \leq 3. \) The second and the third reaction wheels are symmetric about its spin axis:
\[ I_2 = I_3 = 0.3\times1, \quad J_2 = J_3 = \text{diag}[0.086, 0.01, 0.01] \text{kgm}^2. \]

The first reaction wheel has an additional mass particle \( m_a = 0.1 \text{kg} \) at \( \xi_a = 0.1e_2 \text{m} \) that represents reaction wheel imbalance. The resulting first and second mass moment of inertia are given by
\[ I_1 = m_a \xi_a, \quad J_1 = J_2 - m_a \xi_a^2. \]

The initial angle and the angular velocity of the reaction wheels are \( \theta(0) = 0 \) and \( \dot{\theta}(0) = 1000 \times [-0.3, 0.3, 1] \text{ rpm}. \)

No potential energy or external force/moment is considered, i.e., \( U = \tau_1 = 0 \) and \( M_e = f_e = 0 \times 1. \) Simulation period is \( T = 500\text{ sec} \) and the time step is \( h = 0.01 \text{ sec}. \) The corresponding computational results are illustrated at Figure 2, where the presented Lie group variational integrator (LGVI) is compared with numerical results obtained from a variable step size, 4-5th order Runge–Kutta method (RK).

As there is no external torque or moment, the total energy is preserved. Figure 2(a) illustrates that RK fails to preserve the total energy accurately. The total energy computed by LGVI oscillates near its initial value, but there is no secular change over a long time period. This is due to symplectic properties of LGVI [12]. At Figure 2(b), LGVI preserves the orthogonal structure of rotation matrices up to the level of machine precision. However, the orthogonality error of RK increases quite rapidly.

The error in preservation of symplecticity and characteristics of configuration space causes error in computation of trajectories of angular velocity of spacecraft, spin rate of reaction wheels, and the velocity of spacecraft, as illustrated at Figures 2(c)–2(e). These numerical results show that LGVI preserves the geometric properties of spacecraft with imbalanced reaction wheels accurately for the presented complex maneuver that exhibits nontrivial coupling between translations and rotations. More quantitative analysis of computational properties of LGVI compared with other integrators are also available at [7], [13].

REFERENCES