A Modal Decomposition Approach to Distributed Observation and Control for Grid Integration of Renewable Generation

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Abstract—To enable renewable generation to fully contribute to reliable grid operation with higher levels of penetration, it is important that such resources contribute to grid control, including faster time scale active power control that is valuable to small-signal electromechanical stability. However, given the small size and distributed nature of many renewables, it is unrealistic to assume that they will employ the type of centrally coordinated control schemes typical of large, central station synchronous generators. This paper proposes a distributed control design approach that allows each distributed renewable generator or energy storage device to be responsible for improving damping of a single oscillatory electromechanical mode. The consequence is that one may employ a static state feedback design (a classical LQ design is employed here) that requires information of state behavior only within a two-dimension invariant subspace associated with the mode of interest. Using a modal-focused Hautus matrix test of observability, we develop simple tests to establish grid buses from which a desired mode may be most effectively observed from either local frequency measurement, or from a local measurement plus a single remote phasor measurement unit (PMU) signal. In this framework, the state information required may then be recovered by a Luenberger observer of as little as second order. That is, the local, distributed controller on a given renewable generator or energy storage device uses only one or two measurements, may be as simple as second order, and yet contributes to the system objective of improving modal damping.

I. INTRODUCTION

Among the challenges to integrating renewable generation into the electric power system is that of enabling renewables to contribute to a wide range of grid control objectives. On the relatively fast time scales for which closed loop control is typically exercised, minutes to milliseconds, synchronous rotating generators have long been the dominant control “actuators” for the grid. Much of architecture of power system control is predicated upon either local frequency measurement, or local measurement plus a single remote phasor measurement unit (PMU) signal. In this framework, the state information required may then be recovered by a Luenberger observer of as little as second order. That is, the local, distributed controller on a given renewable generator or energy storage device uses only one or two measurements, may be as simple as second order, and yet contributes to the system objective of improving modal damping.

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make power electronically coupled generation (notably wind) mimic the control characteristics and physical inertia of synchronous machines (e.g., [1] - [5]).

The work here will seek a different approach to the objective of active power/frequency grid control from renewables. We seek to exploit the natural modal characteristics of the grid to obtain a distributed observer/controller design. This distributed design will offer the possibility of wholly local implementation on individual actuators (i.e., renewable generators), while also allowing enhanced performance when supplemental remote measurement signals are available. High quality, high bandwidth measurement devices are becoming more prevalent in the power grid, suggesting control designs should have the flexibility to incorporate these improved output measurements where available.

The design methodology adopts the philosophy of assigning control responsibility “mode-by-mode” to individual actuators. The active power/frequency dynamics of the power grid have a structure that is nearly Hamiltonian, with relatively low loss, giving rise to lightly damped, oscillatory natural response. A common objective of feedback control is to improve damping of these oscillatory modes. We select the objective for an individual actuator, at a given network location, to be that of improving damping of just one mode. For this objective, the state feedback control requires information only about state behavior in the subspace associated with the mode of interest. Therefore, a local observer to “feed” this control may be very low order, estimating state behavior only in this subspace. In many power system examples, such a low order observer can operate successfully with only a single local measurement of frequency.

Prior work by the authors had explored this general philosophy in models for power grid electromechanical behavior that were of standard state space form. However, such formulations are undesirable in many power systems applications, because they require algebraic elimination of electrical variables associated with load buses, thereby masking the network structure naturally occurring in the model, and creating a very dense state matrix. Power system dynamic models are most naturally formulated as mixed set of differential-algebraic equations, or DAE models. In a linearized, small signal model, this gives rise to a generalized eigenvalue problem. A key objective of this paper will be to extend the prior control results to the DAE case. We will demonstrate that standard Hautus matrix tests for modal observability (and controllability) generalize in a straightforward fashion to the generalized eigenvalue/DAE formulation.
II. MODAL DECOMPOSITION BASED APPROACH TO CONTROLLER DESIGN

A. Summary of Previous Work

In previous work [6] and [7], a distributed control design was proposed to enhance the electromechanical stability of the grid utilizing actuators with complementary saturation-bandwidth characteristics. The objective of the control there was to regulate grid frequency on time scale of electromechanical oscillations excited by wind power variations, with a state matrix formed based on a linearized model of the system, including an exogenous system model to represent wind power variations. The algebraic variables associated with power balance at load buses were eliminated, yielding a “dense” state matrix.

The design is performed on a system transformed into its modal coordinates. The transformation square matrix $P^{-1}$ is formed such that every real eigenvector, $q_i$, of the system matrix $A$ forms a column of $P^{-1}$ and every complex eigenvector of $A$, $q_j$, results in two columns of $P^{-1}$ such that one column is the real part of $q_j$ and the other column is the imaginary part of $q_j$. The modal coordinate representation is given by $A = PA^{-1}$.

An optimal control design method to form controllers for linear systems subject to input amplitude limits (i.e., actuator saturation) demonstrated in [8], and in our work is tailored to exploit the complementary characteristics of wind generators and electric storage as actuators of interest. These actuators can be classified into two broad classes: (i) low bandwidth, “slow” actuators with broad saturation limits (e.g., power control available by varying blade pitch in wind generators) and (ii) high bandwidth, “faster” actuators with narrow saturation limits (e.g., power delivery/absorption available from battery or flywheel energy storage). The objective is output regulation in the face of disturbances produced by an “exosystem,” where the exosystem is explicitly constructed as part of the overall model. Under appropriate solvability conditions, a linear quadratic regulator (LQR) problem yields the desired state feedback. The $Q$ matrix in the design is partitioned into two blocks, introducing two scaling parameters that are selected to allow the actuators’ complementary saturation and bandwidth characteristics to be effectively utilized.

In order to make the design practically feasible for real-time control in large power systems, a distributed observer based design is adopted. In this design, multiple local state observers are constructed, each estimating states using very limited available measurements. In particular, it is shown that by adopting a modal focused approach, the observer based control performance is satisfactory using locally available measurements, with at most one or two remote measurements. The objective of a given observer is to estimate state behavior only in the subspace associated with a particular mode, whose stability enhancement is the goal of the corresponding local actuator. A key aspect in the proposed controller design is the choice of appropriate measurement signals to be used in the observer design, and choice of appropriate actuator locations. This is addressed by evaluating properties of a corresponding Hautus matrix for a given mode. Conditioning of the Hautus matrix is employed as the measure of degree of observability/controllability of a given mode.

B. Relation between Hautus Matrix and ease of Observation and Control

To limit the number of remote PMU measurements required at any controller location, thereby limiting the cost of securely and reliably communicating these signals, a methodology to select the best set of measurement signals at any controller location is important. Below we review standard relations between the “effort” required to observe the states and the observability gramian, and then, in the development most relevant to this application, show the relation to properties of the observability Hautus matrix in order to quantify the degree of observability of an individual mode.

Let $\lambda_1, \lambda_2, ..., \lambda_m$ be the eigenvalues of the system state matrix $A$, and $c^k$ be the output matrix for a chosen measurement signal $k$. Then, the observability Hautus matrix for mode $i$ is given by

$$H_i = \begin{bmatrix} \lambda_i I - A \\ c^k \end{bmatrix}$$

(1)

Standard textbook results guarantee the system observable if and only if the Hautus matrices associated with every eigenvalue of the system are full rank. To quantify how close an observable system is to loss of observability, a singular value decomposition-based method is typically employed [9], with the proximity to zero of the least singular value of $H_i$ providing the measure.

Let $\sigma^k_i$ be the least singular value associated with eigenvalue/mode $i$ (i.e., associated with $H_i$) for the chosen output channel $k$. Then, the measurement location that results in the largest $\sigma^k_i$ would be the best location to observe the mode $i$. Thus, one might hypothesize that a measurement location that results in “best” singular values across all the modes of concern would yield the best observer performance. A related approach, as adopted here, examines the condition number of $H_i$, recognizing that process of reconstructing state from measurements is essentially an inversion. For a given mode, the measurement location that results in the smallest condition number among all candidate locations is selected.

The appropriateness of this Hautus matrix-based algorithm can be validated with the results obtained as part of the observer design. The observability gramian, $W_o$, is given by

$$W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau$$

(2)

where, $C$ is the system output matrix. For a linear system represented using the standard state space representation, the zero-input response is given by

$$y(t) = C e^{A t} x(0)$$

(3)
where, \( x(0) \) represents the initial state. The system is observable if the initial state \( x(0) \) can be determined uniquely from its zero-input response \( y(t) \) over a finite time interval \([0, t_1]\) [10]. An energy-like expression for effort needed to reconstitute the initial state may be quantified as

\[
E_o(t_1) = \int_0^{t_1} ||y(\tau)||^2 d\tau = \int_0^{t_1} y(\tau)^T y(\tau) d\tau. \tag{4}
\]

Substituting (3) in the above equation, one obtains

\[
E_o(t_1) = \int_0^{t_1} x(0)^T e^{A^T C^T C e^{A^T} x(0)} d\tau
\]

\[
= x(0)^T \left[ \int_0^{t_1} e^{A^T C^T C e^{A^T} x(0)} d\tau \right] x(0)
\]

\[
= x(0)^T W_o(t_1)x(0). \tag{7}
\]

As \( t_1 \to \infty \),

\[
E_o = x(0)^T W_o x(0) \tag{8}
\]

where, \( W_o \) is the unique solution of the equation

\[
A^T W_o + W_o A = -C^T C. \tag{9}
\]

Thus, (8) provides an expression for the “observability effort” in terms of the gramian. Next, we relate the observability Hautus matrix to the observability gramian, within the subspace spanned by the eigenvector(s) of interest.

The observability Hautus matrix for an eigenvalue \( \lambda \) is given by

\[
\mathcal{H}_o = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}
\]

\[
\Rightarrow \lambda^* \mathcal{H}_o = \lambda I - A + C^T C
\]

\[
= \lambda I - A \lambda^* (\lambda I - A) = 0. \tag{10}
\]

Substituting (10) in (9), we get

\[
A^T W_o + W_o A + \lambda^* \mathcal{H}_o - (\lambda I - A)^* (\lambda I - A) = 0. \tag{11}
\]

Let \( e \) represent the right eigenvector associated with a real \( \lambda \) (for a mode described by a complex conjugate eigen-pair, it is straightforward to generalize to the case of two vectors \( e_1 \) and \( e_2 \) spanning the subspace associated with this mode’s natural response). Pre-multiplying and post-multiplying (11) with \( e^* \) and \( e \) respectively, we get

\[
e^* A^T W_o e + e^* W_o A e + e^* \lambda^* \mathcal{H}_o e = 0
\]

\[
\Rightarrow e^* (\lambda I - A)^* (\lambda I - A) e = 0. \tag{12}
\]

But, from the definition of right eigenvector, we have

\[
(\lambda I - A) e = 0
\]

so, (12) reduces to

\[
e^* A^T W_o e + e^* W_o A e + e^* \mathcal{H}_o^* \mathcal{H}_o e = 0
\]

\[
\Rightarrow e^* \lambda^* W_o e + e^* W_o A e + e^* \mathcal{H}_o^* \mathcal{H}_o e = 0
\]

\[
\Rightarrow 2 \text{Re}(\lambda) e^* W_o e + e^* \mathcal{H}_o^* \mathcal{H}_o e = 0
\]

\[
\Rightarrow e^* W_o e = e^* \left[ \begin{bmatrix} \mathcal{H}_o^* \mathcal{H}_o \\ -2 \text{Re}(\lambda) \end{bmatrix} \right] e. \tag{13}
\]

Thus, (8) and (13) provide a relation among the observability gramian, the observability Hautus matrix, and the observability effort within the subspace associated with the mode of interest.

So, we can conclude that the candidate measurement set that yields the smallest singular values of observability Hautus matrices across the modes of concern would result in the best state estimates on the subspace of modes of concern. Thus, this observability Hautus matrix-based algorithm can be used to identify the best set of measurement signals that would result in most appropriate state estimation on the subspace of critical modes. In the context of the power system application, this selection of measurement set will dictate the network locations (i.e., which bus) at which an observer may effectively operate. While comparing different measurement sets for a given mode, the eigenvalue in the denominator of \( \left[ \mathcal{H}_o^* \mathcal{H}_o \right] \) does not impact the condition number comparisons (since the comparison is for two different measurement sets, but for the same given mode). Hence, the comparison of condition numbers of the Hautus matrices across various candidate sets can be used to determine the most effective one.

Derivation of the dual of the above to treat the controllability case is straightforward. The best actuator locations are naturally those that require the least amount of control energy in order to meet the control objective. As in the observability problem, the properties of the controllability Hautus matrix determine the “ease” of control within the invariant subspace associated with the mode of interest. For a given mode whose damping one seeks to improve by feedback, our approach selects the bus location that results in the smallest condition number among the controllability Hautus matrices for candidate actuator locations.

C. Demonstration on Example System

Next, we demonstrate the controllability Hautus matrix algorithm to identify the best actuator locations on the IEEE 39 bus test system, and validate the results through examination of control energy observed in time-domain simulations of the design. The IEEE 39 bus system is a widely used standard test system for power system dynamic studies. The test system consists of 10 generators and 29 loads. For the purpose of demonstrating the proposed algorithm on this test system, 3 of the synchronous generators are replaced with standard Type-3 wind generator models. The actuators considered for this test case are battery storage systems. As described above, for a given battery location, the condition number of the Hautus matrix is used to predict the relative control energy required to provide actuation at that bus. This
algorithm allows efficient identification of locations that result in effective utilization of the batteries, without requiring exhaustive time-domain simulation comparison among all candidate locations.

The proposed Hautus matrix-based algorithm can also be used to compare effectiveness of different configurations for battery operation (e.g., current injection versus power injection), as well as the mode most amenable to damping enhancement from a given location. We first compare cases: one for which the battery is operating in current injection mode connected at a synchronous generator bus (bus 32); another one in which the battery is operating in power injection mode connected at a wind generator bus (bus 30). The resulting condition numbers for the two cases are shown in Table I. From this table it can be seen that the current injection scheme yielded a smaller condition number across all modes, relative to the power injection scheme. This agrees with widely-held “rules of thumb” that judge current injection schemes to be superior. To further validate this conclusion, we examine the absolute value of energy expended/absorbed by the battery for the current injection versus the power injection scheme, over a 25 second time interval. The resulting control energy is 0.0033 p.u. for the current-based scheme, and 0.0083 p.u. for the power-based scheme. The control energy over 25 successive one-second integration intervals is also displayed in Fig. 1. These confirm that the current-based scheme provides regulation with lower control energy than the power-based scheme over each time interval, consistent with the Hautus matrix-based ranking. Next, we compare two different battery connection points, bus 39, versus bus 35. Table II below shows the resulting condition numbers of the Hautus matrix for these two cases. From the table it can be seen that the condition numbers for the case where the battery is connected at bus 39 are smaller than those for the case where the battery is connected at bus 35. The resulting control energies for the two cases, again over a 25 second time interval, are 0.0013 p.u. and 0.013 p.u. respectively. Here again, the time domain control energy performance results are consistent with the ranking of the Hautus matrix-based algorithm.

III. HAUTUS TEST FOR GENERALIZED EIGENVALUE PROBLEM

In this section, we demonstrate that the Hautus test for controllability in a standard linear time invariant state space model extends to the generalized eigenvalue case in a natural fashion. In large-scale power systems, the generalized eigenvalue framework offers the significant advantage that the associated matrices are very sparse. Furthermore, the eigenvectors associated with the algebraic variables, which form a part of the generalized eigenvector, are very useful to identify the role of algebraic network variables in system dynamic response.

Consider the state space formulation of a DAE model

\[
\dot{x} = Ax + B_y y + B_u u
\]

\[
0 = Cx + Dy
\]

\[
y_{out} = C_{out} x
\]

where,

- \(x\) represents the dynamic state variables,
- \(y\) represents the algebraic state variables,
- \(u\) represents the input signals,
- \(y_{out}\) represents the measurement signals.

A generalized eigenpair \((\mu, v)\) for this model is given by

\[
\begin{bmatrix}
A & B_y \\
C & D
\end{bmatrix} v = \mu E v
\]

where,

\[
E = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Mode</th>
<th>Current Injection at Bus 32</th>
<th>Power Injection at Bus 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.84</td>
<td>17.12</td>
</tr>
<tr>
<td>2</td>
<td>21.58</td>
<td>35.48</td>
</tr>
<tr>
<td>3</td>
<td>33.87</td>
<td>51.13</td>
</tr>
<tr>
<td>4</td>
<td>35.10</td>
<td>64.85</td>
</tr>
<tr>
<td>5</td>
<td>49.18</td>
<td>104.80</td>
</tr>
<tr>
<td>6</td>
<td>50.03</td>
<td>121.39</td>
</tr>
<tr>
<td>7</td>
<td>74.03</td>
<td>288.88</td>
</tr>
<tr>
<td>8</td>
<td>366.29</td>
<td>790.52</td>
</tr>
<tr>
<td>9</td>
<td>623.12</td>
<td>1042.58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Power Injection at Bus 39</th>
<th>Power Injection at Bus 35</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.09</td>
<td>16.09</td>
</tr>
<tr>
<td>2</td>
<td>25.72</td>
<td>27.75</td>
</tr>
<tr>
<td>3</td>
<td>33.34</td>
<td>48.46</td>
</tr>
<tr>
<td>4</td>
<td>47.89</td>
<td>51.52</td>
</tr>
<tr>
<td>5</td>
<td>74.22</td>
<td>103.54</td>
</tr>
<tr>
<td>6</td>
<td>185.87</td>
<td>358.10</td>
</tr>
<tr>
<td>7</td>
<td>362.36</td>
<td>840.45</td>
</tr>
<tr>
<td>8</td>
<td>1037.30</td>
<td>1159.04</td>
</tr>
<tr>
<td>9</td>
<td>4762.80</td>
<td>2776.40</td>
</tr>
</tbody>
</table>
Let $v = [v_1^T \ v_2^T]^T$. Then,

$$Av_1 + Byv_2 = \mu v_1, \ Cv_1 = -Dv_2$$

$$\Rightarrow v_2 = -D^{-1}Cv_1$$

$$\Rightarrow (A - ByD^{-1}C)v_1 = \mu v_1$$

where, $(A - ByD^{-1}C)$ is the reduced system matrix $A_{\text{red}}$ obtained by eliminating the algebraic variables. Recall that for the generalized eigenvalue problem, and for a finite eigenvalue of this problem, the $v_1$ components of the full eigenvector are the “same” (up to the inherent scaling non-uniqueness of eigenvectors) as the reduced system’s eigenvector.

The observability Hautus matrix corresponding to this reduced system for an eigenvalue $\mu$ is given by

$$\mathcal{H}_{\text{red}} = \begin{bmatrix} (\mu I - A_{\text{red}}) \\ C_{\text{out}} \end{bmatrix}$$

(18)

The corresponding Hautus matrix for the full generalized problem is given by

$$\mathcal{H}_{\text{full}} = \begin{bmatrix} (\mu E - A) \\ (C_{\text{out}} 0) \end{bmatrix}$$

(19)

**Proposition:** $(A_{\text{red}}, C_{\text{out}})$ is observable $\Leftrightarrow \mathcal{H}_{\text{full}}$ is full rank.

**Proof:** (i) We first show that $(A_{\text{red}}, C_{\text{out}})$ is observable $\Rightarrow \mathcal{H}_{\text{full}}$ is full rank.

To do so, we adopt a “proof-by-contradiction” approach. Assume that $\mathcal{H}_{\text{full}}$ is not full rank,

$$\Rightarrow \exists v \neq 0 : (\mu E - A)v = 0 \text{ and } (C_{\text{out}} 0)v = 0$$

$\Rightarrow v$ is a right eigenvector of $A$. Let $v = [v_1^T \ v_2^T]^T$,

$$\Rightarrow C_{\text{out}}v_1 = 0.$$  

But, $(A_{\text{red}}, C_{\text{out}})$ is observable $\Rightarrow C_{\text{out}}v_1 \neq 0$ for all right eigenvectors of $A_{\text{red}}$. $\Rightarrow$ contradiction.

Hence, $(A_{\text{red}}, C_{\text{out}})$ observable $\Rightarrow \mathcal{H}_{\text{full}}$ is full rank.

(ii) Next, we show that $\mathcal{H}_{\text{full}}$ is full rank $\Rightarrow (A_{\text{red}}, C_{\text{out}})$ is observable.

Here again, we adopt a proof-by-contradiction approach. Assume that $(A_{\text{red}}, C_{\text{out}})$ is not observable,

$$\Rightarrow \exists v_1 \neq 0 : (\mu I - A_{\text{red}})v_1 = 0 \text{ and } C_{\text{out}}v_1 = 0$$

$$\Rightarrow (\mu E - A)[v_1^T \ v_2^T]^T = 0 \text{ (since, } [v_1^T \ v_2^T]^T \text{ is a generalized eigenvector.)}$$

Also, $(C_{\text{out}} 0)[v_1^T \ v_2^T]^T = C_{\text{out}}v_1 = 0$

$$\Rightarrow \mathcal{H}_{\text{full}}[v_1^T \ v_2^T]^T = 0$$

But, $\mathcal{H}_{\text{full}}$ is full rank $\Rightarrow$ contradiction.

Hence, $\mathcal{H}_{\text{full}}$ full rank $\Rightarrow (A_{\text{red}}, C_{\text{out}})$ is observable. $\blacksquare$

It is straightforward to extend the above result to the controllability problem, that is, the reduced order system is controllable if and only if the full order system’s controllability Hautus matrix is full rank.

### IV. GENERALIZED EIGENVALUE PROBLEM FOR POWER SYSTEM DYNAMICS

We consider a generalized eigenvalue problem in the context of small signal, linearized power system dynamics, along with a simple change of coordinates, that reveals structure in the state matrix that can be exploited in developing control applications for large power systems. In particular, consider an augmented network of $n$ buses, with those numbered 1 through $m$ representing the internal voltage of synchronous generators, and the remainder representing generator terminals and loads. The network is augmented in the sense that generator internal bus voltages are explicitly represented, with an appropriate series reactance connecting the internal bus to the terminal bus. For the analysis to follow in this paper, a classical representation of constant internal bus voltage magnitude is assumed. The terminal bus is subject to complex power balance constraints in the same way as a load bus. The dynamic equations for the system possess a modified Hamiltonian form, which yields a closed form relationship between real eigenvalues of a reduced dimension, symmetric problem, and eigenvalues of the full dimension, linearized dynamic model [11].

The standard non-linear model for the rotational dynamics of synchronous generators coupled through the power exchange in the network is given by

$$\dot{\omega} = M^{-1}L_1(P^I - P^N(\delta, V))$$

$$L_1\dot{\delta} = \omega$$

$$0 = L_2(P^I - P^N(\delta, V))$$

$$0 = (Q^I(V) - Q^N(\delta, V))$$

where, $\omega$ is a vector of generator frequency deviations, relative to synchronous frequency $\omega_0$, $\delta$ is a vector of all bus voltage phase angles, $M$ is a diagonal matrix of normalized generator inertias, $P^I, Q^I$ are vectors of net active power and reactive power injections at each bus respectively, $P^N, Q^N$ are vector-valued functions of active power absorbed by network at each bus and reactive power absorbed by network at load buses, normalized by voltage magnitude, $L_1$ are rows 1 through $m$ of an $n \times n$ identity matrix, $L_2$ are rows $m + 1$ through $n$ of an $n \times n$ identity matrix. On linearizing the above equations about an operating point, the equations can be written in a DAE form as

$$\dot{E}\Delta \dot{x} = \dot{R}\Delta x$$

(20)

where,

$$\dot{E} = \begin{bmatrix} I_{m \times m} & 0 & 0 \\ 0 & I_{m \times m} & 0 \\ 0 & 0 & 0_{2(n-m) \times 2(n-m)} \end{bmatrix}$$

$$\dot{R} = \begin{bmatrix} 0 & -I_{m \times m} & 0 \\ I_{m \times m} & 0 & 0 \\ 0 & 0 & I_{2(n-m) \times 2(n-m)} \end{bmatrix} S$$

$$S = \begin{bmatrix} I_{m \times m} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & I_{2(n-m) \times 2(n-m)} \end{bmatrix} * \begin{bmatrix} 0 & 0 \\ 0 & J_{11} & J_{12} \\ 0 & J_{21} & J_{22} \end{bmatrix}$$
where,
\[
J = \begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial P_N}{\partial \delta} & \frac{\partial P_N}{\partial V_L} \\
\frac{\partial Q_N}{\partial \delta} & \frac{\partial (Q_N - Q_I)}{\partial V_L}
\end{bmatrix}
\]
The partial derivatives in the above matrix are given in [11]. The partitions 1 and 2 of \( J \) above use the following convention. Index 1 represents active power/angle components associated with the generator buses, while index 2 represents both active power/angle components associated with the load buses and reactive power/voltage components associated with the load buses. The labeling of variables in this scheme distinguishes power flow variables that appear as dynamic states from algebraically determined variables.

Consider a reduced dimension generalized eigenvalue problem defined by \((E, R)\), where
\[
E = \begin{bmatrix}
I_{m \times m} & 0 \\
0 & 0_{2(n-m) \times 2(n-m)}
\end{bmatrix}
\]
and \( R = D^TJD \), with
\[
D = \begin{bmatrix}
M^{-1/2} & 0 \\
0 & I_{2(n-m) \times 2(n-m)}
\end{bmatrix} S
\]
The finite generalized eigenvalue \( \lambda \) and corresponding eigenvector \( v = [v_1^T \ v_2^T]^T \) of \((E, R)\) are given by \( \lambda E v = R v \). Then, in [11], it is shown that the eigenpair \((\gamma, w)\) corresponding to the full dimension generalized problem \((\tilde{E}, \tilde{R})\) given by \( \gamma \tilde{E} w = \tilde{R} w \) is related to the reduced dimension problem by
\[
\gamma = j \sqrt{\lambda} \quad (21)
\]
\[
w = \begin{bmatrix}
j \sqrt{\lambda} M^{-1/2} v_1 \\
M^{-1/2} v_1 \\
v_2
\end{bmatrix} \quad (22)
\]
This result can be used in the controller design where modal decomposition are to be exploited. For the non-symmetric generalized eigenvalue problem of \((E, R)\), we may perform the controller design in the context of the reduced dimension, symmetric generalized eigenvalue problem \((E, R)\) so that the design space is reduced drastically.

V. CONCLUSIONS

This work has extended existing results for low-dimension, local controller design for distributed renewable generation to treat small signal linearized grid models in DAE form, and further reduced the computational burden by developing relationships between the DAE model and a lower dimensional, symmetric generalized eigenvalue problem that maintains the sparse Laplacian structure of the power flow Jacobian matrix. In particular, we have demonstrated that “mode-by-mode” tests for ease of observability, as computed from smallest singular value of an observability Hautus matrix, extend naturally to the generalized eigenvalue problem associated with the DAE method. This formulation will be particularly valuable in computations for distributed design of local controllers for very large numbers of small, power-electronically coupled generators such as wind and photovoltaic, which will naturally lead to DAE models of very high overall state dimension.

REFERENCES