Gradient Flows For Organizing Multi-Agent System

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\textbf{Abstract}---In this paper, we consider a class of gradient flows that model rules by which a multi-agent system might approach to an equilibrium. The rules are quite simple to state, in fact they depend on a single attraction/repulsion function, but in the generality assumed here the analysis of the resulting flow presents several challenges. In part, these challenges arise from the natural invariance with respect to the Euclidean group of an equilibrium state, implying that it is just the shape of the configuration and not the Euclidean coordinates of the individual agents that matters. We establish, among other things, a metric property of the gradient flow and give conditions under which the paths of the individual agents remain bounded as the flow evolves. We give a parametrized definition of clustering which induces a partial order that reflects the granularity of the clustering and establish important properties of the lattice defined in this way. We also explain significant properties of the clusters related to the attraction/repulsion function. Finally, we note some generic properties of the class of attraction/repulsion functions considered here.

I. INTRODUCTION

Motivated by what is seen in nature and the hopes for manmade systems, over the last two decades there has been a flood of papers dealing with multi-agent systems. Questions concerning the level of interaction that is necessary for the organization of such systems, questions about stability, robustness, etc. have all been treated to some degree (see, e.g., [2]–[11], [13], [14], [17]–[21]). In this paper the dynamics of assembling a configuration is treated along with a characterization of the configurations that are achievable using a simple class of rules based on gradient descent. The importance of gradient descent for finding solutions is widely appreciated both in mathematics and in the real world, added to the usual reasons, in the context of multi-agent control it can be interpreted as providing a decentralized solution for a problem involving an arbitrary collection of agents [13], [18]. The novelty of this paper lies not in the use of this technique but rather in describing some of its strengths and limitations in the the context of multi-agent control.

In its most elementary form, the flow defined by a descent equation comes from a potential function with isolated critical points. Here we deal with what might be thought of as a system with no landmarks. This means that the equations of motion depend only on the pairwise distances between agents and thus remain solutions if all the agents are translated or the entire configuration is rotated or reflected in the Euclidean space. This adds a degree of complication. Moreover, it is not clear at the outset that the descent law will ensure that one or more of the agents will not “escape” off to infinity. Finally, because higher level considerations may dictate the clustering of subgroups we investigate how the choice of the distance function relates to the formation of clusters. This is a rich question relating to classic techniques such as the \(k\)-means algorithm and its variants.

The paper is organized around a few definitions and three theorems. These are, our definitions

\textbf{Shape Space}. Shape of a configuration is determined by the pairwise distances between agents and is invariant under translation, rotation and reflection.

\textbf{Clustering Agents}. A clustering is a partition of agents in a configuration into disjoint sets with certain conditions on radii of clusters and inter-cluster distances.

The theorems in this paper describe the properties of a certain gradient flow and the associated equilibria. We now state the theorems below, a few notions including the definition of the kinematic model, the flow map \(\varphi\) will be clear after section 2.

\textbf{Theorem 1 (Bounded Size of Equilibria)}. Consider the kinematic model described by equation (1). There exist positive numbers \(a\) and \(b\) such that the distance between any two agents in an equilibrium lies in the closed interval \([a,b]\).

\textbf{Theorem 2 (Convergence of The Flow)}. Consider the kinematic model described by equation (1). For any initial condition \(p\), the flow \(\varphi_i(p)\) exists for all \(t > 0\) and converges to an invariant set consisting exclusively of equilibria of the gradient flow.

\textbf{Theorem 3 (Clustering on Sequence of Diverging Configurations)}. If there were a configuration \(p\) such that along the gradient flow \(\varphi_{\geq 0}(p)\) the diameter of the configuration can’t be bounded above, then there would be a monotone sequence of times \(\{t_i\}_{i \in \mathbb{N}}\) approaching to infinity such that

1) there is a nontrivial clustering of agents shared by all configurations \(\{\varphi_i(p)\}_{i \in \mathbb{N}}\).

2) all inter-cluster distances approach to infinity along the sequence of configurations \(\{\varphi_i(p)\}_{i \in \mathbb{N}}\).

3) radii of all clusters are less than a fixed number \(R\) for any configuration in the sequence \(\{\varphi_i(p)\}_{i \in \mathbb{N}}\).

After this introduction, we proceed in steps. In section 2, we will define the kinematic model, work out the potential function and discuss the group action of rigid motion. In section 3, we will introduce the shape space and investigate the induced vector field on it. In section 4, we will establish

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a metric property of equilibria, investigate the stabilization of equilibria orbits and discuss the genericity of equivariant Morse functions. In section 5, we deal with the convergence property of the gradient flow associated with equation (1) and various aspects of clustering of agents.

II. THE KINEMATIC MODEL, THE POTENTIAL FUNCTION AND THE GROUP ACTION OF RIGID MOTION

Consider the motion of a set of $N$ agents in a purely kinematic model whereby every agent feels the presence of every other agent through an effect depending on the pairwise distances between them. The equations of motion for the $N$ agents located at point $x_i \in \mathbb{R}^k$ with $N > k$ takes the form

$$\dot{x}_i = \sum_{j \neq i} g(d_{ij})(x_j - x_i), \quad i = 1, \ldots, N$$

(1)

where $d_{ij} = |x_j - x_i|$ is the Euclidean distance between $x_i$ and $x_j$. The position of each agent influences the motion of every other one. In the widely referenced paper by Gazi and Passino [6], the authors work with a particular function $g$ which is repulsive at short range and attractive at long range.

Our assumptions on this function are that $g$ satisfies two natural conditions:

- **Strong repulsion:** $\lim_{d \to 0} dg(d) = -\infty$
- **Fading attraction:** $g(d) > 0$ for sufficiently large $d$ and $\lim_{d \to \infty} dg(d) = 0$

The term $dg(d)$ appears because it represents the actual attraction/repulsion between pairs of agents. We denote by $\mathcal{G}$ the collection of all smooth functions that satisfy strong repulsion and fading attraction.

The equations we adopt to describe the formation of the configuration are a gradient flow in $\mathbb{R}^{k \times N}$ corresponding to a potential function that is symmetric in the coordinates of the individual agents.

$$\Psi(x_1, \ldots, x_N) := \sum_{i<j=1}^{N} x g(x) dx$$

(2)

The gradient flow will be defined for all positive time if there are no collisions and there is no escape of agents off to infinity. We will prove that collisions and escapes do not occur but this requires detailed analysis because our solution space, as we will define soon, is not compact.

Observe that the interaction law in the kinematic model is reciprocal, i.e., the attraction/repulsion on any pair of agents sums to zero, as a consequence the centroid of a configuration is invariant under the gradient flow, so we may assume that it is always located at the origin. We also observe that an interaction law $g$ in the class $\mathcal{G}$ produces infinite repulsion at 0 separation. Thus equation (1) is only well defined if we agree to limit our attention to configurations without collisions. Thus our model should be thought of as being defined on an appropriate open subset of $\mathbb{R}^{(k-1) \times N}$.

Define the configuration space as

$$P := \{(x_1, \ldots, x_N) \in \mathbb{R}^{k \times N} | \sum_{i=1}^{N} x_i = 0 \text{ and } x_i \neq x_j, \forall i \neq j \}$$

(3)

Of course $\Psi$ is a symmetric function of $x_1, x_2, \ldots, x_N$, i.e., the potential function is invariant under permutation of indices of agents. More significantly, as the potential $\Psi$ depends only on the mutual distances between agents, so any two configurations of the same shape will lie on the same equipotential surface, but not vice versa. A natural way to identify configurations of the same shape is via the group action of special orthogonal group $SO(k)$ acting on $P$ by sending $\theta \in SO(k)$ and a configuration $p = (x_1, \ldots, x_N)$ to

$$\theta \cdot p := (\theta x_1, \ldots, \theta x_N)$$

(4)

there is a recognition of identifying each orbit $\mathcal{O}_p := SO(k) \cdot p$ with a smooth submanifold which can be shown to be a certain Stiefel manifold. (see, e.g., [1]) But for our purposes, only the fact that it is smooth matters.

A significant observation is then the continuum equilibria. If a configuration $p$ is an equilibrium, i.e., the gradient vector field vanishes at $p$, then so will be the whole orbit. Actually the group action of $SO(k)$ on $P$ is compatible with the gradient flow in the sense that if we use $f$ to denote the gradient vector field

$$f := -\nabla \Psi(x_1, \ldots, x_N)$$

(5)

then,

$$f(\theta \cdot p) = \theta \cdot f(p)$$

(6)

so it is clear that $f(\theta \cdot p) = 0$ if and only if $f(p) = 0$. Equation (5) has a consequence that

$$\varphi_1(\theta \cdot p) = \theta \cdot \varphi_1(p)$$

(7)

where $\varphi_1(p)$ represents the solution of the gradient flow starting from $p$ at $t = 0$. So in this particular kinematic model, orbit is the least unit as an equilibria set which distinguishes from general cases where the presence of isolated equilibria is generic (see, e.g., [22], [23]).

Invariants associated with the flow are always important as they reflect intrinsic properties of the dynamical system. An invariant in our model can be the rank of a configuration, it is defined to be the number of independent vectors in the set $\{x_1, \ldots, x_N\}$. A geometric interpretation of rank is the dimension of the least linear subspace in $\mathbb{R}^k$ where a configuration can be embedded. As an orthogonal transformation doesn’t change the rank of a configuration, so each orbit $\mathcal{O}_p$ is of the same rank. It can be shown that the rank of $\varphi_1(p)$ depends continuously on $t$ and hence has to keep constant along the flow (see [1]). Let

$$P_n := \{p \in P | \text{rank} p = n\}$$

(8)

A partition of $P$ with respect to the rank is then given by $P = \bigcup_{n=1}^{k} P_n$. We can show that the closure of each $P_n$ in $P$ is given by $P_n = \bigcup_{l=1}^{n} P_l$ and a natural identification of $P_n$ is

$$P_n \approx Gr(n, k) \times P | _{\mathbb{R}^n}$$

(9)

where $Gr(n, k)$ is the Grassmannian parameterizing all linear subspaces of dimension $n$ in $\mathbb{R}^k$ while $P | _{\mathbb{R}^n} \subset P$ collects all configurations embedded in $\mathbb{R}^n \subset \mathbb{R}^k$ with the last $(k-n)$ coordinates of agents equal to zero. But in this paper all we need is that each $P_n$ is smooth.
One sees that the potential function $\Psi$ is a function of shapes only, and although the effect of translation has been eliminated, it is still inevitable that the equilibria appear as entire orbits, not isolated configurations. Only by somehow “dividing out” by the special orthogonal group can one expect to get isolated equilibria. This suggests that it would be better to arrange matters so that we have a flow describing the evolution of shapes rather than Euclidean coordinates of agents. In a more technical language, we want to study our problem on a quotient space by recognizing the invariance of $\Psi$ problem on a quotient space by recognizing the invariance properties of $\Psi$ in equation (2).

Let $Q$ be the shape space defined by the quotient map $\pi: P \rightarrow Q$ via the $SO(k)$-action on the configuration space. In this section we will show that there is a well-defined gradient-like vector field on the shape space so that each equilibria orbit in the configuration space one-to-one corresponds to a single equilibrium in the shape space.

Suppose there is a well-defined dynamical system on $Q$ with the flow map denoted by $\phi_t$, we then expect that the following diagram commutes

$$
\begin{array}{ccc}
P & \xrightarrow{\phi_t} & P \\
\downarrow{\pi} & & \downarrow{\pi} \\
Q & \xrightarrow{\phi_t} & Q
\end{array}
$$

Conversely, if the above diagram does commute, then it defines $\phi_t$ in a unique way. So we expect $\phi_t$ to possess
- **Compatibility:** $\pi\phi_t(p') = \pi\phi_t(p)$ for any $p' \in \mathcal{O}_p$.
- **Smoothness:** $\phi_t(q)$ is smooth in $t > 0$ for any $q \in Q$.
- **Non-self-intersection:** Each flow curve in $Q$ has no self-intersection.

We now show that $\phi_t$ meets all expectations. First notice that if the compatibility is satisfied, then the diagram commutes because the lifting of the flow $\phi_{t>0}(q)$ to $\phi_{t>0}(p)$ will be independent of the choice of $p \in \pi^{-1}q$. But the compatibility is just a consequence of equation (6). The smoothness comes from the fact that $Q$ is piecewise smooth, in fact if we let $\pi_n: P_n \rightarrow Q_n$ be the restriction of $\pi$ on $P_n$ and $Q_n$, the image of $\pi_n$, then a fact is that each $\pi_n$ is a smooth submersion. We know $P_n$ is an invariant subspace of $P$ under $\phi_t$ as the rank is invariant, so $Q_n$ will be an invariant subspace of $Q$ under $\phi_t$, this then implies that the induced flow curve is smooth under the map $\pi_n$. For **Non-self-intersection**, we argue that the potential function $\Psi$ is constant on each orbit and hence, if a gradient flow leaves an orbit, then it won’t be back.

All then suggest that there is a well-defined vector field $\tilde{f}$ on $Q$ via the commutative diagram below where $TP_n$ and $TQ_n$ are tangent spaces of $P_n$ and $Q_n$, respectively, and $d\pi_n$ is the derivative of $\pi_n$.

$$
P_n \xrightarrow{\tilde{f}} TP_n \\
\downarrow{\pi_n} \quad \downarrow{d\pi_n} \\
Q_n \xrightarrow{\tilde{f}} TQ_n
$$

Define the induced potential as $\tilde{\Psi}(q) := \Psi(p)$ for any $p \in \pi^{-1}q$, then $\tilde{f}$ is at least a gradient-like field with respect to $\tilde{\Psi}$. An important fact is that

**Fact 1.** Equilibria orbits in $P$ one to one correspond to equilibria shapes in $Q$.

To see it, it suffices to show that if $\tilde{f}$ vanishes at $q$, then $f$ vanishes at the orbit $\pi^{-1}q$, but this comes from the fact that the gradient vector field, if doesn’t vanish, then always chooses the direction along which the potential drops most quickly, so $f(p)$ is in the normal space $N_p\mathcal{O}_p$. On the other hand, the kernel of $d\pi_n$ at $p$ is just the tangent space $T_p\mathcal{O}_p$, and this establishes Fact 1.

The quotient space $Q$ is twisted, for example, the shape space of $N$ agents in the plane, excluding configurations in which all agents are collocated, can be represented as $\mathbb{C}P^{N-2} \times \mathbb{R}^+$ (see, e.g., [11], [14]). We here mention the fact that there is parametrization of $Q$ by union of Euclidean spaces, and reconstruction of $Q$ can be done by specifying an attaching map to glue these spaces together, we refer the readers to [1] for more details.

**IV. Metric Properties of the Gradient Flow and the Equilibria**

In the paper [6] by Gazi and Passino, the authors establish an upper bound on the size of an equilibrium under three classes of interaction laws including interaction with finite repulsion and strong attraction, interaction with strong repulsion and strong attraction and interaction with strong repulsion and constant attraction. They do not consider what we call here fading attraction. We will establish bounds which cover this case.

It is well known that Morse functions are dense so that if all equilibria are located in a bounded set, then generically there are only finitely many isolated equilibria and canonical theorems like Morse inequalities can be applied to count the number of equilibria. But in the case we considered here, equilibria orbits are the least unit, so regular Morse theory doesn’t work, yet this obstacle is removed by introducing equivariant Morse theory (see, e.g., [13]). However, the question about the genericity of equivariant Morse functions doesn’t have a complete answer yet, later in the section (and in the Appendix) we will state some of our results.

**Theorem 1 (Bounded Size of Equilibria).** Consider the kinematic model described by equation (1), there exist positive numbers $a$ and $b$ such that the distance between any two agents in an equilibrium lies in the closed interval $[a, b]$.

A complete proof is in [1], [3] and here is a sketch of it.
Let $\alpha > 0$ and $\beta > 0$ be defined so that
\[
g(d) < 0, \forall d \in (0, \alpha) \tag{12}
\]
\[
g(d) > 0, \forall d \in (\beta, \infty) \tag{13}
\]
The upper bound exists because if the diameter of a configuration is too large, then we will be able to show that there is a bipartition of the configuration into two clusters and meanwhile a hyperplane as a linear classifier. Moreover, the distance between any two agents in different clusters is greater than $\beta$, but such a configuration can’t be an equilibrium as both centers of clusters are unbalanced. The lower bound exists because if there is a pair of agents too close to each other, their mutual repulsion tends to infinity by strong repulsion and we have to locate at least one other agent nearby to counteract its effect, but then to keep the 3-agent system balanced, at least one other agent has to be located around and so on so forth, we then arrive at a situation that all agents are contained in a ball with the radius small enough that the distance between any two agents is less than $\alpha$, so then all interactions are repulsions and such a configuration can’t be an equilibrium either.

The two numbers $\alpha$ and $\beta$ also help us evaluate $a$ and $b$. Fix $\beta$, decrease $\alpha$ if necessary so that $|g(\alpha)| \geq |g(d)|$ for any $d \in [\alpha, \beta]$, then a rough estimation for $a$ and $b$ can be
\[
a = \max \{g^{-1}(N^\perp g(\alpha)) \cap (0, \alpha)\} \tag{14}
\]
\[
b = N\beta \tag{15}
\]
these two equations relate the size of an equilibrium to a basic property of $g$.

Stable equilibria are important as practically they are the only observable ones. An equilibria orbit $O_p$ is stable if its image $q$ is stable in $Q$. A problem of interest is the design of an interaction law $g$ so that the potential $\Psi$ generated by $g$ stabilizes certain specified orbits

**Fact 3** Given a finite set of disjoint orbits $\{O_{p_1}, \ldots, O_{p_t}\}$, there exists an interaction law $g \in \mathcal{G}$ that stabilizes all of them.

Actually there is an explicit recipe for constructing such $g$, let $\{d_1, \ldots, d_n\}$ be the set of distinct distances appearing in configurations $\{p_1, \ldots, p_t\}$, then we may just set
\[
g(d_i) = 0, i = 1, \ldots, n \tag{16}
\]
\[
g'(d_i) < 0, i = 1, \ldots, n \tag{17}
\]
It is easy to check that the induced potential $\Psi$ is a local minima at each $q_i = \pi p_i$.

Conversely, given a potential function $\Psi$ generated by $g$, we would like to know the total number of equilibria orbits and the index associated with each of them, there has been a number of works on counting critical formations (see e.g., [11]–[13], [15], [16]). Yet we have to admit the fact that the counting problem has to be based on an important assumption, namely the potential $\Psi$ is an equivariant Morse function, i.e., there are only finitely many orbits and the Hessian of $\Psi$ is non-degenerate when restricted on the normal bundle of each equilibria orbit. Notice that in our case, the non-degeneracy of the Hessian implies finite number of equilibria orbits as the set of equilibria is compact by Theorem 1. If the potential $\Psi$ generated by $g$ is an equivariant Morse function, then we can apply equivariant Poincare-series to estimate the total number of equilibria orbits as this computation is done explicitly in the paper [13] for 2D and 3D cases. A question is then if we let $\mathcal{G}_\tau \subset \mathcal{G}$ be the collection of interaction laws that produce equivariant Morse potentials, is then $\mathcal{G}_\tau$ open and dense in $\mathcal{G}$? The answer for 2D-case is yes, and in the Appendix we state two theorems dealing with the genericity of equivariant Morse functions.

V. SWARM AGGREGATION AND CLUSTERING AGENTS

The non-escape property has been investigated in many cases and under various assumptions (see, e.g., [6]–[9], [18]–[20]). In this section, we state a theorem which establishes the convergence of the gradient flow associate with our kinematic model and give a sketch of the proof. Part of the proof depends on the clustering of agents which is a key element in the proof as we use it to deal with dilute configurations. On the other hand, clustering can be studied for its own sake and in this section we discuss some of its aspects.

**Theorem 2 (Convergence of The Flow)** Consider the kinematic model described by equation (1). For any initial condition $p$, the flow $\phi_t(p)$ exists for all $t > 0$ and converges to an invariant set consisting exclusively of equilibria of the gradient flow.

A complete proof is in [3] and we will give an outline of the proof in this section.

The proof of non-collision is more or less intuitive by our assumption of strong repulsion on $g$. We will actually show that for each configuration $p$, there is a number $a_p > 0$ such that all distances between agents in $\phi_t(p)$ are greater than $a_p$ and this holds for all $t > 0$. The existence of the lower bound $a_p$ is done by contradiction, i.e., we assume there is a flow along which the minimum distance between agents can’t be bounded below by a positive number. Then a key fact, as we establish in [3], is that for any $\epsilon > 0$, there will be a moment $\tau_\epsilon$ such that the diameter of the whole configuration $\phi_\tau(p)$ is $\epsilon$. We will then be able to find two positive numbers $\epsilon_1$ and $\epsilon_2, \epsilon_1 > \epsilon_2$, together with two moments $\tau_1$ and $\tau_2$, $\tau_1 < \tau_2$, such that $\Psi(\phi_{\tau_1}(p)) < \Psi(\phi_{\tau_2}(p))$. But this is a contradiction since the potential always decreases along a gradient flow.

Before proving the property of non-escape, we introduce the notion of clustering of agents. A clustering $c(l, \epsilon)$ on a configuration $p$ is a partition of $N$ agents into disjoint union of clusters satisfying two conditions specified below. Let $r_i$ be the radius of the $i$-th cluster and let $l_i$ be the distance between the center of the $i$-th cluster and the center of the $j$-th cluster, then

**Distance condition**: each $l_{ij}$ is great than $l$.

**Ratio condition**: each $r_i/l_{ij}$ is less than $\epsilon$.

Any configuration admits the trivial clustering, namely
the one with only one cluster containing all the agents. An important fact is that

**Fact 4 (Nontrivial Clustering).** Given a pair of parameters \((l, \varepsilon)\), there exists a fixed number \(D > 0\) such that if the diameter of a configuration \(p\) is greater than \(D\), then \(p\) admits a nontrivial clustering.

On the other hand, a configuration may admit multiple nontrivial clusterings with respect to one pair of parameters. We then ask whether there is a canonical way of clustering? To answer this question, we establish a partial order on clustering. A clustering \(\tau(l, \varepsilon)\) induces a partition on the index set as \(\{1, \ldots, N\} = \bigcup_{i=1}^{m} S_i\), with each \(S_i\) nonempty, now suppose another clustering \(\tau'(l, \varepsilon)\) induces a different partition as \(\{1, \ldots, N\} = \bigcup_{j=1}^{n} S'_j\), then we say \(\tau(l, \varepsilon)\) is finer than \(\tau'(l, \varepsilon)\) and denote by \(\tau(l, \varepsilon) \succ \tau'(l, \varepsilon)\) if each \(S_i\) is a subset of \(S'_j\) for some \(j\), this partial order reflects the granularity of the partition.

**Fact 5 (Linear Order on Clustering with Fixed Parameters).** Given a pair of parameters \((l, \varepsilon)\) with \(\varepsilon < 1/4\), any two different clustering \(\tau(l, \varepsilon)\) and \(\tau'(l, \varepsilon)\) is comparable and hence, all clustering with fix parameter \((l, \varepsilon)\) form a chain of linear orders, i.e., \(\tau_1(l, \varepsilon) \succ \cdots \succ \tau_m(l, \varepsilon)\) with the last one the trivial clustering.

The first one \(\tau_1(l, \varepsilon)\) is then indecomposable with respect to the pair of parameters \((l, \varepsilon)\). The next statement tells us how the chain of clusterings varies as the pair of parameters changes.

**Fact 6 (Variation of Chain).** Given two parameters \((l, \varepsilon)\) and \((l', \varepsilon')\) with \(l' \geq l\) and \(\varepsilon' \leq \varepsilon < 1/4\), then \(\{\tau_i(l', \varepsilon')\}_{i=1}^{m}\) is a subchain of \(\{\tau_i(l, \varepsilon)\}_{i=1}^{m}\), i.e, for each \(\tau_i(l', \varepsilon')\) there is \(\tau_i(l, \varepsilon)\) such that they induce the same partition on the index set \(\{1, \ldots, N\}\).

More details and properties about this lattice can be found in [1], [3]. So far we haven’t given a reason why we introduce clustering, but here is one, the proof of convergence of the gradient flow is also done by contradiction, we assume there is a certain flow curve that escapes off to infinity and the next statement then describes a significant fact for any of such flow curves.

**Theorem 3 (Clustering on Sequence of Diverging Configurations).** If there were an initial condition \(p\) such that along the gradient flow \(\varphi_{\varepsilon=0}(p)\) the diameter of the configuration can’t be bounded above, then there would be a monotone sequence of times \(\{t_i\}_{i \in \mathbb{N}}\) approaching to infinity such that

1) there is a nontrivial clustering of agents shared by all configurations \(\{\varphi_{\varepsilon_0}(p)\}_{i \in \mathbb{N}}\).
2) all inter-cluster distances approach to infinity along the sequence of configurations \(\{\varphi_{\varepsilon_0}(p)\}_{i \in \mathbb{N}}\).
3) radii of all clusters are less than a fixed number \(R\) for any configuration in the sequence \(\{\varphi_{\varepsilon_0}(p)\}_{i \in \mathbb{N}}\).

A complete proof of Theorem 3 is in [3]. We now sketch the proof of the non-escape property about the gradient flow. Notice that the potential function \(\Psi\) is bounded below while \(\Psi(\varphi_{\varepsilon}(p))\) is non-increasing as a function of \(t\) and hence, the limit exists as \(t\) goes to infinity. This then imposes a restriction on the admissible set of the gradient flow. After long time, the flow curve has to stay within certain “zone of safety” by what we mean a subset of \(P\) where dissipation rate is low. Apply Theorem 3 to get the sequence \(\{\varphi_{\varepsilon}(p)\}_{i \in \mathbb{N}}\), then for any sufficiently large \(t_i\), clusters of agents in \(\varphi_{\varepsilon}(p)\) will stay bounded not only at \(t_i\) but also for any time \(t > t_i\) as long as the inter-cluster distances are large. This is because the fading attraction implies that we may treat each cluster as an isolated subsystem, so in order to keep a low dissipation rate, agents of each cluster have to form a sub-configuration which is close to the set of equilibria of their own, and by Theorem 1 any of such sets is bounded. On the other hand, we can show that if all clusters stay far away from each other while the size of each cluster remains bounded along the flow, then the flow curve won’t diverge. So after all, we have to conclude that for each sufficiently large \(t_i\) there is a moment \(t > t_i\) such that certain inter-cluster distance drops below a fixed threshold, say \(L\). We define a zone of dissipation by

\[
Z := \{p \in P \mid \exists (i, j) \text{ s.t. } d_{ij} \in [L - R, L + R]\}
\]

where \(R\) is the fixed number defined in condition (3) of Theorem 3. There is no harm to assume \(L\) large so that \(Z\) doesn’t intersect the set of equilibria by Theorem 1 and we can show that for any flow curve entering into \(Z\), there will be a certain loss in potential along the flow and the amount is bounded below by a fixed number \(\delta > 0\). As \(\{t_i\}_{i \in \mathbb{N}}\) is a time sequence approaching to infinity, so we may assume that \(t_{i+1} > t_i\) by passing to a subsequence and hence, the decrease of the potential along the flow curve will exceed \(\sum_{i \in \mathbb{N}} \delta = \infty\) which is a contradiction as the potential \(\Psi\) is bounded below.

So far, we have showed that each flow is bounded in a compact set, but this is enough because it brings us to the case where the underlying space is compact and hence, we prove the convergence of the gradient flow in the kinematic model described by equation (1).

**APPENDIX**

Certain technical restrictions on the function \(g\) have been made so that \(\Psi\) will have the properties required to use equivariant Morse theory. The following two theorems, which we do not prove here, assert that in a well defined, but technical, sense "almost all" \(g\) have these properties. A complete story is in [4].

**Theorem 4.** Suppose \(K \subset P_0\) is any closed set in \(P\). Let \(\mathcal{G}_K\) be the subset of \(\mathcal{G}\) such that each potential \(\Psi\) generated by \(g \in \mathcal{G}_K\) is an equivariant Morse function when restricted on \(K\), then \(\mathcal{G}_K\) is open and dense in \(\mathcal{G}\) with respect to the Whitney \(C^r\)-topology for \(r \geq 1\).

**Theorem 5.** Suppose \(k\), the dimension of Euclidean space, is two. Let \(\mathcal{G}_{P}\) be the subset of \(\mathcal{G}\) such that each potential \(\Psi\) generated by \(g \in \mathcal{G}_{P}\) is an equivariant Morse function on \(P\), then \(\mathcal{G}_{P}\) is open and dense in \(\mathcal{G}\) with respect to the Whitney \(C^r\)-topology for \(r \geq 1\).
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