Decentralized Adaptive Disturbance Rejection for Relative-Degree-One Local Subsystems

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Abstract—We present a strictly decentralized adaptive controller for single-input single-output linear time-invariant subsystems that are minimum phase and relative degree one. This decentralized adaptive controller requires only local output measurement and no information is shared between the local controllers. The controller is effective for stabilization and disturbance rejection, where the disturbance is unmeasured but generated from an unknown Lyapunov-stable linear system.

I. INTRODUCTION

Classical output-feedback model reference adaptive control (MRAC) applies to single-input single-output (SISO) linear time-invariant systems that are minimum phase [1]–[5]. The goal of output-feedback MRAC is to design a control such that all closed-loop signals are bounded and the output of the plant asymptotically follows the output of a reference model. Output-feedback MRAC operates under the assumptions that the plant is minimum phase, the sign of the high-frequency gain is known, an upper bound on the order of the plant is known, and the relative degree is known.

Classical output-feedback MRAC has been extended to address decentralized control for SISO subsystems with local output feedback [6]–[8]. The results in [7] address local subsystems that are relative degree one and 2, while [8] addresses local subsystems that are relative degree greater than 2. The approaches of [6]–[8] address stabilization and command following provided that each local subsystem is minimum phase. The controllers in [6]–[8] guarantee bounded tracking errors, but do not drive the tracking errors to zero. In particular, each local tracking error converges to a residual set that depends on the magnitude of the interconnection matrices and on the local adaptive design parameters. While the adaptive controllers in [6]–[8] address command following, none of these techniques address disturbance rejection. Decentralized adaptive control for subsystems with local full-state feedback is addressed in [9]–[14].

In this paper, we present a decentralized MRAC technique for SISO linear time-invariant subsystems that are minimum phase and relative degree one. This controller is strictly decentralized, meaning that no information is shared between local controllers. Moreover, the decentralized adaptive controller presented in this paper is effective for stabilization and disturbance rejection in the presence of an unknown disturbance, provided that the disturbance is generated from a Lyapunov-stable linear system (i.e., the disturbance is a sum of sinusoids). The decentralized adaptive controller operates under the assumption that the magnitude of the subsystem interconnections satisfy a bounding condition.

II. PROBLEM FORMULATION

Let $J \triangleq \{1, 2, \ldots, l\}$, and for all $i \in J$ and all $t \geq 0$, consider the system

\[
\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + B_i \sum_{j \in J \backslash\{i\}} \delta_{i,j} y_j(t) + D_i w(t),
\]

\[
y_i(t) = C_i x_i(t),
\]

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $x_i(0) \in \mathbb{R}^{n_i}$ is the initial condition, $u_i(t) \in \mathbb{R}$ is the control input, $w(t) \in \mathbb{R}^m$ is the exogenous disturbance, $y_i(t) \in \mathbb{R}$ is the output, and $(A_i, B_i, C_i)$ is controllable and observable.

For each $i \in J$, $x_i$ is the local state, $u_i$ is the local control, and $y_i$ is the local output. Moreover, for each $i \in J$, the local control $u_i$ uses feedback of the local output $y_i$ but does not use feedback of the nonlocal outputs $\{y_j\}_{j \in J \backslash\{i\}}$. Unless otherwise stated, all statements in this paper that involve the subscript $i$ are for all $i \in J$.

We make the following assumptions regarding (1)–(2):

(A1) If $\lambda \in \mathbb{C}$ and $\det \begin{bmatrix} \lambda I_{n_i} - A_i & B_i \\ C_i & 0 \end{bmatrix} = 0$, then $\text{Re} \, \lambda < 0$.

(A2) $h_i \triangleq C_i B_i$ is nonzero and the sign of $h_i$ is known.

(A3) There exists a known integer $n_i$ such that $n_i \leq n_i$.

The system (1)–(2) is otherwise unknown. Specifically, $A_1, \ldots, A_l, B_1, \ldots, B_l, C_1, \ldots, C_l, D_1, \ldots, D_l, \delta_{1,1}, \ldots, \delta_{1,l}, \ldots, \delta_{l,1}, \ldots, \delta_{l,l}$, and $x(0), \ldots, x_l(0)$ are otherwise unknown. Assumption (A1) states that each local subsystem of (1)–(2) is minimum phase, that is, the zeros of the transfer function from $u_i$ to $y_i$ lie in the open-left-half complex plane. Assumption (A2) implies that the transfer function from $u_i$ to $y_i$ is relative degree one.

Let $p = d/dt$ denote the differential operator. We make the following assumptions regarding $w$:

(A4) For all $t \geq 0$, $w(t)$ is bounded and satisfies $\alpha_w(p) w(t) = 0$, where $\alpha_w(s)$ is a nonzero monic polynomial with distinct roots that lie on the imaginary axis.

(A5) There exists a known integer $n_w$ such that $n_w \triangleq \text{deg} \, \alpha_w(s) \leq n_w$.

Assumption (A4) implies that $w$ consists of a sum of sinusoids; however, the disturbance $w$ and its spectrum are not assumed to be known.

Next, consider the reference model

\[
\alpha_{m,i}(p) y_{m,i}(t) = h_{m,i} \beta_{m,i}(p) r_i(t),
\]
where \( t \geq 0; r_i(t) \in \mathbb{R} \) is the bounded reference-model command; \( y_{m,i}(t) \in \mathbb{R} \) is the reference-model output; \( \alpha_{m,i}(s) \) is a monic Hurwitz polynomial with degree \( n_{m,i} \); \( \beta_{m,i}(s) \) is a monic Hurwitz polynomial with degree \( n_{m,i} - 1 \); \( \alpha_{m,i}(s) \) and \( \beta_{m,i}(s) \) are coprime; and \( h_{m,i} \) is nonzero. Define \( G_{m,i}(s) = h_{m,i} \beta_{m,i}(s)/\alpha_{m,i}(s) \), let \( \gamma_i > 0 \), and define
\[
F_i(s) \triangleq \frac{G_{m,i}(s)}{1 - \gamma_i G_{m,i}(s)} = \frac{h_{m,i} \beta_{m,i}(s)}{\alpha_{m,i}(s) - \gamma_i h_{m,i} \beta_{m,i}(s)}.
\]
Note that \( F_i(s) \) depends on the parameter \( \gamma_i > 0 \), which is discussed in the following section. We make the following assumption regarding the reference model \( G_{m,i} \):
(A6) \( F_i(s) \) is strictly positive real.

Our goal is to develop an adaptive controller that generates the control \( u_i \) such that \( y_i \) asymptotically follows \( y_{m,i} \) in the presence of \( w \). Thus, our goal is to drive \( z_i(t) \triangleq y_i(t) - y_{m,i}(t) \) to zero. The analysis presented in this paper focuses on the disturbance rejection problem (i.e., \( y_{m,i}(t) \equiv 0 \)).

### III. IDEAL DECENTRALIZED CONTROLLER

We now develop the ideal decentralized controller. To construct this controller, we assume the plant (1)–(2) is known. We relax this assumption in subsequent sections.

Let \( n_{c,i} \) be an integer that satisfies
\[
n_{c,i} \geq \max\{n_{m,i} - 1, \bar{n}_i + \bar{n}_w\},
\]
define \( A_i(s) \triangleq \begin{bmatrix} s^{n_{m,i} - 1} & s^{n_{m,i} - 2} & \cdots & s & 1 \end{bmatrix}^T \), and let \( \rho_i(s) \) be a monic Hurwitz polynomial with degree \( n_{c,i} - (n_{m,i} - 1) \), which is nonnegative. Next, the matrix transfer function \( \frac{1}{\alpha_{m,i}(s) \rho_i(s)} A_i(s) \) has the minimal realization \((A_{i,i}, B_{i,i}, I_{n_{c,i}})\), where \( A_{i,i} \triangleq \begin{bmatrix} 1 & 0 \end{bmatrix} \) and
\[
A_{i,i} = \begin{bmatrix}
I_{n_{c,i} - 1} & a_i \\
0_{(n_{c,i} - 1) \times 1} & 1
\end{bmatrix},
\]
where \( a_i \in \mathbb{R}^{1 \times n_{c,i}} \) such that \( \det(sI - A_{i,i}) = \beta_{m,i}(s) \rho_i(s) \).

For \( t \geq 0 \), consider the system (1)–(2) with \( u_i(t) = u_{*,i}(t) \), where \( u_{*,i}(t) \) is the ideal control generated by an ideal decentralized controller. Specifically, for \( t \geq 0 \), consider the system
\[
\dot{x}_{*,i}(t) = A_{i,i} x_{*,i}(t) + B_{i,i} u_{*,i}(t)
+ B_i \sum_{j \in \mathcal{N}\setminus\{i\}} \delta_{i,j} y_{*,j}(t) + D_i w(t),
\]
\[
y_{*,i}(t) = C_{i,i} x_{*,i}(t),
\]
where
\[
u_{*,i}(t) = \theta_{*,i}^T \phi_{*,i}(t),
\]
and
\[
\theta_{*,i} \triangleq \begin{bmatrix}
L_{*,i} \\
M_{*,i} \\
N_{*,i}
\end{bmatrix},
\phi_{*,i}(t) \triangleq \begin{bmatrix}
U_{*,i}(t) \\
Y_{*,i}(t) \\
r_{*,i}(t)
\end{bmatrix},
\]
where \( L_{*,i} \in \mathbb{R}^{n_{c,i}}, M_{*,i} \in \mathbb{R}^{n_{c,i}}, \) and \( N_{*,i} \in \mathbb{R} \); and
\[
U_{*,i}(t) \in \mathbb{R}^{n_{c,i}}, \text{ and } Y_{*,i}(t) \in \mathbb{R}^{n_{c,i}} \text{ satisfy}
\]
\[
\dot{U}_{*,i}(t) = A_{*,i} U_{*,i}(t) + B_{*,i} u_{*,i}(t),
\]
\[
\dot{Y}_{*,i}(t) = A_{*,i} Y_{*,i}(t) + B_{*,i} y_{*,i}(t),
\]
where \( U_{*,i}(0) \in \mathbb{R}^{n_{c,i}} \) and \( Y_{*,i}(0) \in \mathbb{R}^{n_{c,i}} \).

Therefore, the ideal closed-loop system, which consists of (6)–(11), is given by
\[
\dot{x}_{*,i}(t) = \hat{A}_i \hat{x}_{*,i}(t) + \hat{B}_i r_i(t)
+ \hat{E}_i \sum_{j \in \mathcal{N}\setminus\{i\}} \delta_{i,j} \hat{C}_j \hat{x}_{*,j}(t) + \hat{D}_i w(t),
\]
\[
y_{*,i}(t) = \hat{C}_i \hat{x}_{*,i}(t),
\]
where \( \hat{x}_{*,i}(t) \triangleq \begin{bmatrix} x_{*,i}^T(t) U_{*,i}^T(t) \end{bmatrix}^T \), and
\[
\hat{A}_i \triangleq \begin{bmatrix}
A_{*,i} & B_{*,i} L_{*,i}^T & B_{*,i} M_{*,i}^T \\
B_{*,i} & 0 & A_{*,i} + B_{*,i} L_{*,i}^T & B_{*,i} M_{*,i}^T \\
B_{*,i} C_{*,i} & 0 & A_{*,i}
\end{bmatrix},
\]
\[
\hat{C}_i \triangleq \begin{bmatrix}
C_{*,i} \ 0 \ 0
\end{bmatrix},
\]
\[
\hat{B}_i \triangleq N_{*,i} \begin{bmatrix}
B_{*,i} \\
B_{*,i} \\
0
\end{bmatrix}, \quad \hat{E}_i \triangleq \begin{bmatrix} B_{*,i} \\
0 \\
0 \end{bmatrix}, \quad \hat{D}_i \triangleq \begin{bmatrix} D_i \\
0 \\
0
\end{bmatrix}.
\]

The following result provides properties of (12)–(13). The proof is omitted for space considerations.

**Lemma 1.** Let \( n_{c,i} \) satisfy (4), and let \( N_{*,i} = h_{m,i}/h_{i} \). Then, there exists \( L_{*,i} \in \mathbb{R}^{n_{c,i}} \) and \( M_{*,i} \in \mathbb{R}^{n_{c,i}} \) such that the following statements hold regarding (12)–(13):

(i) \( \hat{A}_i \) is asymptotically stable.

(ii) For all \( \tilde{x}_{*,1}(0), \tilde{x}_{*,2}(0), \ldots, \tilde{x}_{*,0}(0) \), all \( t \geq 0 \), and all \( i \in \mathcal{I} \),
\[
\alpha_{m,i}(p) \rho_i(p) y_{*,i}(t) = h_{m,i} \beta_{m,i}(p) \rho_i(p) r_i(t)
+ h_i \ell_{*,i}(p) \sum_{j \in \mathcal{N}\setminus\{i\}} \delta_{i,j} y_{*,j}(t),
\]
where \( \ell_{*,i}(s) \triangleq \beta_{m,i}(s) \rho_i(s) - L_{*,i}^T A_i(s) \).

(iii) \( \hat{C}_i(sI - \hat{A}_i)^{-1} \hat{B}_i = G_{m,i}(s) \).

Let \( N_{*,i} = h_{m,i}/h_{i} \), and let \( L_{*,i} \in \mathbb{R}^{n_{c,i}} \) and \( M_{*,i} \in \mathbb{R}^{n_{c,i}} \) be the parameters given by Lemma 1. Part (iii) of Lemma 1 implies that \( G_{m,i}(s) = \hat{C}_i(sI - A_i)^{-1} \hat{B}_i \), and thus, it follows from (3) that \( F_i(s) = \hat{C}_i(sI - A_i - \gamma_i \hat{B}_i \hat{C}_i)^{-1} \hat{B}_i \). Since (A6) states that \( F_i(s) \) is strictly positive real, and part (i) of Lemma 1 states that \( \hat{A}_i \) is asymptotically stable, it follows from the Meyer-Kalman-Yakubovich lemma [3] that there exist positive-definite matrices \( P_i \in \mathbb{R}^{(n_{c,i} + 2n_{c,i}) \times (n_{c,i} + 2n_{c,i})} \) and \( Q_i \in \mathbb{R}^{(n_{c,i} + 2n_{c,i}) \times (n_{c,i} + 2n_{c,i})} \) such that
\[
(\hat{A}_i + \gamma_i \hat{B}_i \hat{C}_i)^T P_i + P_i (\hat{A}_i + \gamma_i \hat{B}_i \hat{C}_i) + Q_i = 0,
\]
\[
P_i \hat{B}_i = \hat{C}_i^T.
\]

Next, we invoke an assumption regarding \( \delta_{1,i}, \ldots, \delta_{\ell,i} \):
(A7) For all \( i \in \mathcal{I} \),
\[
\sum_{j \in \mathcal{N}\setminus\{i\}} \delta_{j,i}^2 \leq 2 \gamma_i \left( \min_{j \in \mathcal{I}} \frac{\lambda_{\min}(Q_j)}{\ell \lambda_{\max}(P_j)} \right).
\]
Assumption (A7) limits the magnitude of the interconnections. Note that the upper bound given by (16) depends on \( \gamma_i \), which can be arbitrarily large provided that (A6) is satisfied. However, (A6) also involves the reference model \( G_{m,i} \), which affects \( Q_i \) and \( P_i \), which also appear in the upper bound given by (16). Numerical testing suggests that Assumption (A7) can be satisfied for arbitrarily large interconnections \( \sum_{j \in \mathcal{N}_i} \delta_{j,i}^2 \) by selecting a reference model \( G_{m,i} \) with sufficiently fast poles.

The next result provides additional properties of the ideal closed-loop system (12)–(13) with \( r_i(t) \equiv 0 \). The proof is omitted for space considerations.

**Lemma 2.** Consider (12)–(13), which satisfies assumptions (A1)–(A7). Let \( N_{s,i} = h_{m,i}/h_i \), and let \( L_{s,i} \in \mathbb{R}^{n_{c,i}} \) and \( M_{s,i} \in \mathbb{R}^{n_{c,i}} \) be given by Lemma 1. Assume that \( r_i(t) \equiv 0 \). Then, the following statements hold:

(i) If \( w(t) \equiv 0 \), then the equilibrium \( (\hat{x}_{s,1}^*(t), \ldots, \hat{x}_{s,r}^*(t)) \equiv 0 \) of (12) is asymptotically stable.

(ii) For all \( \hat{x}_{s,1}(0), \ldots, \hat{x}_{s,r}(0) \in \mathbb{R}^{n_{c,i}+2n_{c,i}} \),

\[
\lim_{t \to \infty} y_{s,i}(t) = \cdots = \lim_{t \to \infty} y_{s,i}(t) = 0.
\]

(iii) There exists \( \hat{x}_{s,1}(0), \ldots, \hat{x}_{s,r}(0) \in \mathbb{R}^{n_{c,i}+2n_{c,i}} \) such that for all \( t \geq 0 \),

\[
y_{s,i}(t) = \cdots = y_{s,i}(t) = 0.
\]

**IV. DECENTRALIZED ADAPTIVE CONTROL**

In this section, we address decentralized adaptive stabilization and disturbance rejection for relative-degree-one subsystems. Let \( U_i(t) \in \mathbb{R}^{n_{c,i}} \) and \( Y_i(t) \in \mathbb{R}^{n_{c,i}} \) satisfy

\[
\hat{U}_i(t) = A_{i,t} U_i(t) + B_{i,t} u_i(t),
\]

\[
\hat{Y}_i(t) = A_{i,t} Y_i(t) + B_{i,t} y_i(t),
\]

where \( U_i(0) \in \mathbb{R}^{n_{c,i}} \) and \( Y_i(0) \in \mathbb{R}^{n_{c,i}} \), and \( A_{i,t} \) and \( B_{i,t} \) are given by (5). Define \( \phi_i(t) = [U_i^T(t) \ Y_i^T(t) \ r_i(t)]^T \), and consider the controller

\[
u_i(t) = \theta_i^T(t) \phi_i(t),
\]

where \( \theta_i : [0, \infty) \to \mathbb{R}^{2n_{c,i}+1} \) is given by

\[
\dot{\theta}_i(t) = -\text{sgn}(h_i) z_i(t) \Gamma_i \phi_i(t),
\]

where \( \Gamma_i \in \mathbb{R}^{(2n_{c,i}+1) \times (2n_{c,i}+1)} \) is positive definite. The control architecture is shown in Figure 1.

Let \( \theta_{s,i} \in \mathbb{R}^{2n_{c,i}+1} \) be given by (9), where \( N_{s,i} \overset{\triangle}{=} h_{m,i}/h_i \), and \( L_{s,i} \in \mathbb{R}^{n_{c,i}} \) and \( M_{s,i} \in \mathbb{R}^{n_{c,i}} \) are the ideal controller parameters given by Lemma 1. Define

\[
\dot{\theta}_{s,i}(t) = \theta_i(t) - \theta_{s,i}.
\]

Thus, it follows from (1)–(2) and (17)–(19) that the closed-loop system is given by

\[
\dot{x}_i(t) = \hat{A}_i x_i(t) + \hat{B}_i \theta_i^T(t) \phi_i(t) + \hat{B}_i r_i(t)
\]

\[
+ \hat{E}_i \sum_{j \in \mathcal{N}_i} \delta_{i,j} C_j \hat{x}_j(t) + \hat{D}_i w(t),
\]

\[
y_i(t) = \tilde{C}_i \hat{x}_i(t).
\]

where \( \hat{x}_i(t) = [x_i^T(t) \ U_i^T(t) \ Y_i^T(t)]^T \).
(13) from (24) and (25), respectively, yields
\[
\dot{e}_i(t) = \tilde{A}_i e_i(t) + \frac{1}{N_{*,i}} \tilde{B}_i \tilde{\theta}_i(t) \phi_i(t) + \tilde{E}_i \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{C}_j e_j(t),
\]
\[
y_i(t) = \tilde{C}_i e_i(t),
\]
where part (iii) of Lemma 2 implies that \(y_i(t) - y_{*,i}(t) = y_i(t)\).

Let \(P_i \in \mathbb{R}^{(n_i+2n_{*,i}) \times (n_i+2n_{*,i})}\) be the positive-definite solution to (14), and define the partial Lyapunov-like function
\[
V_i(e_i, \tilde{\theta}_i) = e_i^T P_i e_i + \frac{1}{\varepsilon_i} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i / |N_{*,i}|.
\]

Evaluating the derivative of \(V_i\) along the trajectory of (20), (26) with \(r_i(t) \equiv 0\), and using (15), (27) yields
\[
\dot{V}_i(e_i, \tilde{\theta}_i) = e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i E_i \tilde{C}_j e_j.
\]

Next, it follows from (15) that \(h_{m,i} = \tilde{C}_i \tilde{B}_i = \tilde{B}_i^T P_i \tilde{B}_i > 0\). Since \(N_{*,i} = h_{m,i}/h_{i}\), it follows that \(\text{sgn}(h_{i}) = \text{sgn}(h_{m,i}/h_{i}) = \text{sgn}(N_{*,i}).\) Then, it follows from (28) that
\[
\dot{V}_i(e_i, \tilde{\theta}_i) = e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i E_i \tilde{C}_j e_j.
\]

Define \(e_i \triangleq \lambda_{\text{min}}(Q_i)/\ell \lambda_{\text{max}}(P_i E_i \tilde{E}_i^T P_i)\), and note
\[
0 \leq \sum_{j \in \mathcal{J} \setminus \{i\}} \left[ \sqrt{\xi_i \tilde{E}_i^T P_i e_i - \frac{1}{\xi_i} \delta_{i,j} \tilde{C}_j e_j} \right]^T
\times \left[ \sqrt{\xi_i \tilde{E}_i^T P_i e_i - \frac{1}{\xi_i} \delta_{i,j} \tilde{C}_j e_j} \right]
= (\ell - 1) e_i^T P_i \tilde{E}_i \tilde{E}_i^T P_i e_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\xi_i} \delta_{i,j}^2 e_i^T \tilde{C}_j^T \tilde{C}_j e_j
- 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j
\leq (\ell - 1) \lambda_{\text{min}}(Q_i) e_i^T e_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\xi_i} \delta_{i,j}^2 e_i^T \tilde{C}_j^T \tilde{C}_j e_j
- 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j,
\]

which implies that
\[
\sum_{j \in \mathcal{J} \setminus \{i\}} 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j \leq (\ell - 1) \lambda_{\text{min}}(Q_i)/\ell e_i^T e_i
+ \frac{1}{\xi_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_i^T \tilde{C}_j^T \tilde{C}_j e_j.
\]

Next, using (30), it follows from (29) that
\[
\dot{V}_i(e_i, \tilde{\theta}_i) \leq e_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\text{min}}(Q_i)}{\ell} I \right) e_i
+ \frac{1}{\xi_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_i^T \tilde{C}_j^T \tilde{C}_j e_j.
\]

Define \(V(e_1, \ldots, e_k, \tilde{\theta}_1, \ldots, \tilde{\theta}_k) \triangleq \sum_{i \in \mathcal{I}} V_i(e_i, \tilde{\theta}_i)\), and it follows from (31) that the derivative of \(V\) along the trajectory of (20) and (26) is given by
\[
\begin{align*}
\dot{V}(e_1, \ldots, e_k, \tilde{\theta}_1, \ldots, \tilde{\theta}_k) &= \sum_{i \in \mathcal{I}} V_i(e_i, \tilde{\theta}_i) \\
&\leq \sum_{i \in \mathcal{I}} \left[ e_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\text{min}}(Q_i)}{\ell} I \right) e_i \\
&+ \frac{1}{\xi_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_i^T \tilde{C}_j^T \tilde{C}_j e_j \right] \\
&= \sum_{i \in \mathcal{I}} e_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\text{min}}(Q_i)}{\ell} I \right) e_i \\
&+ \sum_{i \in \mathcal{I}} \frac{1}{\xi_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_i^T e_j \\
&\leq \sum_{i \in \mathcal{I}} e_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\text{min}}(Q_i)}{\ell} I \right) e_i \\
&+ \frac{1}{\xi_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_i^T e_j.
\end{align*}
\]
where \(\varepsilon_i \triangleq \lambda_{\text{min}}(Q_i)/\ell\), which is positive. Therefore, \(V\) is nonpositive and \(V \leq -\sum_{i \in \mathcal{I}} \xi_i e_i^T e_i\) implies that \(0 \leq \sum_{i \in \mathcal{I}} \xi_i e_i^T e_i \leq -V\). Moreover, integrating from 0 to \(\infty\) yields \(0 \leq \int_0^\infty \sum_{i \in \mathcal{I}} \xi_i e_i^T e_i(t) \leq V(0) - \lim_{t \to \infty} V(t) \leq V(0)\), where the upper and lower bounds imply that \(\int_0^\infty \sum_{i \in \mathcal{I}} \xi_i e_i^T e_i(t)\) exists. Thus, it follows that \(V\) is bounded, which implies that \(e_i\) and \(\tilde{\theta}_i\) are bounded. Since \(w\) is bounded, it follows from part (ii) of Lemma 2 that \(\hat{x}_{*,i}\) is bounded. Since \(e_i\) and \(\hat{x}_{*,i}\) are bounded, it follows from (19) and (21) that \(\tilde{\theta}_i\) and \(u_i\) are bounded, which confirms (i).

Next, since \(e_i\), \(\tilde{\theta}_i\), and \(\phi_i\) are bounded, (26) implies that \(e_i\) is bounded. Since \(e_i\) and \(\tilde{\theta}_i\) are bounded, it follows that \(|\sum_{i \in \mathcal{I}} \xi_i d_i^T e_i(t) e_i(t)| = 2 |\sum_{i \in \mathcal{I}} \xi_i e_i^T e_i(t)\) exists and is uniformly continuous. Since \(f(t) \triangleq \sum_{i \in \mathcal{I}} \xi_i e_i^T e_i(t)\) is uniformly continuous, Barbalat’s Lemma implies that \(\lim_{t \to \infty} f(t) = 0\). Thus, \(\lim_{t \to \infty} y_i(t) = 0\), and it follows from (27) that \(\lim_{t \to \infty} y_i(t) = 0\), which confirms (ii).
We now specialize Theorem 1 to the case where the reference-model commands and the disturbances are zero. The proof is omitted for space considerations.

**Theorem 2.** Consider the closed-loop system (20) and (22), where \( n_{e,i} \) satisfies (4), the open-loop system (1)–(2) satisfies assumptions (A1)–(A7), \( u(t) \equiv 0 \), and \( r_i(t) \equiv 0 \). Then, the equilibrium \( (\hat{x}_1, \ldots, \hat{x}_n, \hat{\theta}_1, \ldots, \hat{\theta}_n) = 0 \) is Lyapunov stable. Furthermore, for all initial conditions \( x_i(0) \in \mathbb{R}^{n_i} \), \( U_i(0) \in \mathbb{R}^{n_{e,i}} \), \( Y_i(0) \in \mathbb{R}^{n_{c,i}} \), and \( \theta_i(0) \in \mathbb{R}^{2n_{c,i}+1} \), the following statements hold:

1. \( x_i(t), u_i(t), \theta_i(t), U_i(t) \), and \( Y_i(t) \) are bounded.
2. \( \lim_{t \to \infty} \hat{x}_i(t) = 0 \).

In principle, MRAC robustness modifications (e.g., dead zones, \( \sigma \)-modification, parameter projection) could be used to modify the decentralized adaptive controller (17)–(20). However, analyzing the stability properties of (17)–(20) with robustness modification is an open problem.

**V. Numerical Examples**

**Example 1.** Decentralized adaptive stabilization. Consider the system (1)–(2), where \( \ell = 4 \),

\[
A_1 = A_3 = \begin{bmatrix} -5 & 14 \\ 1 & 0 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix}, \quad (33)
\]

and \( B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \) and \( C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix} \), which satisfies (A1) and (A2). The interconnections are given by \( \delta_{1,2} = \delta_{2,3} = \delta_{3,4} = \delta_{4,2} = \delta_{4,3} = 1 \), and \( \delta_{1,3} = \delta_{1,4} = \delta_{3,1} = \delta_{4,1} = 0 \). The origin of the unforced system (1)–(2) and (33) is unstable. We let \( \bar{n}_i = n_i = 2 \), which satisfies (A3). We let \( w(t) \equiv 0 \) and \( r_i(t) \equiv 0 \) and consider the stabilization problem. The reference model is \( G_{m,i}(s) = (s+7)/(s^2+21s+88) \), and \( \gamma_i = 10 \). Thus, \( F_i(s) \), given by (3), satisfies (A6). Next, let \( n_{e,i} = 2 \), which satisfies (4), and let \( A_i \) be given by (5), where \( a_i = \begin{bmatrix} -8 & -7 \end{bmatrix} \), which has eigenvalues at \(-7 \) and \(-1\).

The adaptive controller (17)–(20) is implemented in feedback with (1)–(2) and (33), where \( \Gamma_i = 10^2 I_5 \), \( w(t) \equiv 0 \), and \( r_i(t) \equiv 0 \). Figure 2 provides a time history of \( x_i \) and \( u_i \), where \( x_1(0) = x_3(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \), \( x_2(0) = x_4(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T \), and \( U_1(0) = Y_1(0) = 0 \). The state \( x_i \) converges to zero.

**Example 2.** Decentralized adaptive disturbance rejection. Consider the system (1)–(2), where \( \ell = 4 \),

\[
A_1 = A_3 = \begin{bmatrix} -10 & -21 \\ 1 & 0 \end{bmatrix}, \quad (34)
\]

\[
A_2 = A_4 = \begin{bmatrix} -12 & -27 \\ 1 & 0 \end{bmatrix}, \quad (35)
\]

and \( B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \) and \( C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix} \), and

\[
D_1 = D_3 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_2 = D_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad (36)
\]

which satisfies (A1) and (A2). The interconnections \( \delta_{1,1}, \delta_{1,4}, \delta_{4,4} \) are the same as in Example 1. The origin of the unforced system (1)–(2) and (34)–(35) is asymptotically stable. We let \( \bar{n}_i = n_i = 2 \), which satisfies (A3). We let \( r_i(t) \equiv 0 \) and consider the disturbance rejection problem. The reference-model parameters satisfying (A6) are the same as in Example 1. The disturbance is given by \( w(t) = \sin(0.25 \pi t) \sin(0.5 \pi t) \), which satisfies (A4). Note that the disturbance spectrum is unknown and the disturbance is unmeasured. We let \( \bar{n}_w = n_w = 4 \), which satisfies (A5). Next, let \( n_{e,i} = 6 \), which satisfies (4), and let \( A_i \) be given by (5), where \( a_i = \begin{bmatrix} -12 & -45 & -80 & -75 & -36 & -7 \end{bmatrix} \), which has eigenvalues at \(-7 \) and \(-1\).

The adaptive controller (17)–(20) is implemented in feedback with (1)–(2) and (34)–(36), where \( \Gamma_i = 10^2 I_{13} \) and \( r_i(t) \equiv 0 \). Figure 3 provides a time history of \( y_i \) and \( u_i \), where \( x_1(0) = x_3(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \), \( x_2(0) = x_4(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T \), and \( U_1(0) = Y_1(0) = 0 \). The system is allowed to run open-loop for 10 seconds, then the decentralized adaptive control is turned on. The output \( y_i \) converges asymptotically to zero, thus rejecting the disturbance \( w \).

**Example 3.** Decentralized adaptive command following. Consider the system (1)–(2), where \( \ell = 2 \),

\[
A_1 = \begin{bmatrix} -10 & -16 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -11 & -28 \\ 1 & 0 \end{bmatrix}, \quad (37)
\]
and $B_i = [1 \ 0]^T$ and $C_i = [1 \ 2]$, and
\[ D_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \] (38)

which satisfies (A1) and (A2). The interconnections are $\delta_{1,2} = 2$ and $\delta_{2,1} = 1$. The origin of the unforced system (1)–(2) and (37)–(38) is asymptotically stable. We let $\bar{y}_i = y_i = 2$, which satisfies (A3). For this example, we let $w(t) \equiv 0$, and consider the command following problem. Although Theorems 1 and 2 do not address command following, we use this example to explore the command following properties of the decentralized adaptive controller.

Next, we consider the reference model $G_{m,i}(s) = (s + 7)/(s^2 + 20s + 79)$, and let $\gamma_i = 10$. Thus, $F_i(s)$, given by (3), satisfies (A6). The reference-model commands are $r_1(t) = 0.5 \sin 0.25 \pi t$ and $r_2(t) = 0.4 \sin 0.5 \pi t$. Next, let $n_{c,i} = 2$, which satisfies (4), and let $A_{t,i}$ be given by (5), where $a_i = \begin{bmatrix} -8 & -7 \end{bmatrix}$, which has eigenvalues at $-7$ and $-1$. Note that $A_{t,i}$ has an eigenvalue equal to the zero of $G_{m,i}(s)$, which is $-7$.

The adaptive controller (17)–(20) is implemented in feedback with (1)–(2) and (37)–(38), where $\Gamma_i = 10^2 I_5$ and $w(t) \equiv 0$. Figure 4 provides a time history of $y_i$, $y_{m,i}$, $z_i$, and $u_i$, where the initial conditions are zero. The system is allowed to run open-loop for 5 seconds, then the decentralized adaptive control is turned on. The performance $z_i$ does not converge to zero but does remain bounded.

REFERENCES