Network Reconstruction from Intrinsic Noise: Minimum-Phase Systems

David Hayden†, Ye Yuan and Jorge Gonçalves

Abstract—This paper considers the problem of inferring the structure and dynamics of an unknown network driven by unknown noise inputs. Equivalently we seek to identify direct causal dependencies among manifest variables only from observations of these variables. We consider linear, time-invariant systems of minimal order and with one noise source per measured state. If the transfer matrix from the inputs to manifest states is known to be minimum phase, this problem is shown to have a unique solution irrespective of the network topology. This is equivalent to there being only one spectral factor (up to a choice of signs of the inputs) of the output spectral density that satisfies these assumptions. Hence for this significant class of systems, the network reconstruction problem is well posed.

I. INTRODUCTION

Many phenomena are naturally described as networks of interconnected dynamical systems and the identification of the dynamics and structure of a network has recently become an important problem. A variety of model classes exists including differential equation, probabilistic, information theoretic and Boolean. Given a model class, the problem is typically underdetermined and additional assumptions on the network structure are made, such as sparsity or restriction to particular topologies. We focus here on Linear, Time-Invariant (LTI) systems, for which there are still many interesting theoretical questions outstanding, and leave the network structure unrestricted.

Previous work characterised identifiability in the deterministic case where targeted, known inputs may be applied to the network [1]. This also extends to biologically-inspired "knock-out" manipulations [2], [3]. In practice however, and often in biological applications, these types of experiments are not possible or are expensive to conduct. One may simply be faced with the outputs of an existing network driven by its own intrinsic variation. Noise is endemic in biological networks and its sources are numerous [4]; making use of this natural variation as a non-invasive means of identification is an appealing prospect, for example in gene regulatory networks [5].

This problem has been considered in various forms in the literature. In [6] for example, networks of known, identical subsystems are considered, which can be identified using an exhaustive grounding procedure similar to that in [3]. A solution is presented in [7] for identifying the undirected structure for a restricted class of polytree networks; and in [8] for "self-kin" networks. In [9] the problem is posed as a closed-loop system identification problem for a more general, but known, topology; in [10], the authors claim that their method can also be applied to networks with unknown topology.

In all of the above-cited work, the intrinsic variation is modelled as unknown noise sources applied only to the states that are measured. Whilst being an unrealistic assumption in some applications, this input requirement has been shown to be necessary for solution uniqueness even in the deterministic case [1]. In addition, for the more general case of full process noise the problem has been shown to be ill posed [11]. Here we also make the assumption that the transfer matrix is minimum phase, which is shown to be sufficient for both the network topology and dynamics to be uniquely identifiable. The non-minimum-phase case is treated in a separate paper [12], in which case the solution is not unique in general.

Section II provides necessary background information on spectral factorization, structure in LTI systems and the network reconstruction problem. The main result is then presented in Section III, followed by some discussion and a numerical example. Conclusions are drawn in Section IV.

Notation

Denote by $A(i,j)$, $A(i,:)$ and $A(:,j)$ element $(i,j)$, row $i$ and column $j$ respectively of matrix $A$. Denote by $A^T$ the transpose of $A$ and by $A^*$ the conjugate transpose. We use $I$ and $0$ to denote the identity and zero matrices with implicit dimension, where $e_i := I(:,i)$, and diag($a_1, ..., a_n$) to denote the diagonal matrix with diagonal elements $a_1, ..., a_n$.

We use standard notation to describe linear systems, such as the quadruple $(A, B, C, D)$ to denote a state-space realization of transfer function $G(s)$, $x(t)$ to describe a time-dependent variable and $X(s)$ its Laplace transform and we omit the dependence on $t$ or $s$ when the meaning is clear. We also define a signed identity matrix as any square, diagonal matrix $J$ that satisfies $J(i,i) = \pm 1$.

II. PRELIMINARIES

A. Spectral Factorization

Consider systems defined by the following Linear, Time-Invariant (LTI) representation:

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \] (1)

with input $u(t) \in \mathbb{R}^m$, state $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathbb{R}^p$, system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ and transfer function from $u$ to $y$:

$G(s) = C(sI - A)^{-1}B + D$. 

Make the following assumptions:

**Assumption 1.** The matrix $A$ is Hurwitz.

**Assumption 2.** The system is driven by unknown white noise $u(t)$ with covariance $\mathbb{E}[u(t)u^T(t)] = I\delta(t-\tau)$.

**Assumption 3.** The system $(A, B, C, D)$ is globally minimal.

The meaning of Assumption 3 is explained below. From $y(t)$, the most information about the system that can be obtained is the output spectral density:

$$\Phi(s) = \mathbb{E}[Y(s)Y^*(s)] = G(s)G^*(s)$$

The spectral factorization problem (see for example [13]) is that of obtaining spectral factors $G'(s)$ that satisfy:

$$G'G'^* = \Phi.$$  

Note that the degrees of two minimal solutions may be different; hence make the following definition.

**Definition 1** (Global Minimality). For a given spectral density $\Phi(s)$, the globally-minimal degree is the smallest degree of all its spectral factors.

Any system of globally-minimal degree is said to be globally minimal. Anderson [14] provides an algebraic characterisation of all realizations of all spectral factors as follows. Given $\Phi(s)$, define the positive-real matrix $Z(s)$ to satisfy:

$$Z(s) + Z^*(s) = \Phi(s)$$  \hspace{1cm} (2)

Minimal realizations of $Z$ are related to globally-minimal realizations of spectral factors of $\Phi$ by the following lemma.

**Lemma 1** ([14]). Let $(A, B, C, D)$ be a minimal realization of the positive-real matrix $Z(s)$ of (2), then the system $(A, B, C, D)$ is a globally-minimal realization of a spectral factor of $\Phi$ if and only if the following equations hold:

$$RA^T + AR = -BB^T$$

$$RC^T = B_z - BD^T$$

$$2D_z = DD^T$$

for some positive-definite and symmetric matrix $R \in \mathbb{R}^{n \times n}$.

This result was used by Glover and Willems [15] to provide conditions of equivalence between any two such realizations. In this paper we make the additional assumption:

**Assumption 4.** The transfer function $G(s)$ is square and minimum phase.

By which we mean that $G(s)$ is full rank for all $s$ with $\text{Re}(s) > 0$. In this case, any two spectral factors $G$ and $G'$ are related by: $G' = GU$ for some orthogonal matrix $U \in \mathbb{R}^{p \times p}$. Realizations of $G$ and $G'$ are then related by the following lemma.

**Lemma 2.** If $(A, B, C, D)$ and $(A', B', C', D')$ are globally-minimal realizations of square, minimum-phase transfer functions, then they have equal output spectral density if and only if:

$$A', B', C', D') = (TAT^{-1}, TBU, CT^{-1}, DU)$$ \hspace{1cm} (4)

for some invertible $T \in \mathbb{R}^{n \times n}$ and orthogonal $U \in \mathbb{R}^{p \times p}$.

...
We also define a state-space realization of a particular DSF $(Q_0, P_0)$ as any realization for which the (unique) DSF is $(Q_0, P_0)$. The relationship between state space, DSF and transfer function representations is illustrated in Figure 1, which shows that a state-space realization uniquely defines both a DSF and a transfer function. However, multiple DSFs are consistent with a given transfer function and a given DSF can be realized by multiple state-space realizations.

All realizations of a particular $G$ are parameterized by the set of invertible matrices $T \in \mathbb{R}^{n \times n}$. A subset of these will not change the DSF as follows.

**Definition 4** ($(Q,P)$-invariant transformation). A state transformation $T$ of system $(A_0, B_0, C_0, D_0)$ with DSF $(Q_0, P_0)$ is $(Q,P)$-invariant if the transformed system $(TA_0T^{-1}, T B_0, C_0 T^{-1}, D_0)$ also has DSF $(Q_0, P_0)$.

The blue region in Figure 1(a) is the set of all $(Q,P)$-invariant transformations of $(A_0, B_0, [I \ 0], 0)$.

**C. Network Reconstruction**

The network reconstruction problem was cast in [1] as finding exactly $(Q_0, P_0)$ from $G_0$. Since in general multiple DSFs are consistent with a given transfer function, some additional *a priori* knowledge about the system is required for this problem to be well posed. It is common to assume some knowledge of the structure of $P$, as follows.

**Assumption 6.** The matrix $P$ is diagonal and full rank.

Note that $P$ must also be square by Assumption 4. This is a standard assumption in the literature [1], [6], [8], [9] and equates to knowing that each of the manifest states is directly affected only by one particular input. By direct we mean that there is a link or a path only involving latent states from the input to the manifest state.

The following theorem is adapted form Corollary 1 of [1]:

**Theorem 1** ([1]). There is at most one DSF $(Q,P)$ with $P$ square, diagonal and full rank that is consistent with a transfer function $G$.

Given a transfer function $G_0$ for which the generating system is known to have $P_0$ square, diagonal and full rank, one can therefore uniquely identify the “true” DSF $(Q_0, P_0)$.

**Example 1.** Consider the following stable, minimal system with two manifest states and one latent state:

\[
A_0 = \begin{bmatrix}
-1 & 0 & 4 \\
0 & -2 & 5 \\
-6 & 0 & -3
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix},
\]

with $C_0 = [I \ 0]$ and $D_0 = 0$. The system transfer matrix, which is minimum phase, is given by:

\[
G_0(s) = \begin{bmatrix}
\frac{s^3 + 3}{s^2 + 4s + 5} & 0 \\
0 & 1
\end{bmatrix}
\]

and may be realized by an infinite variety of $A$ and $B$ matrices. The DSF is given by:

\[
Q_0(s) = \begin{bmatrix}
0 & 0 \\
\frac{1}{s+2} & 0
\end{bmatrix}, \quad P_0(s) = \begin{bmatrix}
\frac{s^3 + 3}{s^2 + 4s + 5} & 0 \\
0 & \frac{1}{s+2}
\end{bmatrix}
\]

Fig. 1: Pictorial representation of relationship between state space, DSF space and transfer function space for a particular system $(A_0, B_0, [I \ 0], 0)$. In (a) is contained the set of state transformations of this system by matrices $T$ that preserve $C_0 = [I \ 0]$; in red is the particular realization with $T = I$ and in blue the set of realizations with the same DSF $(Q_0, P_0)$. In (b) is the set of all DSFs that have realizations in (a); in blue is the particular DSF $(Q_0, P_0)$. In (c) is the single transfer function $G_0$, with which are consistent all DSFs in (b) and which can be realized by all realizations in (a).

The DSF describes causal interactions among manifest states that may occur via latent states in the underlying system.

**D. A realization for diagonal $P$**

The presence of latent states allows some freedom in the choice of realization used to represent a particular DSF. It will be convenient to use a particular form for systems with $P$ square and diagonal, defined here. We start with the following lemma.

**Lemma 3.** The matrix $V$ (and hence $P$) is diagonal if and only if the matrices:

\[
B_1 \quad \text{and} \quad A_{12}A_{22}^kB_2
\]

for $k = 0, 1, \ldots, l - 1$ are diagonal, where $l = \text{dim}(A_{22})$.

The proof is omitted but follows from expressing $V$ in (6) as a Neumann series and making use of the Cayley-Hamilton Theorem. It is clear that $P$ is diagonal if and only if $V$ is. Hence, without loss of generality, order the manifest states such that $B_1$ can be partitioned:

\[
B_1 = \begin{bmatrix}
0 & 0 \\
0 & B_{22}
\end{bmatrix}
\]

where $B_{22}$ is square, diagonal and full rank. Any system $(A_0, B_0, C_0, D_0)$ with $P_0$ square and diagonal can be transformed using $(Q,P)$-invariant transformations into one in the following form (see [11] for details).
Definition 5 (V-Diagonal Canonical Form). Any DSF \((Q,P)\) with \(P\) square and diagonal has a realization with \(A_{12}, A_{22}, B_1, \ldots\) signed identity. First note that in V-diagonal canonical form:
\[
\begin{bmatrix}
\hat{c}_{11} & 0 & 0 \\
0 & \hat{c}_{22} \times & 0 \\
0 & 0 & B_{22}
\end{bmatrix}
\]  
(10)

where \(\times\) denotes an element that is determined by \(Q\). The following is a canonical realization of \(V\):
\[
\begin{bmatrix}
\hat{a}_{11} & 0 & 0 \\
0 & \hat{a}_{22} \times & 0 \\
0 & 0 & B_{22}
\end{bmatrix}
\]
(11)

where terms are defined as follows. First obtain a minimal SISO realization of \(V(i,i)\) in controllable canonical form:
\[
V(i,i) = \delta_i + \gamma_i(sI - \alpha_i)^{-1}\beta_i
\]
(12)

Hence for a significant class of systems – those with stable interactions among manifest variables – that are driven by filtered noise, where the filters are minimum phase, the matrix \(G\) is minimum phase.

Theorem 2. Two systems \((A, B, C, D)\) and \((A', B', C', D')\) under Assumptions 1-6 with DSFs \((Q, P)\) and \((Q', P')\) have equal output spectral density:
\[
\Phi(s) = G(s)G^*(s) = G'(s)G'^*(s)
\]
if and only if \(G' = GJ\), for some signed identity matrix \(J\). This is equivalent to having \(Q' = Q\) and \(P' = PJ\). Given a particular \(\Phi_0\), the minimum-phase spectral factor \(G_0J\) is therefore unique up to some choice of \(J\), after which the solution for the DSF \((Q_0, P_0J)\) is unique.

Proof. From Lemma 2, two systems under Assumptions 1-6 have equal output spectral density if and only if they satisfy (4) for some invertible \(T \in \mathbb{R}^{n \times n}\) and orthogonal \(U \in \mathbb{R}^{p \times p}\). We shall derive necessary conditions for (4) to hold and show that these imply that \(U\) must be a signed identity matrix.

First, \(C' = CT^{-1}\) from (4) is satisfied if and only if \(T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix}\), partitioned as \(C\) for some \(T_1\) and \(T_2\). Then \(B' = TBU\) gives:
\[
\begin{align*}
B'_1 &= B_1U \\
B'_2 &= (T_1B_1 + T_2B_2)U
\end{align*}
\]
(13)

Take \((A, B)\) and \((A', B')\) to be in V-diagonal canonical form (10); then from (13a) the size of the partitioning of \(B_1\) is the same as that of \(B'_1\). Since \(B'_1\) must be diagonal, (13a) implies
\[
U = \begin{bmatrix} U_{11} & 0 \\ 0 & J_{22} \end{bmatrix}
\]
partitioned as \(B_1\) for some orthogonal \(U_{11}\) and signed identity \(J_{22}\). In the case that \(B_1\) is invertible \((B_1 = B_{22})\), it is clear that \(U = J_{22}\) and the result holds.

In general, since the second block column of \(B'_2\) must be zero, we require: \(T_1B_1 = 0\). From (4) we now have:
\[
\begin{align*}
A'_{12} &= A_{12}T_2^{-1} \\
A'_{22} &= T_2A_{22} + T_1A_{12}T_2^{-1} \\
B'_2 &= T_2B_2U
\end{align*}
\]
(14)

Define \(\hat{T}_1 := T_2^{-1}T_1\) for clarity. The matrices \(V\) and \(V'\) are both required to be diagonal by Assumption 6, where:
\[
V' = B'_1 + A'_{12}(sI - A'_{22})^{-1}B'_2
\]
\[
= B_1J_{22} + A_{12}(sI - A_{22} - \hat{T}_1A_{12})^{-1}B_2U
\]

Since \(B_1J_{22}\) is diagonal, then from Lemma 3, \(V'\) is diagonal if and only if
\[
A_{12}(A_{22} + \hat{T}_1A_{12})kB_2U
\]
(15)
is diagonal for \(k = 0, \ldots, l - 1\) where \(l = \dim(A_{22})\). Given that \(V\) is diagonal, we will now show that a necessary condition for \(V'\) to be diagonal is that \(U_{11}\) is a signed identity. First note that in V-diagonal canonical form:
\[
A_{12}(A_{22})^kB_2U = \begin{bmatrix} c_{11}d_{11}^k & 0 \\ 0 & 0 \end{bmatrix}
\]
(16)
\[ \hat{c}_{11} \hat{a}_{11}^k \hat{b}_{11} = \text{diag}(\gamma_1 \alpha_{11}^k \beta_1, \ldots, \gamma_{p_1} \alpha_{p_1}^k \beta_{p_1}) \] (17)
where \( p_1 = \text{dim}(U_{11}) \). Recall that \( r_i = \text{dim}(\alpha_i) \) is the order of the transfer function \( V(i, i) \).

We now prove by induction that the following statement holds for \( i = 1, \ldots, p_1 \) and for all \( k \geq 0 \) if (15) is diagonal:

| If \( \{ k < r_i \text{ and } \gamma_{i,1} = \ldots = \gamma_{i,k+1} = 0 \} \) |
| For \( \{ j = 0, \ldots, k \} \) |
| \( A_{12} A_{22}^k B_2 U \) \( (;, i) = 0 \) |
| \( A_{12} A_{22}^k B_2 U \) \( (i,:) = 0^T \) |
| Else |
| \( U_{11}(i,:) = (U_{11}(i,:))^T = \pm e_i \) |

Base case: \( k = 0 \)

Note that \( r_i > 0 \) for \( i = 1, \ldots, p_1 \) (otherwise \( V(i, i) = 0 \)) and hence \( k < r_i \) for \( k = 0 \). Then (15) requires \( A_{12} B_2 U \) to be diagonal, which is equivalent to:

\[ \hat{c}_{11} \hat{b}_{11} U_{11} = \text{diag}(\gamma_{1,1}, \ldots, \gamma_{p_1,1}) U_{11} \] (19)

being diagonal. Consider row \( i \): if \( \gamma_{i,1} = 0 \) then clearly \( (A_{12} B_2 U)(i,:) = 0^T \) and (since it is diagonal) \( (A_{12} B_2 U)(i,:) = 0 \) too. Conversely, if \( \gamma_{i,1} \neq 0 \) then \( U_{11}(i,j) = 0 \) \( \forall j \neq i \) (since (19) must be diagonal) and therefore \( U_{11}(i,i) = \pm 1 \) such that \( U \) is orthogonal. We therefore have \( U_{11}(i,:) = \pm e_i^T \) (by a standard property of orthogonal matrices) \( U_{11}(i,:) = U_{11}(i,:)^T \). Hence (18) holds for \( k = 0 \) for all \( i = 1, \ldots, p_1 \).

Induction

Assume (18) holds for \( k-1 \) for \( i = 1, \ldots, p_1 \) and show that if the \( k^{th} \) term of (15) is diagonal then (18) holds for \( k \). The \( k^{th} \) term of (15) is:

\[ A_{12} A_{22}^k B_2 U = \]

\[ A_{12} A_{22}^k B_2 U + \sum_{h=0}^k \zeta_h A_{12} A_{22}^h (\hat{T}_1 A_{12}^k) \hat{T}_1 (A_{12} B_2 U) + \sum_{h=1}^k \eta_h A_{12} (\hat{T}_1 A_{12}^h) \hat{T}_1 (A_{12} A_{22}^h B_2 U) \] (20a)

(20b)

(20c)

for some \( \zeta_h \) and \( \eta_h \). Consider any \( i \) for which \( \gamma_{i,1} = \ldots = \gamma_{i,k} = 0 \) and note that \( \gamma_{i,k+1} = 0 \Rightarrow k < r_i \), otherwise \( V(i,i) = 0 \). Hence we must show (a) that if \( \gamma_{i,k+1} = 0 \), the \( i^{th} \) column and row of (20a) are zero vectors and (b) that if \( \gamma_{i,k+1} \neq 0 \), the \( i^{th} \) column and row of \( U_{11} \) are (signed) unit vectors. Note also that since \( \gamma_{i,1} = \ldots = \gamma_{i,k} = 0 \), the \( i^{th} \) row of \( \hat{c}_{11} \hat{a}_{11}^k \hat{b}_{11} \) is given by:

\[ \hat{c}_{11} \hat{a}_{11}^k \hat{b}_{11}(i,:) = \gamma_{i,k+1} e_i^T \] (21)

from (17) and the canonical form of (12).

(a) Suppose \( \gamma_{i,k+1} = 0 \), then from (16) and (21):

\[ (A_{12} A_{22}^k B_2 U)(i,:) = [\gamma_{i,k+1} U_{11}(i,:)]^T = 0^T \]

as desired. To show \( (A_{12} A_{22}^k B_2 U)(;i) = 0 \) note that the \( i^{th} \) columns of the bold parts of (20b) and (20c) are all zero from (18) for \( k-1 \) and hence the \( i^{th} \) column of (20a) must also be zero such that (20) is diagonal.

(b) Suppose \( \gamma_{i,k+1} \neq 0 \) and consider element \((i,j)\) of (20) for \( j \neq i \), which must be equal to zero. If \( \gamma_{j,1} = \ldots = \gamma_{j,k} = 0 \) then from (18) for \( k-1 \), the \( j^{th} \) columns of (20b) and (20c) are zero vectors and element \((i,j)\) is determined only by (20a), giving:

\[ \left( \hat{c}_{11} \hat{a}_{11}^k \hat{b}_{11} U_{11} \right)(i,j) = \gamma_{i,k+1} U_{11}(i,j) = 0 \] (18)

from (16) and (21), and hence \( U_{11}(i,j) = 0 \). Otherwise, if \( \gamma_{j,h} \neq 0 \) for some \( 1 \leq h \leq k \), we have \( U_{11}(i,j) = \pm e_j \) and hence \( U_{11}(i,j) = 0 \) directly from \( U_{11}(i,:) = \pm e_i^T \) which requires \( U_{11}(i,:) = \pm e_i^T = U_{11}(i,:)^T \) as desired.

Therefore by induction (18) holds for \( i = 1, \ldots, p_1 \) for all \( k \geq 0 \). In particular, it holds for \( k = \max_i(r_i) \), in which case the “it” condition is never satisfied and \( U_{11}(i,:) = \pm e_i \) for \( i = 1, \ldots, p_1 \) and hence \( U_{11} = J_{p_1} \) is a signed identity matrix. We must therefore have \( U = J \) for some signed identity matrix \( J \) in order for (4) to be satisfied. From (9), equality of spectral densities implies:

\[ G^r = (I - Q'^{-1})^{-1} P' = (I - Q)^{-1} P J = G J \]

Inverting the above and equating diagonal elements yields \( P' = PJ \) and hence \( Q' = Q \).

Given only the spectral density \( \Phi_0 \), the reconstruction problem for stable systems with minimum-phase \( P_0 \) therefore has a unique solution for \( Q_0 \) irrespective of topology. We find this to be a surprising and very positive result. The sign ambiguity in \( P_0 \) is entirely to be expected as only the variance of the noise is known.

B. Discussion

In practice it may be difficult to identify exactly either the minimum-phase spectral factor or the underlying DSF directly. One approach is to first estimate the spectral density, using for example Welch’s method, then find factorizations of this as in Lemma 1. However, given only an estimate of the transfer function, one must robustly estimate the DSF [17], necessitating a second estimation stage.

A second approach is to use blind system identification to estimate a state-space realization in V-diagonal canonical form, for example using subspace identification. It is necessary to use the state space because in general an estimated transfer matrix would not have a DSF with diagonal \( P \) consistent with it. The estimated model would uniquely specify a DSF with \( P \) diagonal, but the structure of \( Q \) would in general not be correct and some form of regularization would be needed to correct this as in [17].

A third route would be to simply estimate \( Q \) and \( P \) directly from data, in the knowledge the “true” parameters are identifiable. Methods exist for this purpose in the case where the topology is known [9]; however, even for the case of deterministic inputs, this is still an unsolved problem for an unknown topology.
C. Example

We give an example for which all solutions for the DSF are constructed from a given spectral density using Lemma 1. Solutions to (3) with diagonal $P$ are characterised by a single Algebraic Riccati Equation (ARE), the details of which can be found in [11] and [12]. The result of [12] is that for almost all systems this ARE has a finite number of solutions, hence by solving it we can construct all DSFs with equal spectral density. By Theorem 2 only one of them can be consistent with a minimum-phase transfer matrix.

**Example 2.** Start with the output spectral density $\Phi(s)$ for the system of Example 1 and from it construct the positive-real matrix $Z(s)$, such that $Z(s) + Z^*(s) = \Phi(s)$. Construct any minimal realization of $Z$ by standard methods, such as:

$$A = \begin{bmatrix} -3.9 & -0.97 & 1.9 \\ -3.6 & -2.2 & 2.4 \\ -15.5 & -1.5 & 1.1 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0.17 & 0.032 \\ 0.032 & 0.57 \\ 0.092 & 0.60 \end{bmatrix}$$

with $C = [I \ 0]$ and $D_z = 0$. Partition $B_z = \begin{bmatrix} B_{1z} \\ B_{2z} \end{bmatrix}$ as in (5), then all solutions to (3) can be shown to be:

$$R = \begin{bmatrix} B_{1z} \\ B_{2z} \end{bmatrix}^T \begin{bmatrix} B_{1z} \\ B_{2z} \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

where $R_2 = 1.02$ or $1.65$, and correspondingly

$$B_2 = \begin{bmatrix} 1.49 & 0.5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0.28 & -1.01 \end{bmatrix}$$

and $D = D_z = 0$. Choose the signs of $B_1$ to be positive for simplicity, then there are exactly two solutions ($A, B, C, D$), where $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ for each of the possible choices of $B_2$.

Transform both of these by $T = \begin{bmatrix} I & 0 \\ -B_2B_1^{-1} & 1 \end{bmatrix}$ to put them in V-diagonal canonical form. Then these transformed systems are both realizations of DSFs with diagonal $P$. The first corresponds to the system of Example 1, which has a minimum-phase transfer matrix, and the second to the following stable, minimal system:

$$A' = \begin{bmatrix} -3.3 & -2.9 & 4 \\ -2.9 & -5.7 & 5 \\ -8.3 & -3.7 & 3 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $C' = [I \ 0]$, $D' = 0$ and DSF:

$$Q'(s) = \frac{s^3 + 1.3}{s^2 + 2.7s + 1.3}, \quad \quad P'(s) = \frac{s^3 + 0.34s + 2.3}{s^2 + 0.34s + 2.3}$$

The transfer matrix for the second system is given by:

$$G'(s) = \begin{bmatrix} s^3 + 0.66 \\ s^3 + 0.34s + 2.7 \end{bmatrix}, \quad \quad \frac{s^3 + 1.3}{s^2 + 2.7s + 1.3}$$

which (from Theorem 2) is necessarily non-minimum phase – it has a transmission zero at $s = 3$. These are the only two solutions for $Q$ that have corresponding $P$ diagonal, and only one of them has a minimum-phase transfer function.

IV. CONCLUSIONS

We considered the problem of identifying the structure and dynamics of an unknown network driven by unknown intrinsic noise. For stable, minimum-phase LTI systems with standard assumptions on the noise, we prove that there is only one minimum-phase spectral factor of the output spectral density (up to a choice of signs) that satisfies these assumptions. From this spectral factor the network topology and the dynamics of its links can be found uniquely, regardless of the topology. To our knowledge this is the first such result that places no restrictions on the structure of the network. The non-minimum-phase case is treated in a separate paper [12] for which the solution is not necessarily unique but for almost all systems is a member of a finite set that can be computed. This paper establishes the identifiability of the network; we provide some discussion as to how to actually find the solution in practice, but this remains the subject of substantial future work.

**REFERENCES**


