Adaptive Control of Uncertain Systems with Gain Scheduled Reference Models and Constrained Control Inputs

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Abstract—This paper develops a new state feedback model reference adaptive control approach for uncertain systems with gain scheduled reference models in a multi-input multi-output (MIMO) setting with constrained control inputs. A single Lyapunov matrix is computed for multiple linearizations of the nonlinear closed-loop gain scheduled reference system, using convex optimization tools. This approach guarantees stability of the closed-loop gain scheduled reference model. Adaptive state feedback control architecture is then developed, and its stability is proven for the case with constrained control inputs. The resulting closed-loop system is shown to have bounded solutions with bounded tracking error, with the proposed stable gain scheduled reference model. Sufficient conditions for ultimate boundedness of the closed-loop system are derived. A semi-global stability result is proved with respect to the level of saturation for open-loop unstable plants while the stability result is shown to be global for open-loop stable plants. Simulation results show that the developed adaptive controller can be used effectively to control a degraded turboshaft engine for large thrust commands, with guaranteed stability and proper tracking performance.

I. INTRODUCTION

Adaptive control of systems which are operating in multiple operating points has been of interest to researchers recently; some of the related literature is briefly reviewed subsequently. Adaptive control of piecewise linear systems has been developed in [1]. In these kinds of adaptive control systems, multiple linear time invariant (LTI) systems are used and transitions between these models are modeled as switches. These switchings introduce discontinuities and jumps in the control inputs. Adaptive control of time varying systems with gain scheduling is done in [2], [3]. The stability analysis for these systems is also performed using a time varying quadratic Lyapunov function, with some conditions on time varying Lyapunov matrix \( P(t) \) and its rate \( \dot{P}(t) \). Various adaptive control approaches for systems with input saturation are described in [4], [3], [5]. The stability proofs in these works are shown for adaptive control systems with LTI reference models.

To fulfill the need for a stable gain scheduled model reference model for systems with constrained control inputs, this paper develops a gain scheduled reference model without switching problem; which, in case of the gas turbine engine example its stability can be shown by finding a single Lyapunov function. Gain scheduled reference model design and stability analysis is performed using the method presented in [6], [7]. The scheduling variable in our reference model design process is an endogenous parameter [8], which is a function of the measurable plant outputs. Then stability analysis is conducted for a state feedback adaptive control system with gain scheduled reference model and constrained control inputs. The analysis uses some results from [4], [3]. The constraints on the control inputs are implemented using a multi-dimensional rectangular saturation function. This controller, then, has been implemented on a high fidelity physics-based JetCat SPT5 turboshaft engine model for large throttle commands with constraints on the control inputs to keep the engine in its safe operating envelope.

The contribution of this paper is the development of a stable state feedback model reference adaptive control algorithm for systems with gain scheduled reference models and constrained control inputs in a MIMO setting; the approach is applicable to systems, such as gas turbine engines, which the stability of their gain scheduled reference model is guaranteed by computing a single Lyapunov function [9], [10], [11]. A detailed stability analysis is performed for the proposed controller which can be used towards control software verification and certification [12].

The rest of this paper is organized as follows. In section II, a linear parameter dependent representation of the nonlinear system dynamics is presented, and stability analysis of the closed-loop system is shown. In section III, a model reference adaptive control with a gain scheduled reference model and constrained control inputs is designed with detailed stability proof. In section IV, simulations are performed for two different engine models including the nominal and degraded engine models. Simulation results show that the developed adaptive controller can be used effectively for the entire flight envelope of the turboshaft engine with guaranteed stability. Section V, concludes the paper.

II. LINEAR PARAMETER DEPENDENT REFERENCE MODEL DESIGN AND STABILITY

Consider the nonlinear dynamical system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
y(t) &= g(x(t), u(t)),
\end{align*}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the control input vector, \( y(t) \in \mathbb{R}^m \) is the output vector, \( f(\cdot) \) is an \( n \)-dimensional differentiable nonlinear vector function which represents the plant dynamics, and \( g(\cdot) \) is an \( m \)-dimensional differentiable nonlinear vector function which generates the plant outputs. We intend to design a feedback
control such that $y(t)$ properly tracks a reference signal $r(t)$ as $t$ goes to infinity, where $r(t) \in D_r \subset \mathbb{R}^m$, and $D_r$ is a compact set. Assume that for each $r \in D_r$, there is a unique pair $(x^e_r, u_e)$ that depends continuously on $r$ and satisfies the equations

$$0 = f^p(x^p_r, u_e), \quad r = g^p(x^p_r, u_e),$$

where $x^p_r$ is the desired equilibrium point and $u_e$ is the steady-state control that is needed to maintain equilibrium at $x^p_r$. It is often useful to parameterize the family of system equilibria as follows:

**Definition 1:** The functions $x^p_r(\alpha(t)), u_e(\alpha(t))$, and $r_e(\alpha(t))$ define an equilibrium family for the plant (1) on the set $\Omega$ if

$$f^p(x^p_r(\alpha(t)), u_e(\alpha(t))) = 0,$$

$$g^p(x^p_r(\alpha(t)), u_e(\alpha(t))) = r_e(\alpha(t)), \quad \alpha \in \Omega.$$

In order to make the design process easier [10], [11], we control the system via filtered inputs, rather than the input themselves, so there is no need for equilibrium control value other than zero (i.e. $x^e_r(\alpha(t)) = 0, v_e(\alpha(t)) = 0, \forall \alpha$). The filter is defined as $u^{(k)} = \frac{u}{s + \zeta}$, where $\zeta > 0$. The plant (1) with the filtered inputs, and its general controller can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \\ \dot{\delta x}(t) \\ \dot{\delta v}(t) \end{bmatrix} = \begin{bmatrix} f^p(x^p(t), u(t)) \\ -\zeta v(t) \\ f^c(x(t), p^c(x(t), u(t)), r(t)) \\ g^c(x(t), r(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \eta c \\ 0 \\ \eta c \end{bmatrix} v(t),$$

and the closed-loop nonlinear system is

$$\dot{x}(t) = F(x(t), r(t)),$$

where $x(t) \in D_x \subset \mathbb{R}^{n+2m}$, and $r(t) \in D_r \subset \mathbb{R}^m$. The controller is defined to be

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} -\epsilon c I \\ -\zeta \end{bmatrix} \begin{bmatrix} 0 \\ K^c_t(\alpha(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ \delta y(t) \end{bmatrix}.$$

Note that $\delta y(t) = \delta x^p(t)$, (i.e. $C^p(\alpha(t)) = I, D^p(\alpha(t)) = 0$). Linearizing (4), with the controller (6), results in linearized closed-loop system

$$\begin{bmatrix} \delta x^p(t) \\ \delta u(t) \\ \delta \dot{x}(t) \\ \delta \dot{v}(t) \end{bmatrix} = \begin{bmatrix} A^p(\alpha(t)) & B^p(\alpha(t)) & 0 \\ 0 & -\zeta c I & \eta c K^c_t(\alpha(t)) \\ I & 0 & -r c I \end{bmatrix} \begin{bmatrix} \delta x^p(t) \\ \delta u(t) \\ \delta \dot{x}(t) \\ \delta \dot{v}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \delta r(t), \quad \forall \alpha \in \Omega,$$

where for all $\alpha \in \Omega$

$$A_m(\alpha(t)) := \begin{bmatrix} A^p(\alpha(t)) & B^p(\alpha(t)) & 0 \\ 0 & -\zeta c I & \eta c K^c_t(\alpha(t)) \\ I & 0 & -r c I \end{bmatrix}.$$

Note that $\delta x^p(t) = x^p(t) - x^p_e(\alpha(t)), \delta y(t) = y(t) - y_e(\alpha(t)), \delta u(t) = u(t) - u_e(\alpha(t)), A^p(\alpha(t)), B^p(\alpha(t)), C^p(\alpha(t)), D^p(\alpha(t))$ are the parameterized plant linearization family matrices and $x^p_e(\alpha(t)), u_e(\alpha(t))$, and $y_e(\alpha(t))$ are the parameterized steady-state variables for the states, inputs and outputs of the plant, which form the equilibrium manifold of plant (1). The subscript “e” stands for “steady-state” throughout this paper. The parameter $\alpha$ is called the scheduling variable and should be measurable in real time. $\alpha(t)$ is a function of the plant states, and it is defined to be the Euclidean norm of the output vector $\alpha(t) = \|y(t)\|$.

**Remark 1:** Using pre-designed linear controllers, $K^c_t(\alpha_j), j \in \{1, 2, ..., q\}$, available for $q$ important operating points of the system, $K^c_t(\alpha(t))$ can be obtained for all $\alpha \in \Omega$ based on a stability preserving interpolation approach described in [13] with respect to the scheduling parameter $\alpha$ in a smooth, continuous way. An approach by which the interpolated controller stabilizes the linearized plant for all $\alpha \in \Omega$. Another approach is to compute $K^c_t(\alpha(t))$ by polynomial approximation as a function of $\alpha$.

**Assumption 1:** The matrix $A_m(\alpha(t))$ is bounded, i.e., for all $t > 0, \|A_m(\alpha(t))\| \leq k_A$, where $k_A < \infty$ is a constant.

**Remark 2:** This assumption can be investigated by extensive numerical simulation studies of the physical system that is being investigated, using a high fidelity dynamic model. For systems such as gas turbine engines, it already has been investigated in [9], [10]. Consider the closed-loop system (5), and assume there is a family of equilibrium points $(x_e(t), r_e(t))$ such that $F(x_e(t), r_e(t)) = 0$. Define $A^m_{nl} = \frac{\partial F}{\partial x} \in \mathcal{S}, \forall x \in D_x$, where $\mathcal{S} := \{A^m_{nl} \forall x \in D_x\}$ is the set of linearizations of system (5). Assume there exist symmetric positive definite matrices $P$ and $Q$, such that

$$PA^m_{nl} + A^{nlT}m P \preceq -Q, \quad \forall A^m_{nl} \in \mathcal{S},$$

then the system (5) is stable for all the trajectories defined by $A^m_{nl} \in \mathcal{S}$. In other words, assuming the initial state is sufficiently close to some equilibrium, then the closed-loop system remains in a neighborhood of the equilibrium manifold for all $t \geq 0$.

**Remark 3:** In practice we can not obtain $\mathcal{S}$, since it contains infinite number of linearizations of the system; instead, we can linearize system (5) for a large number of operating points $x_i \in D_x, i = 1, ..., L$, which we claim is sufficient to cover the set of actual operating conditions, to show the stability of the closed-loop system. Define $S$ as a matrix polytope described by its vertices, $S := \text{Co} \{A^m_{nl_1}, \ldots, A^m_{nl_L}\}$, where $A^m_{nl_i} = \frac{\partial F}{\partial x} \bigg|_{x=x_i} \in S$, for all $i \in \{1, 2, ..., L\}$. Note that $A^m_{nl_i}$ can be obtained by linearizing the nonlinear system (5) at non-equilibrium points (transient condition), and also at equilibrium points (steady state condition), which in this paper, are represented by $A_m((\alpha_i))$. Then using convex optimization tools [14], [15], for some matrix $Q = Q^T$, we compute a single symmetric positive definite matrix $P$, such that

$$PA^m_{nl_i} + A^{nlT}m P \preceq -Q, \quad \forall i \in \{1, 2, ..., L\}.$$

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With assumption 1 satisfied, and the claim that \( A_m(\alpha) \in S \), for all \( \alpha \in \Omega \), then system (7) is also stable. In the next section, we will show how to verify the above claim.

**Remark 4:** For the purpose of stability analysis, there is a need for multiple linearizations of the closed-loop system to construct a feasible set \( S \). The minimum number of required linearizations, \( L \), depends on the physical system; and it changes for different dynamical systems. This knowledge usually can be obtained through an extensive numerical simulation study of the dynamical system, using a high fidelity model [7].

**Lemma 1:** If matrices \( P \) and \( Q \) exist, such that LMI (10) is satisfied, and \( A_m(\alpha(t)) \in S \), for all \( \alpha \in \Omega \), then system (7) is stable for all trajectories defined by \( A_m(\alpha(t)) \in S \), and

\[
P A_m(\alpha(t)) + A_m^T(\alpha(t))P \leq -Q, \quad \forall \alpha \in \Omega. \tag{11}
\]

**Remark 5:** The existence of a single matrix \( P \) which guarantees the stability of a closed-loop system over some operating envelope has already been shown for dynamical systems such as gas turbine engines [9], [10], [11] and high performance aircraft [16]. The numerical verification of the assumption in Lemma 1 is that the linearized plant lives in the convex hull of the linearization matrix samples, for gas turbine engines can be found in [9], [10].

### III. Adaptive Control with Constrained Control Inputs

#### A. Mathematical Preliminaries

The definitions and lemmas presented here are mainly adopted from [17], [18], [3].

**Definition 2:** [17] A variant of the projection algorithm, \( \Gamma \)-projection, updates the parameter along a symmetric positive definite gain \( \Gamma \) as defined below

\[
\text{Proj}_{\Gamma}(\theta, y) = \begin{cases} 
\frac{\Gamma_f - \frac{\Gamma_f f(\theta)}{f(\theta)} \frac{f(\theta)}{f(\theta)}^T \Gamma_f(f, \theta)}{f(\theta) > 0 \land f(\theta) > 0}, & \text{if } f(\theta) > 0, \\
\Gamma_y, & \text{otherwise}.
\end{cases} \tag{12}
\]

**Lemma 2:** Given \( \theta^* \in \Omega \), then \( \theta - \theta^* \) is a convex function of \( \theta \).

**Lemma 3:** Let \( \text{Proj}_\Gamma(\Theta, Y) \) be defined like \( \text{Proj}_{\Gamma}(\theta, y) \) with \( \Theta \) being a symmetric matrix in \( \mathbb{R}^{m \times 1} \) and \( \Theta^* \) being a symmetric matrix in \( \mathbb{R}^{m \times m} \).

The constraints on the control inputs will be defined as a rectangular saturation function of \( v \). The saturation function is given by \( R_s(v) \), where the elements of \( R_s \) are defined by

\[
R_s(v) = \text{sat}(v) = \begin{cases} 
v_i, & \text{if } |v_i| \leq v_{i,\text{max}}, \quad i = 1, \ldots, m, \\
v_{i,\text{max}} \text{sgn}(v_i), & \text{if } |v_i| > v_{i,\text{max}}.
\end{cases} \tag{13}
\]

This saturation function can be expressed as the sum of a direction preserving component and an error component, so that

\[
R_s = \text{sat}(v) = \begin{cases} 
v, & \text{if } |v| \leq h(v), \\
\bar{v} = v_d + \bar{v}, & \text{if } |v| > h(v),
\end{cases} \tag{14}
\]

where \( v_d = \bar{v}, \bar{v} = v/|v| \) is the unit vector in the direction of \( v \), and \( h(v) \) returns the magnitude of the projection of \( v \) onto the hyper-rectangle. In this formulation \( v_d \) is in the same direction as \( v \) and \( \bar{v} \) is an error vector.

**Definition 3:** The function \( R_s(\cdot) \), is a multi-dimensional rectangular saturation function defined by

\[
R_s(v) = \begin{bmatrix} v_{1,\text{max}} \text{sat}(\frac{v_1}{v_{1,\text{max}}}) \\ \vdots \\ v_{m,\text{max}} \text{sat}(\frac{v_m}{v_{m,\text{max}}}) \end{bmatrix}, \tag{15}
\]

#### B. Problem Formulation

Consider the dynamical system

\[
\begin{bmatrix} \dot{\delta x}(t) \\ \dot{\delta y}(t) \\ \delta z(t) \end{bmatrix} = \begin{bmatrix} A^p(\alpha(t)) & B^p(\alpha(t)) & 0 \\ C^p(\alpha(t)) & D^p(\alpha(t)) & -\epsilon c \times I \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta r(t) \end{bmatrix}, \quad \forall \alpha \in \Omega. \tag{16}
\]

Scheduling parameter is defined to be \( \alpha(t) = |\delta x(\alpha(t))| \) (i.e. scheduling parameter is the norm of the plant states). Note that, here \( \delta y(t) = \delta x(\alpha(t)), \) i.e. \( C^p(\alpha(t)) = I, D^p(\alpha(t)) = 0 \). For simplicity from now on we rename the variables \( \delta x(t), \delta y(t) \) and \( \delta r(t) \) as \( \delta z(t) := x(t), \delta y(t) := y(t) \) and \( \delta r(t) := r(t) \). Plant (16) can be written as

\[
\dot{x}(t) = A(\alpha(t))x(t) + Bv(t) + B_r r(t), \quad \forall \alpha \in \Omega. \tag{17}
\]

The nominal control for this system is \( v_{nom}(t) = K^T(\alpha(t))x(t) \), for all \( \alpha \in \Omega \), where \( K^T(\alpha(t)) = [0, 0, K_i^T(\alpha(t))] \). The time-varying reference model is defined as

\[
\dot{x}_m(t) = A_m(\alpha(t))x_m(t) + B_r r(t), \quad \forall \alpha \in \Omega. \tag{18}
\]

In the previous section we showed the stability of this reference model. Note that \( r(t) \in \mathbb{R}^m \) is the command signal such that \( ||r(t)|| \leq r_{\text{max}} \).

**Assumption 2:** There exists an ideal gain matrix \( K^*T(\alpha(t)) = [0, 0, K_i^T(\alpha(t))] \), that results in perfect matching between the reference model (18) and the plant (17) such that

\[
A_m(\alpha(t)) = A(\alpha(t)) + BK^*T(\alpha(t)), \quad \forall \alpha \in \Omega. \tag{19}
\]

where \( A_m(\alpha(t)) \) is a Hurwitz matrix for all \( \alpha \in \Omega \).

**Remark 6:** The feasibility of this assumption has already been verified in [9], [10], [11], for gas turbine engine applications which we consider as the main application of this paper. For other systems, modeling and numerical studies are needed for such verification.
Assumption 3: Let $K^*(\alpha(t)) \in \Theta_K$ for all $\alpha \in \Omega$, where $\Theta_K$ is a known convex compact set. We also assume that $K^*(t)$ is continuously differentiable, and the derivative is uniformly bounded, $\|\dot{K}^*(\alpha(t))\| \leq d_k < \infty$ for all $\alpha \in \Omega$.

Remark 7: $\alpha(t)$ is defined to be $\alpha(t) = \|y(t)\| = \|x^p(t)\|$; since it is a function of endogenous variables (i.e., the plant states), its boundedness is guaranteed by boundedness of the plant states. As a result, its derivative $\dot{\alpha}(t) = \frac{x^p(t)^T x^p(t)}{\|x^p(t)\|^2}$ is also bounded. More details can be found in [6], [7], [9], [10], [8].

Remark 8: Compact set $\Theta_K$ can be obtained by extensive numerical simulation studies of the system that the controller is being designed for. Smoothness, continuity, and differentiability of $K(\alpha(t))$, and also uniform boundedness of $K(\alpha(t))$, can be guaranteed, by using proper design and computation process for $K(\alpha(t))$ (see Remark 1).

The adaptive controller is defined as

$$v(t) = v_{ad}(t) = K^T(t)x, \quad \forall \alpha \in \Omega. \quad (20)$$

Combining equations (17) and (20), for all $\alpha \in \Omega$ we obtain closed loop system

$$\dot{x}(t) = A_m(\alpha(t))x(t) + B\hat{K}^T(t)x(t) + B_r(t), \quad (21)$$

where $\hat{K}(t) = \hat{K}(t) - K^*(t)$. Defining $e(t) = x(t) - x_m(t)$, the error dynamics are

$$\dot{e}(t) = A_m(\alpha(t))e(t) + B\hat{K}^T(t)x(t), \quad \forall \alpha \in \Omega. \quad (22)$$

With the knowledge of lower and upper bounds of the parameters $K^*(t)$, the parameter projection adaptive law is

$$\tilde{K}(t) = \text{Proj}_\Gamma \left( \hat{K}(t), -x(t)e^T(t)PB \right), \quad (23)$$

where $\Gamma = \Gamma^T > 0$, $P = P^T > 0$ is a solution of LMI (11), and $\text{Proj}_\Gamma(\cdot, \cdot)$ is the $\Gamma$-projection operator defined in Definition 2.

C. Stability Analysis

In order to avoid the adaptive controller parameters to be adjusted improperly by the saturation error, we use the augmented error method in the adaptive control design developed in [4], [3] to provide the stability analysis for a gain scheduled model reference adaptive control system. The plant (17) with saturated control inputs can be written as

$$\dot{x}(t) = A(\alpha(t))x(t) + BR(v(t)) + B_r(t), \quad \forall \alpha \in \Omega. \quad (24)$$

where $v(t)$ is the adaptive control input which introduced in equation (20), the ultimate goal is to determine adaptive parameters such that all signals in the plant (24) are guaranteed to be bounded, and $y(t)$ tracks $r(t)$. The deficiency of $v(t)$ is defined as $\Delta v(t) = v(t) - R_s(v(t))$. Now, plant (24) can be written as

$$\dot{x}(t) = A(\alpha(t))x(t) + BR(v(t)) - B\Delta v(t) + B_r(t), \quad \forall \alpha \in \Omega. \quad (25)$$

Plant (25) with controller (20), for all $\alpha \in \Omega$, can be written as

$$\dot{x} = A(\alpha(t))x(t) + B\hat{K}^T(t)x(t) - B\Delta v(t) + B_r(t). \quad (26)$$

Subtracting the reference model (18) and the plant (26), a closed-loop error dynamics equation is obtained as

$$\dot{e}(t) = A_m(\alpha(t))e(t) + B\hat{K}^T(t)x(t) - B\Delta v(t), \quad \forall \alpha \in \Omega. \quad (27)$$

In order to eliminate the adverse effect of the disturbance $\Delta v(t)$, for all $\alpha \in \Omega$ we generate a signal $e_\Delta(t)$ as

$$\dot{e}_\Delta(t) = A_m(\alpha(t))e(t) - K(t)\Delta v(t), \quad e_\Delta(t_0) = 0 \quad (28)$$

where $K(t) = B - K(t)$. Let $K(t) \in \Theta_\Delta$, where $\Theta_\Delta$ is a convex compact set. We now choose adaptive laws for adjusting the parameters

$$\dot{\hat{K}}(t) = \text{Proj}_\Gamma \left( \hat{K}(t), -x(t)e^T(t)PB \right), \quad (29)$$

where $P = P^T > 0$ is a solution of LMI (11). The gains in adaptive laws $\Gamma = \Gamma^T > 0$ and $\Gamma = \Gamma^T > 0$ are positive definite matrices $\Gamma = \Gamma^T > 0$. $\Delta v(t)$ is bounded when the control inputs are constrained under rectangular saturation.

Theorem 2: The error $e_\Delta(t)$ in equation (29) is bounded, $\|e_\Delta(t)\|_{L_\infty} \leq \sqrt{\frac{k_m \|\Gamma^{-1}\|}{\lambda_{\max}(P)}} + 4\lambda_{\max}(P)\|\Gamma^{-1}\| \sum_{i=1}^m \max_{K_{ij}} \|k_{ij}^p(t)\| |K_{ij}(t)| |K_{ij}(t)| dK_{ij}$. 

Proof: See [19] for detailed proof.

Remark 9: The proof of Theorem 2 showed the boundedness of $e_\Delta(t)$, however it can not guarantee the boundedness of the tracking error $e(t)$. To prove the boundedness of $e(t)$, we must prove that $x(t)$ is bounded when the control inputs are constrained under rectangular saturation.

We define $\Theta^*_{\max}$ and $\Theta_{\max}$ to be $\Theta^*_{\max} = \sup \|K^*(t)\|$, $\Theta_{\max} = \max \sup \|K(t)\|, \sup \|K(t)\|$. Since we assumed the control gains belong to a known compact set, then $\Theta^*_{\max}$ and $\Theta_{\max}$ are positive and finite, hence there exists a smallest $n \in \mathbb{N}$ such that $\theta_{\max}^* < \eta \Theta_{\max}$. For efficiency of notation we define $\gamma_{\max}^* := \max \|\Gamma^{-1}\|, \|\Gamma^{-1}\|$, $\gamma_{\min} := \min \|\Gamma^{-1}\|, \gamma_{\max} := \max \|\Gamma^{-1}\|, \gamma_{\min} := \min \|\Gamma^{-1}\|$, and $\rho := \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, where $\rho > 0$ is the limit of the $i$th element of $v(t)$ and $Z_B \in \mathbb{R}$ is defined using the induced norm by the vector 2-norm such that the property is described by $\|x^p(t)P[B, B_r]\| \leq Z_B \|x(t)\|$. We also define the following constants for simplicity, $x_{\min} :=$
\[
\lambda_{\min}(Q) - (3n+2)ZB\Theta_{\min}, \quad x_{\max} := \frac{ZB\nu_{\min}}{\lambda_{\min}(Q) - (3n+2)ZB\Theta_{\min}}, \quad Z_{\max} := \frac{ZB(\nu_{\min} + 2r_{\max})}{\lambda_{\min}(Q) - (3n+2)ZB\Theta_{\min}}.
\]

**Theorem 3:** Under Assumptions 2 and 3 for the system (24) with the controller (20) and the adaptive laws in (30), \( x(t) \) has a semi-globally bounded trajectory with respect to the level of saturation for all \( t > 0 \) if (i) \( ||x(0)|| < \frac{2\rho}{\lambda_{\max}} \), and (ii) \( \sqrt{\nu(0)^2} < \frac{2\rho}{\lambda_{\max}} \). Further \( ||x(t)|| < x_{\max} \), \( \forall t > 0 \), and error \( e(t) \) is in the order of \( ||e(t)|| = O(\sup_{\tau \leq t} ||\Delta \nu(\tau)||) \).

**Proof:** See [19] for detailed proof.

**Remark 10:** Theorem 3 implies that if the initial conditions of the state and the parameter error lie within certain bounds, then the adaptive system will have bounded solutions. The local nature of the result for unstable systems is because of the saturation limits on the control input. For open-loop stable systems the results are global.

**IV. TURBOSHAFT ENGINE EXAMPLE**

We apply the developed adaptive controller to a high-fidelity physics-based model of JetCat SPT5 turboshaft engine driving a variable pitch propeller developed in [20]. The effect of engine degradation due to aging is modeled in the nonlinear simulation by modifying the efficiencies and flow capacities of key engine components such as: High Pressure Compressor (HPC), High Pressure Turbine (HPT) and Low Pressure Turbine (LPT). The values of these parameters used in this simulation are \( \eta_{\text{hpc}} = -1.470\% \), \( W_{\text{c,hpc}} = -2.455\% \), \( \eta_{\text{hpt}} = -1.315\% \), \( W_{\text{c,hpt}} = +0.880\% \), \( \eta_{\text{hpt}} = -0.269\% \), and \( W_{\text{c,lpt}} = +0.1294\% \), where their nominal values are zero. To show the stability of the closed-loop reference system, 40 different linearizations of the system have been used, to solve inequality (10), in Matlab with the aid of YALMIP [14] and SeDuMi [15] packages. The numerical value for the common matrix \( P \) is

\[
P = \begin{bmatrix}
0.491 & 0.079 & 0.102 & -0.004 & -0.072 & -0.039 \\
0.079 & 0.446 & 0.053 & 0.007 & -0.097 & -0.013 \\
0.102 & 0.053 & 0.181 & -0.041 & -0.028 & -0.022 \\
-0.004 & 0.007 & -0.041 & 0.130 & 0.023 & 0.013 \\
-0.072 & -0.097 & -0.028 & 0.023 & 0.321 & 0.045 \\
-0.039 & -0.013 & -0.022 & 0.013 & 0.045 & 0.332
\end{bmatrix},
\]

where its condition number is \( \kappa(P) = 6.6303 \), and \( Q = 0.1 \times I_6 \). Other controller parameters are \( \epsilon_c = 1, \eta_c = 3 \). The numerical values for the adaptive controller are set as \( \Gamma = \text{diag}(\{50, 50, 50, 50, 50, 50\}) \), \( \Gamma_{\Delta} = \text{diag}(\{30, 30\}) \), \( v_{\text{max}} = 0.12 \), \( v_{\text{max}} = 0.15 \), and the initial conditions and the compact sets are \( \Theta_{K_c} = \{[-2, 0] [-2, 0]: [-2, 0] [-2, 0]\}, \Theta_{\Delta} = \{\{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \}, \Theta_{\Delta} = \{\{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \{0\} \}, \Theta_{\Delta} = \{0 0 0 0 0 0 0 0 0 2 7 0 0 0\}, \Theta_{\Delta} = \{0 0 0 0 0 0 0 0 0 2 7 0 0 0\}\} \}

\]

Simulations are conducted for two different cases including the control of the nominal model (NomEng), and the control of the deteriorated engine due to aging (AgedEng). These case studies, simulate the engine acceleration from the idle to the cruise condition and then its deceleration back to the idle condition for a standard day at sea level condition. Simulation results are shown in figures 1 to 6. Figures 1 shows the high pressure spool speed tracking its reference trajectory closely. Figure 2, shows the evolution of the infinity norm of the errors \( ||e(t)||_\infty, ||e_{\Delta}(t)||_\infty, ||e_{\Delta}(t)||_\infty \). The steady-state error in the Aged Engine (AgedEng) simulation case is because of the effect of the aging on the engine health parameters, and this causes a change in the equilibrium manifold of the aged engine in comparison to the nominal engine (NomEng). In other words, since we are using nominal engine equilibrium manifold to design a linear parameter dependant reference model, and the aged engine linear model has a different equilibrium manifold \( x_{\text{p,nom}}(\alpha(t)) \neq x_{\text{p,aged}}(\alpha(t)) \), and \( u_{\text{e,nom}}(\alpha(t)) \neq u_{\text{e,aged}}(\alpha(t)) \), then \( \delta x_{\text{aged}}(t) = x_{\text{aged}}(t) - x_{\text{aged}}(\alpha(t)) \neq 0 \), and \( \delta u_{\text{aged}}(t) = \delta u_{\text{aged}}(t) - \delta u_{\text{aged}}(\alpha(t)) \neq 0 \), and this means \( ||\delta x_{\text{aged}}(t)|| > \delta x_{\text{min}} \neq 0 \) for all \( t > 0 \), hence, there will be a steady-state error value greater than zero. Figure 3 shows the evolution of the control inputs \( u(t) = [v_1(t), v_2(t)]^T \), which are inputs to the augmented system, each element is corresponding to one of the control inputs to the original system. For better performance and also to keep the engine in the safe range of operation, hard limits have been defined for both augmented control inputs, \( |v_i| \leq v_{\text{max}} \) for \( i = 1, 2 \).

Figure 4 shows the histories of fuel flow and propeller pitch angle as the control inputs to the plant. Figures 5 shows the evolution of the gain scheduling controller integral gain matrix \( (K_c(\alpha)) \) and also adaptive controller gain matrix \( (K_{\Delta}(\alpha)) \). Figure 6 shows the evolution of the nonzero elements of the augmented adaptive parameter for the saturated system \( (K_{\Delta}(\alpha)) \).
Stability analysis was performed for the developed adaptive control of the saturated system ($K_{\Delta}(t)$) with constrained control inputs architecture by proving the ultimate boundedness of the error signal. Sufficient conditions for ultimate boundedness of the closed-loop system were derived. A semi-global stability result was proved with respect to the level of saturation for open-loop unstable plants while the stability result becomes global for open-loop stable plants. Simulations result for adaptive control of JetCat SPT5 turboshift engine physics-based model with some degradation due to aging, shows that the proposed adaptive controller tracks the reference model.

ACKNOWLEDGMENTS

This material is based upon the work supported by the AFRL and also the NSF.

REFERENCES