Performance Optimization Over Positive $l_\infty$ Cones

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Abstract—In this paper, we first consider the case where the input to a system is restricted to be in the positive cone of $l_\infty$, denoted by $l_\infty^+$, and seek to characterize the system’s induced norm from $l_\infty^+$ to $l_\infty$. We obtain an exact characterization of this norm which is particularly easy to calculate in the case of LTI systems. Furthermore, we consider the model matching problem to show that time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems. In the second part of the paper, we consider the case when the output is forced to be in the positive $l_\infty$ cone when the input is in this cone. We show if internal positivity is sought, a dynamic optimal controller offers no advantage over a static one.

I. INTRODUCTION

There are many physical problems in which some variables are restricted to be non-negative (or non-positive); examples can be found in biology, economics, and many other areas [1] [2] [3]. Motivated by such problems, the theory of positive systems has been the focus for many researchers. Notions such as stability, stabilizability, positive realization, and (distributed) control synthesis of such systems have been the subject of research [4]-[13].

In this paper, we are interested in characterizing and optimizing the $l_\infty$ gain of linear systems that contain positivity type of constraints. We first consider the case where the input is restricted to be in the positive cone of $l_\infty$, denoted by $l_\infty^+$, and seek to characterize the induced norm from $l_\infty^+$ to $l_\infty$. We stress here, that no positivity constraint is imposed on the system itself. We obtain an exact characterization of this norm (the induced norm from $l_\infty^+$ to $l_\infty$) in terms of standard $l_\infty$ induced norms of appropriately defined subsystems which is particularly easy to calculate in the case of LTI systems. As an application (and motivation), we consider a filtering problem in which the signal to be estimated, $s$, is known to live in a positive cone, i.e. $s \in l_\infty^+$. In general, just designing a filter to minimize the standard $l_\infty$ induced norm of the operator from signal to the estimation error will be conservative. Instead, we can use the apriori knowledge of positiveness of the signal by considering the same problem with $l_\infty^+$ to $l_\infty$ induced norm.

Furthermore, based on this development, we consider the model matching problem to show that:

1) Time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems.

2) The optimization is a linear programming problem.

We further generalize the results in the case of mixed input signals when there are inputs both in $l_\infty^+$ and $l_\infty$. As an example, we consider the aforementioned filtering problem and solve it when the signal is positive and bounded but there also exists noise which is only bounded and not necessarily positive.

In the second part of the paper and in the context of $l_\infty$ optimization, we also consider the case when the output is forced to be in the positive $l_\infty$ cone when the input is in this cone. This reflects as, so-called, an external positivity constraint on the system. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is a linear programming and hence a tractable problem [14]. If, on the other hand, the constraint known as internal positivity is sought, we show that a dynamic controller offers no advantage over a static one. The results in [13] or [15] can be readily used to obtain an optimal (static) state feedback controller. We note that, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. We extend certain results of [13] to the case when part of the state is measurable (with measurement noise present).

II. PRELIMINARIES

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{n \times m}$ denote the sets of positive integers, non-negative integers, real numbers, positive real numbers, $n$-dimensional real vectors, and $n \times m$ dimensional real matrices respectively. For any $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, its $l_1$ and $l_\infty$ norm is defined as $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_{\infty} = \max |x_i|$. For any $M = [m_{ij}] \in \mathbb{R}^{n \times m}$, $\|M\|_1 = \max_i \sum_{j=1}^{m} |m_{ij}|$ and $\|M\|_{\infty} = \max_j \sum_{i=1}^{n} |m_{ij}|$. Given a sequence $y = (y(n))_{n=1}^{\infty}$, where $y(k) \in \mathbb{R}^n$, for $k \in \mathbb{Z}_+$, its $l_1$ and $l_\infty$ norm is given by $\|y\|_1 = \sum_{i=1}^{n} \|y(k)\|_1$ and $\|y\|_{\infty} = \sup_{k \in \mathbb{N}} \|y(k)\|_{\infty}$, whenever they are finite. The space of such sequences whose $l_1$ or $l_\infty$ norm is finite is denoted by $l_1^n$ and $l_\infty^n$, respectively.

Note that $l_\infty^n$ is the space of bounded sequences. In this paper, we are also interested in a certain subset of $l_\infty^n$ which is denoted by $l_\infty^+$. This set is characterized as

$$l_\infty^+ = \{ (y(k))_{k=0}^{\infty} \in l_\infty^n : y_i(k) \geq 0, \forall i \in \{1, \ldots, n\}, k \in \mathbb{Z}_+ \},$$

where $y_i(k)$ is the $i^{th}$ entry of vector $y(k) \in \mathbb{R}^n$. In other words, $l_\infty^+$ is the set of bounded non-negative sequences.

Let $\mathbb{L}^{n \times m}_{TV}$ be the space of all linear, causal, and bounded operators, $T : l_\infty^n \to l_\infty^n$. That is, for any $x, y \in l_\infty^n$, $T(x+y) = Tx + Ty$, $P_k T P_k u = TP_k u$, for $k \in \mathbb{Z}_+$, and

$$\|T\| := \sup_{u \neq 0} \frac{\|Tu\|_{l_\infty}}{\|u\|_{l_\infty}} < +\infty,$$  

(1)
where $P_k$ is the truncation operator defined by
\[ P_kx = (x_0, x_1, \ldots, x_{k-1}, 0, 0, \ldots). \]
Also, denote by $\mathcal{L}^{m \times n}_{TV}$ the subspace of all $T \in \mathcal{L}^{m \times n}_{TV}$ such that $\Lambda T = T \Lambda$, where $\Lambda$ is the delay operator
\[ \Lambda x = \Lambda(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots), \text{ for } \forall x \in L^m. \]
It is well-known that any $T \in \mathcal{L}^{m \times n}_{TV}$ can be represented by a lower triangular infinite dimensional matrix
\[ T = \begin{bmatrix}
T(0, 0) & 0 & 0 & \cdots \\
T(1, 0) & T(1, 1) & 0 & \cdots \\
T(2, 0) & T(2, 1) & T(2, 2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad (2) \]
where $T(i, j) \in \mathbb{R}^{m \times n}$ for all $i, j \in \mathbb{Z}_+$, $i \geq j$. Moreover, (1) defines a norm on $\mathcal{L}^{m \times n}_{TV}$ and
\[ \|T\| = \sup_{i \in \mathbb{Z}_+} \left\| \begin{bmatrix} T(i, 0) & T(i, 1) & \cdots & T(i, i) \end{bmatrix} \right\|_1. \quad (3) \]
In [16], the authors introduced the normed space $\mathcal{L}^{m \times n}_0$ whose elements, $G \in \mathcal{L}^{m \times n}_0$, can be represented by upper triangular infinite dimensional matrices
\[ G = \begin{bmatrix}
G(0, 0) & G(0, 1) & G(0, 2) & \cdots \\
0 & G(1, 1) & G(1, 2) & \cdots \\
0 & 0 & G(2, 2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \]
where $G(i, j) \in \mathbb{R}^{m \times n}$ for all $i, j \in \mathbb{Z}_+$ and $j \geq i$. Moreover, $\mathcal{L}^{m \times n}_0$ is equipped with a norm, $\|G\|_m$, where $[G]_i$ is the $i^{th}$ column of $G$. It was shown that $\mathcal{L}^{m \times n}_0$ is the pre-dual of $\mathcal{L}^{m \times n}_{TV}$ with pairing $\langle T, G \rangle := \text{Trace}(TG)$. Furthermore, $\|T\| = \sup_{i \in \mathbb{Z}_+} \|G\|_m$. 

**Definition 1:** An operator $T \in \mathcal{L}^{m \times n}_{TV}$ is said to be externally positive if for all $i, j \in \mathbb{Z}_+$, $i \geq j$, $T(i, j) \in \mathbb{R}^{m \times n}_+$, where $\mathbb{R}^{m \times n}_+$ is the closure of $\mathbb{R}^{m \times n}$ in standard topology. The set of such operators is denoted by $\mathcal{L}^{m \times n}_{TV+}$. In analogous way, we also define $\mathcal{L}^{m \times n+}_{TV}$.

We also use the notation $a \vee b := \max\{a, b\}$ for real numbers $a$ and $b$.

### III. Systems with Positive Inputs

In this section, we are interested in linear systems whose input is positive. More precisely, for $T \in \mathcal{L}^{m \times n}_{TV}$, define the functional (norm) $\|\cdot\|_+ : \mathcal{L}^{m \times n}_{TV} \rightarrow \mathbb{R}$ as
\[ \|T\|_+ = \sup_{u \neq 0} \frac{\|Tu\|_m}{\|u\|_m}. \quad (4) \]
Intuitively speaking, this functional (induced norm), similar to $l_1$ norm for LTI systems, gives the peak to peak ratio of the output to input when input is restricted to a positive cone. We are interested in characterizing this functional. To this end, we have the following:

**Proposition 2:** The space $(\mathcal{L}^{m \times n+}_{TV}, \|\cdot\|_+)$ is a normed space. Furthermore, $\|\cdot\|_+$ is dominated by $\|\cdot\|$. That is, $\|T\|_+ \leq \|T\|$, for all $T \in \mathcal{L}^{m \times n}_{TV}$.

**Proof:** The proof easily follows from the definition of norm.

For simplicity of presentation, we mainly focus on multi-input single-output systems. By doing so, we will not lose any generality for our purposes as any $T \in \mathcal{L}^{m \times n}_{TV}$ can be written as
\[ T = \begin{bmatrix}
T_1 \\
\vdots \\
T_m
\end{bmatrix}, \quad (5) \]
where $T_i \in \mathcal{L}^{1 \times n+}_{TV}$ for $i \in \{1, 2, \ldots, m\}$. Furthermore, it is straightforward to show that $\|T\| = \max_i \|T_i\|$, and $\|T\|_+ = \max_i \|T_i\|_+$. In what follows, one of our goals is to characterize this newly defined norm (4) and find an expression for it similar to (3). To this end, we notice that any $T \in \mathcal{L}^{m \times n}_{TV}$ can be decomposed uniquely as $T = T^+ - T^-$, where $T^+, T^- \in \mathcal{L}^{1 \times n+}_{TV}$ contain only positive and absolute value of negative entries of $T$. For example, consider $T \in \mathcal{L}^{1 \times m}_{TV}$ with matrix representation (2), then the entries of $T^+$ and $T^-$ are respectively given by $T^+(i, j) = T(i, j) \vee 0$ and $T^-(i, j) = -T(i, j) \vee 0$ for $i, j \geq 1$.

**Lemma 3:** Let $T \in \mathcal{L}^{m \times n}_{TV}$ which is decomposed as in (5). Then
\[ \|T\|_+ = \max \|T_i\|_+ = \max \max \left\{ \|T^+_i\|, \|T^-_i\| \right\}. \]

**Proof:** We will show this for SISO systems as the generalization to more general case is immediate. Notice that,
\[ \|T\|_+ = \sup_{t \in \mathbb{Z}_+} |y(t)|, \]
where $y = Tu$ and $u \in \mathcal{B}(l_\infty^+, 1)$. Clearly, $|y(t)| = |\sum_{k=0}^\infty T(t, k)u(k)|$ and since $0 \leq u(k) \leq 1$, it is easy to see that $|y(t)| \leq \max \{\sum_{k=0}^\infty |T^+(t, k)|, \sum_{k=0}^\infty |T^-(t, k)|\}$. Therefore,
\[ \|T\|_+ \leq \max \{\sup_{t \in \mathbb{Z}_+} |y(t)| \} \leq \max \{\sup_{t \in \mathbb{Z}_+} \sum_{k=0}^\infty |T^+(t, k)|, \sup_{t \in \mathbb{Z}_+} \sum_{k=0}^\infty |T^-(t, k)|\}, \quad (6) \]
and the result follows.

This lemma provides an exact expression for computation of $\|T\|_+$. Another expression for $\|T\|_+$ which fits our purposes in later sections is presented next.

**Proposition 4:** Let $T \in \mathcal{L}^{m \times n}_{TV}$. Then,
\[ \|T\|_+ = \sup_i \frac{1}{2} \left( \sum_{j=0}^{m} \left| t_k(i, j) \right| + \sum_{j=0}^{m} \left| t_k(i, j) \right| \right). \]
where $t_k(i, j)$ is the $k^{th}$ entry of row vector $T(i, j) = [t_1(i, j), t_2(i, j), \ldots, t_m(i, j)]$.

**Proof:** The proof follows similar to the proof of Theorem 5 and is omitted here.

In dealing with LTI systems, (6) can be simplified and linked to the usual $l_1$ ($l_\infty$ induced) norm of the system. Before presenting the results for LTI case, we need to recall the $\lambda$-transform for $T \in \mathcal{L}^{m \times n}_{TV}$ with impulse response $(T(k))_{k=0}^\infty$ is defined by $\hat{T}(\lambda) = \sum_{k=0}^\infty \lambda^k T(k)$, for $\lambda$’s such that the series converges. The following is one of the main results.
Theorem 5: Let $T \in \mathcal{L}_{II}^{m \times m}$ and decompose it as in (5) such that $T_i \in \mathcal{L}_{I}^{1 \times m}$. Then, 

$$\|T\|_+ = \max_i \frac{1}{2} \left[ \|T_i\| + \|\hat{T}_i(1)\| \right],$$

(7)

where the absolute value, $|\hat{T}_i(1)|$, is taken entry wise.

Proof: We will prove this theorem in a simpler case of SISO systems. By Lemma 3, we have $\|T\|_+ = \max \{\|T^+\|, \|T^-\|\}$. Without loss of generality, suppose $\|T^+\| \geq \|T^-\|$. That is,

$$\sum_{k=0}^{\infty} T^+(k) - T^-(k) \geq 0,$$

(8)

and consequently, $\|T\|_+ = \|T^+\|$. We will show this equals the right hand side of (7). To this end, one an rewrite (7) as

$$\frac{1}{2} \left[ \sum_{k=0}^{\infty} T^+(k) + T^-(k) + \sum_{k=0}^{\infty} T^+(k) - T^-(k) \right]$$

Notice that $T^+$ and $T^-$ is chosen such that $\min\{T^+(k), T^-(k)\} = 0$ for all $k \in \mathbb{Z}_+$. Therefore, $\|T^+(k) - T^-(k)\| = T^+(k) + T^-(k)$ and using (8), (9) simplifies to

$$\frac{1}{2} \left[ \sum_{k=0}^{\infty} (T^+(k) + T^-(k)) + \sum_{k=0}^{\infty} (T^+(k) - T^-(k)) \right]$$

$$= \sum_{k=0}^{\infty} T^+(k) = \|T^+\|,$$

which completes the proof.

IV. MODEL MATCHING PROBLEMS

In this section, we consider a generic model matching problem

$$\inf_Q \|H - UQV\|_+,\quad (10)$$

where $H$, $U$, and $V$ are stable LTI systems and show that this problem with the norm $\|\cdot\|_+$ is indeed convex and tractable. Moreover, we will show that time varying compensation, $Q \in \mathcal{L}_{TV}$, can not outperform time invariant compensation, $Q \in \mathcal{L}_{II}$. That is,

$$\inf_{Q \in \mathcal{L}_{II}^{m \times m}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \|H - UQV\|_+.$$

Let $H = \begin{bmatrix} H_1 & \vdots & U_1 \\ \vdots & \ddots & \vdots \\ H_m & \vdots & U_m \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}$, where $U_i, H_i \in \mathcal{L}_{II}^{1 \times n}$ for some integers $m$ and $n$. The following corollary is a direct consequence of Theorem 5:

Corollary 6: For the model matching problem (10), we have

$$\inf_{Q \in \mathcal{L}_{II}^{m \times m}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \max_{Q \in \{1, 2, \ldots, m\}} \frac{1}{2} \left[ \|H_i - U_iQV\| + \|\hat{H}_i(1) - \hat{U}_i(1) \hat{Q}(1) \hat{V}(1)\| \right].$$

(11)

Remark 7: Note that (11) is a linear programming problem and the optimal value can be found with arbitrary accuracy using methods in [14].

Next, we presents another important feature of the norm $\|\cdot\|_+$ which is similar to that of $L_1$ ($L_\infty$ induced) norm.

Theorem 8: Let $H, U,$ and $V$ be LTI systems. Then,

$$\inf_{Q \in \mathcal{L}_{II}^{m \times m}} \|H - UQV\|_+ = \inf_{Q \in \mathcal{L}_{TV}} \|H - UQV\|_+.$$

Proof: We start the proof by showing for any given stable $Q \in \mathcal{L}_{TV}$

$$\|H - UQV\|_+ \leq \|H - UQV\|_+,$$

which holds since

$$\|H - UQV\|_+ = \sup_{u \in \ell^m_{\infty}} \|\langle H - UQV\rangle u\|_{L_\infty}$$

$$\geq \sup_{u \in \ell^m_{\infty}, u \neq 0} \|\Lambda^k (H - UQV) \Lambda^k u\|_{L_\infty}$$

$$= \sup_{u \in \ell^m_{\infty}, u \neq 0} \|u\|_{L_\infty}$$

$$= \left\| \Lambda^{-k} (H - UQV) \Lambda^k \right\|_+,$$

as $H, U$, and $V$ are LTI and commute with the delay operator. Now, define

$$Q_N = \frac{1}{N} \sum_{k=0}^{N-1} \Lambda^{-k}Q\Lambda^k.$$ 

Using triangle inequality, it follows that for any $N \in \mathbb{Z}_+$,

$$\|H - UQ_NV\|_+ \leq \|H - UQV\|_+.$$ 

It is argued in [16], [17], and [18] that $(Q_N)_{N=0}^\infty$ has a weak * convergent subsequence, denote it by $(Q_{N_k})_{k=0}^\infty$. That is,

$$Q_{N_k} \rightharpoonup Q_{LT1},$$

where $Q_{LT1} \in \mathcal{L}_{II}^{m \times m}$ is stable. Obviously, for any $X \in \mathcal{L}_{II}^{m \times m}$ with $\|X\|_{\mathcal{L}_{II}} \leq 1$ it holds that

$$\langle H - UQ_{N_k}V, X \rangle \to \langle H - UQ_{LT1}V, X \rangle.$$ 

It can be easily verified that

$$\|H - UQ_{LT1}V\|_+ = \sup_{X \in \mathcal{L}_{II}^{m \times m}} \langle H - UQ_{LT1}V, X \rangle,$$

which from lower-semi continuity of sup implies

$$\|H - UQ_{LT1}V\|_+ \leq \|H - UQ_{N_k}V\|_+ \leq \|H - UQV\|_+,$$

(12)

and this completes the proof.

Remark 9: Similarly, as in [19], one can show that nonlinear smooth $Q$ cannot outperform linear $Q.$
V. MIXED SIGNALS

In the previous section, we focused on the $l_\infty$ gain of the output when the input is restricted to the positive cone $l_\infty^+$. In this section, we consider a more general case when only part of the input is positive, i.e., $u \in l_{\infty+}^1 \times l_{\infty+}^2$. To motivate this problem, we give the following example related to filtering:

Consider the problem depicted in Figure 1, where $s \in \mathcal{B}(l_\infty^1, 1)$ is the input to the plant $P$ and $n \in \mathcal{B}(l_\infty^1, b)$, for some $b \geq 0$, is the measurement noise. The interest is to design a filter $Q$ such that the difference between the input signal, $s$, and its estimate $\hat{s}$ is minimized in the $l_\infty$ sense. That is, the problem amounts to

$$\inf_{Q \in \mathcal{B}(l_\infty^1, 1)} \sup_{n \in \mathcal{B}(l_\infty^1, b)} \left\| \begin{bmatrix} I - QP & -Q \end{bmatrix} \begin{bmatrix} s \\ n \end{bmatrix} \right\|_{l_\infty},$$

or equivalently

$$\inf_{Q \in \mathcal{B}(l_\infty^1, 1)} \sup_{n \in \mathcal{B}(l_\infty^1, 1)} \left\| \begin{bmatrix} I - QP & -Q \end{bmatrix} \begin{bmatrix} s \\ n \end{bmatrix} \right\|_{l_\infty}.$$

The next theorem deals with such problems in a more general case. The proof immediately follows from the definition of the norm and is omitted here.

**Theorem 10:** Let $H_1 \in \mathcal{L}_{II}^{1 \times m_1}$ and $H_2 \in \mathcal{L}_{II}^{1 \times m_2}$, for some positive integers $m_1$ and $m_2$. Then,

$$\sup_{u_1 \in l_{\infty+}^1, u_2 \neq 0} \left\| \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|_{l_\infty} = \sup_{u_1 \in \mathcal{B}(l_\infty^1, 1), u_1 \neq 0} \left\| \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\|_{l_\infty} = \|H_1\|_{l_\infty} + \|H_2\|_{l_\infty}.$$

Using the above theorem, we have:

**Corollary 11:** For the filtering problem depicted in Figure 1, it holds that

$$\inf_{Q \in \mathcal{B}(l_\infty^1, 1)} \sup_{n \in \mathcal{B}(l_\infty^1, 1)} \|s - \hat{s}\|_{l_\infty} = \inf_{Q \in \mathcal{L}_{II}} b\|Q\| + \frac{1}{2} \left[ \|I - QP\| + \|I - Q(1)P(1)\| \right].$$

It should be noted that, as before, it can be similarly argued that nonlinear smooth $Q$ offers no advantage over LTI $Q$. However, if non-smooth $Q$’s are allowed, there is a possibility of improving performance, e.g. see [20] and [21] using the invariant set methods. Note for example, any LTI solution $Q$ obtained by our methods can be used to generate a simple non-smooth (thresholding) estimator $Q_N = Y \circ Q$ where

$$\text{for } x \in l_{\infty}, (Yx)(k) = \begin{cases} x(k), & \text{if } x(k) \geq 0 \\ 0, & \text{if } x(k) < 0 \end{cases}.$$

Such a $Q_N$ keeps the LTI estimate $\hat{s}(k)$ if it is non-negative and sets it to zero if negative. Clearly, it does not perform worse than $Q$ in $\|\cdot\|_{l_\infty}$ sense and, if anything, it may outperform $Q$ since it always provides a positive estimate. Threshholding such LTI filters is a subject of our current research.

VI. POSITIVE SYSTEMS

In this section, we focus on positive systems and present some observations and results regarding the control synthesis for such systems. Clearly, an operator is externally positive (Definition 1) if and only if the output to any input in the positive cone belongs to the positive cone when the initial condition is set to zero. One can also think in terms of state-space realization of $T$,

$$T : \begin{cases} x(t + 1) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases},$$

where $x$, $u$, and $y$ are state, input and output, respectively; and $A$, $B$, $C$, and $D$ are matrices of appropriate dimensions.

**Definition 12:** An operator $T$ with state-space realization of the form (13) is said to be internally positive if the matrices $A$, $B$, $C$, and $D$ have non-negative entries.

An operator $T$ is internally positive if and only if the output and the states are non-negative whenever the input and the initial condition are non-negative. Obviously, internal positivity implies external positivity but the converse is not true, in general. Our first remark, in this section, is that designing a stabilizing controller such that the closed loop system is externally positive can be cast as a convex optimization. More precisely, consider a general control problem where $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} : \begin{bmatrix} w \\ u \end{bmatrix} \rightarrow \begin{bmatrix} z \\ y \end{bmatrix}$ is the generalized plant; $w$ and $u$ are the exogenous and control input; $z$ and $y$ are the regulated and measured output, respectively. The problem of interest is to find a controller $K : y \rightarrow u$ that stabilizes the plant, minimizes the effects of $w$ on $z$, and makes the map from $w$ to $z$ externally positive. Such a problem can be converted to the following linear program:

$$\mu := \inf_{Q \text{\; stable}} \|H - UQV\|,$$

for some stable $H$, $U$, and $V$ [22], subject to

$$H - UQV \geq 0,$$

where the inequality in (14) is taken component-wise on the impulse response of $H - UQV$ or its lower triangular representation. Although it is an infinite dimensional optimization, one can argue for any $\varepsilon > 0$, there exists a positive integer $k$ such that

$$\mu \leq \min_{Q \in (Q(0))_{l_{\infty}}^+} \|P_k(H - UQV)\| < \mu + \varepsilon,$$

$$P_k(H - UQV) \geq 0,$$
and
\[(H - U\tilde{Q}V) - P_t (H - UQ^tV) > -\varepsilon,\]
where $Q^* = (Q^* (t))_{t=0}^{k-1}$ is the optimal solution of (15) and
\[\tilde{Q} := (Q^* (0), ..., Q^* (k-1), 0, 0, ...).\]
For problems of this sort, we refer to [14] and [23].

In state-space, there is a simple way to calculate the $l_1$

norm ($l_1$ induced norm) of an LTI system, $G$, with state-space matrices $(A, B, C, D).$ As reported in [13], one has
\[\|G\| = \|C(I-A)^{-1}B1 + D1\|_{l_1},\]
where $1$ is a column vector of compatible dimension with all entries equal to one.

**Lemma 13:** [13, continuous-time counterpart] If $G$ is externally positive then $\|G\| < \gamma$ for some $\gamma > 0$ if and only if there exists $\nu \in \mathbb{R}_+^n$ such that
\[A\nu + B1\nu < \nu, CV + D1\nu < \gamma\nu,\]
where $n_\nu, n_\gamma,$ and $n$ are the number of inputs, outputs, and states, respectively.

Let
\[G = \begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} \end{pmatrix}, \quad K = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix},\]
(16)
then, the map $T$ from $w$ to $z$ if given by
\[T(G, K) = \begin{pmatrix} A + B_2 D_{12} C_2 & B_2 C_2 \\ B_2 C_2 & A_k \\ C_1 + D_{12} D_{12} C_2 & D_{12} C_k \end{pmatrix} \begin{pmatrix} B_1 + B_3 D_{21} \\ B_1 D_{21} \end{pmatrix}, \]
(17)
Now, we present a new result regarding the optimal control synthesis for such systems. The next theorem addresses a problem which was previously reported as an open problem in [24].

**Theorem 14:** For $\gamma > 0$, if there exists a controller (16) of order $n_\nu$ such that the closed loop system (17) is internally positive, stable, and has $l_1$ norm less than $\gamma (\|T(G, K)\| < \gamma),$ then there exists a static controller $\bar{K}$ such that $T(G, \bar{K})$ is also positive, internally stable, and $\|T(G, \bar{K})\| < \gamma.$

**Proof:** Suppose a controller $K$ with state-space matrices as in (16) yields to a positive closed loop system $T(G, K)$ with $\|T(G, K)\| < \gamma.$ The result follows by direct calculations of showing $T(G, \bar{K})$ has the desired properties where
\[\bar{K} = \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix}.\]
Indeed, since $\|T(G, K)\| < \gamma,$ by Lemma 13, there should exists $\nu_1, \nu_2 \in \mathbb{R}_+^n$ such that
\[\begin{pmatrix} A + B_2 D_{12} C_2 & B_2 C_2 \\ B_2 C_2 & A_k \\ C_1 + D_{12} D_{12} C_2 & D_{12} C_k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} < \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix},\]
(18)
and
\[\begin{pmatrix} B_1 + B_3 D_{21} \\ B_1 D_{21} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} > 0,\]
(19)
Since the closed loop (more precisely $B_2 C_k$ and $D_{12} C_k$) and $\nu_2$ are non-negative, from the top inequalities in (18) and (19) we have
\[(A + B_2 D_{12} C_2) \nu_1 + (B_1 + B_3 D_{21}) \nu_1 \nu_2 < \nu_1,\]
\[(C_1 + D_{12} D_{12} C_2) \nu_1 + (D_{11} + D_{12} D_{21}) \nu_1 < \gamma \nu_1,\]
which are necessary and sufficient conditions for $\|T(G, \bar{K})\| < \gamma,$ by Lemma 13.

Finding a static controller $K \in \mathbb{R}^{n_u \times n_y}$ where $n_u$ and $n_y$ are the number of control inputs and measured outputs such that $\|T(G, K)\| < \gamma$ can be written as the following bilinear program
\[\exists K \in \mathbb{R}^{n_u \times n_y}, \nu \in \mathbb{R}_+^n : \]
\[\begin{pmatrix} A & B_1 \\ C_1 & D_1 \end{pmatrix} + \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} K \begin{pmatrix} C_2 & D_2 \end{pmatrix} \geq 0,\]
\[\begin{pmatrix} A & B_1 \\ C_1 & D_1 \end{pmatrix} + \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} K \begin{pmatrix} C_2 & D_2 \end{pmatrix} \begin{pmatrix} \nu \nu_1 \end{pmatrix} \leq \begin{pmatrix} \nu \gamma \nu_1 \end{pmatrix},\]
(20)
We comment here that (20) can be reduced to linear programming in some special cases [13]. For example, in the case of state feedback when $C_2 = I, D_{12} = 0,$ and $D_{21} = 0$ defining new variable $\mu = K \Pi,$ where $\Pi = \text{diag} (v_1, v_2, ..., v_n)$ and $v_i$ is the $i^{th}$ component of $\nu,$ reduces the problem to the existence of diagonal $\Pi \in \mathbb{R}^{n \times n} \text{ and } \mu \in \mathbb{R}^{n_u \times n}$ such that
\[\begin{pmatrix} A & B_1 \\ C_1 & D_1 \end{pmatrix} \Pi + \begin{pmatrix} 2 \nu_1 \\ 0 \end{pmatrix} \mu \geq 0,\]
\[A B_1 C_1 D_1 \begin{pmatrix} \nu \nu_1 \end{pmatrix} + \begin{pmatrix} 2 \nu_1 \\ 0 \end{pmatrix} \sum_{i=1}^n \nu_i \mu_i \leq \begin{pmatrix} \nu \gamma \nu_1 \end{pmatrix},\]
where $\nu_i \mu_i$ is the $i^{th}$ column of $\mu.$ As another special case, one can think of generalizing the same idea to the case when only part of the state is measured but it is possibly noisy, that is $C_2 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $D_{21} \neq 0$ but diagonal.

Finally, we would like to remark that results similar to Theorem 14 can be shown for some other performance measures. For instance, the next theorem deals with the case when the performance is measured in $L_2$ induced sense.

Before stating the final theorem, we need the following two lemmas.

**Lemma 15:** Let $G$ and $H$ be non-negative matrices with $0 \leq G \leq H.$ Then, $\tilde{G} (G) \leq \tilde{H} (H),$ where $\tilde{G} (.)$ denotes the maximum singular value.

**Proof:** Notice that $0 \leq G^* \leq H^*,$ where $*$ denotes the matrix transpose. Therefore, $0 \leq G^* G \leq H^* H,$ [25, Lemma 3], and hence $\rho (G^* G) \leq \rho (H^* H),$ [26], where $\rho (.)$ is the spectral radius.

**Lemma 16:** Given an internally positive $G$ with state-space matrices $(A, B, C, D), \| G \|_{l_2 \text{- ind}} < \gamma$ for some $\gamma > 0$ if and only if there exists a positive matrix $Z$ of compatible dimension such that
\[AZ + B < Z,\]
\[Z^2 C^* + D^* \begin{pmatrix} C Z + D \gamma^2 I \end{pmatrix}\]
is positive definite. (22)
Proof: Notice that since $G$ is internally positive, $\|G\|_{l^2-ind} < \gamma$ if and only if $\|\hat{G}(1)\|_{l^2-ind} = \bar{\sigma}(\hat{G}(1)) < \gamma$, where $\hat{G}(1)$ is the dc gain of $G$. That is, $\|G\|_{l^2-ind} < \gamma$ if and only if

$$\hat{\sigma}[C((I-A)^{-1}B + D)] < \gamma.$$  \(\text{(23)}\)

First, suppose (23) holds. Since $A$ is non-negative and stable, $(I-A)^{-1}$ is non-negative as well. Therefore, for any positive matrix $X, Y := (I-A)^{-1}X \geq 0$. Moreover, one can choose $X > 0$ such that $Y > 0$. Now, since (23) is strict inequality, there exists $\varepsilon > 0$ such that

$$\hat{\sigma}[C((I-A)^{-1}B + \varepsilon Y + D)] < \gamma.$$  \(\text{II.6}\)

Let $Z := (I-A)^{-1}B + \varepsilon Y$. Then, $(I-A)Z - B = \varepsilon Y > 0$, and $\hat{\sigma}[CZ + D] < \gamma$ which are equivalent to (21) and (22), respectively.

Conversely, suppose (21) and (22) hold. Notice that (21) implies $A$ is Schur stable and $(I-A)^{-1}B < Z$. Therefore, $C((I-A)^{-1}B + D < CZ + D$. By Lemma 15, this implies $\hat{\sigma}[C((I-A)^{-1}B + D)] < \hat{\sigma}[CZ + D]$. Furthermore, (22), invoking Schur complement type of argument, implies $\hat{\sigma}[CZ + D] < \gamma$ which completes the proof of the converse.

Remark 17: Similarly, one can show $\|G\|_{l^2-ind} < \gamma$ for some $\gamma > 0$ if and only if there exists a positive matrix $Z$ of compatible dimension such that $ZA + C < Z$ and

$$ZB + D > \gamma \bar{\sigma}.$$  \(\text{II.6}\)

Theorem 18: Let $\gamma > 0$. Suppose in the generalized plant (16) $D_{12} = 0$ or $D_{21} = 0$. If there exists a dynamic controller (16) such that the closed loop system (17) is internally positive, stable, and has $l_2$ induced norm less than $\gamma$ (\(\|T(G,K)\|_{l^2-ind} < \gamma\)), then there exists a static controller $K$ such that $T(G,K)$ is also positive, internally stable, and $\|T(G,K)\|_{l^2-ind} < \gamma$.

Proof: Using the results of the last lemma, the proof follows similarly to that of Theorem 14.  \(\text{II.6}\)

VII. CONCLUSION

In this paper, we presented an exact characterization of the induced norm from $l^+_\infty$ to $l_\infty$. As an application, a filtering problem was studied. Furthermore, based on this development, we considered the model matching problem to show that i) time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems and ii) the optimization is a linear programming problem.

In the second part of the paper and in the context of $l_\infty$ optimization, we considered the positive systems (internal and external). We showed, if external positivity constraint is imposed on the closed loop map, finding an optimal controller is a linear programming and hence a tractable problem. If, on the other hand, internal positivity is sought, we show that a dynamic controller offers no advantage over a static one.