Unfalsified Adaptive Control with MPC Candidates

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Abstract— In this paper, we consider unfalsified adaptive control with candidate controllers that are in the structure of model predictive control (MPC) and are associated with parametrized models. Thus, each candidate controller corresponds to a parameter of the models and a switching corresponds to an update of the parameter selection. We introduce a new switching algorithm in order to pursue a performance criterion imposed on an output signal of an underlying system as well as identification of the system. This switching algorithm achieves an asymptotic boundedness of the performance criterion while the parameter update may be performed endlessly. At the beginning of an experiment, the switchings can be set to be performed mainly focused on identification of the system. Thus, we implement an input design scheme in the MPC structure for a better transient behavior of the closed-loop system via fast system identification.

I. INTRODUCTION

Unfalsified adaptive control [1] has been established in the various forms of switching controllers, e.g. [2], that, for a given uncertain system, search for a stabilizing controller. These control schemes guarantee that if there is at least one such controller in a given candidate controller set, one of them is selected and remains connected to the system after a finite number of switchings and, hence, stability of the closed-loop system is unfalsified. The stabilizing properties of the candidate controllers are measured by a cost function and some other performance criteria can also be implemented in the cost function. Unfalsified adaptive control is especially useful in the case where there is a severely uncertain component in the closed-loop system, such as model mismatch and a disturbance signal. On the other hand, as uncertainty decreases, the merit of unfalsified adaptive control fades away and many model-based control schemes stand out, e.g. model predictive control [3].

In this paper, we consider the situation where much prior knowledge of a given system is available but it is suspected that the system has slowly changed over time or has abruptly changed due to a fault. Models of the system are parametrized around a finite number of potential operating points and are available beforehand. The set of parameters may be continuous and contain various model structures. And, we place an output-feedback MPC controller with these parametrized models in the framework of unfalsified adaptive control as candidate controllers. Thus, a switching implies an update of a parameter used in the MPC and can also be called adaptive MPC. For many decades, adaptive MPC has been studied in many fields and a recent work can be found, for example, in [4]. The biggest difference of this paper is that, unlike other adaptive MPC designs, we employ the unfalsified adaptive control scheme for the parameter update in the case where many informations on the given system are not available such as a bound on a disturbance signal and whether the system is in a given model set.

A unique feature of MPC is that all the performance criteria on input and output of a system are explicitly implemented in it. For example, if an input signal of the system is desired to be bounded in amplitude, we can guarantee it by putting it as a hard constraint in an optimization problem inside an MPC controller. By doing this, we may operate the system at an operating point with a small signal and approximate it by a linear model of the system with an expectation that the MPC controller with the linear model will work efficiently. Thus, for the closed-loop stability, which is often defined as bounded-input bounded-output (BIBO) stability from external input signals to any produced signal in the loop, the BIBO relationship between external signals to the system input signal is practically not necessary. And, only the BIBO relationship between external signals to the output signal of the system needs to be guaranteed. This requirement combined with a performance criterion on the system-output signal, is the main focus of unfalsified adaptive control in this paper.

In the case that the models are continuously parametrized, the number of the candidate controllers is infinite. Although the switching control in [5] guarantees a finite number of switchings and unfalsification of closed-loop stability, our interest lies on an unlimited number of switchings to search for a potentially better model of the system. Thus, we first modify the switching algorithm in [6] to accommodate an MPC structure associated with infinite number of models and, then, employ this modified switching algorithm with a hope that a more accurate model can be found. This switching algorithm uses two cost functions. The first one measures the BIBO relationship between external signals to the output signal of the system and some other performance criteria such as reference tracking. The other cost function measures model mismatch between the models and the system. By guaranteeing a finite number of switchings via the first cost function, desired performance criteria are satisfied conservatively. And, we keep pursuing a better model with a possibly endless switchings via the second cost function.

In addition, the switching algorithm in this paper allows the cost function of model mismatch to be more active than the other cost function at the beginning of experiments. Thus, early switchings are focused on identifying the current candidates, to identify the current model mismatch.
operating point and, hence, faster identification of the current operating point leads to a better transient behavior of the closed-loop system. In order to achieve fast identification, we implement an input design scheme in the MPC.

Input design problems for the purpose of system identification has been studied in many aspects, e.g. [7], [8], and [9]. And, the aspect of the input design scheme in this paper is to maximize the differences between operating points measured by the cost function of model mismatch, which is called model discrimination. By discriminating operating points, we have a higher possibility of finding the current operating point and searching the vicinity of the current operating point for a better model. We find a sufficient condition on the input signal of the system that leads to model discrimination at an early stage of experiments. This condition is implemented in the MPC as an additional constraint. This additional constraint makes the optimization problem in the MPC a nonconvex optimization problem and, thus, a suboptimal solution is sought for a fast computation via semidefinite relaxation techniques followed by a random search algorithm, e.g. [10] and [11].

II. ADAPTIVE MODEL PREDICTIVE CONTROL

The norms \( \| \cdot \|_2 \) and \( \| \cdot \| \) are the Euclidean norm and the \( \ell_2 \)-norm, respectively. Denote by \( \ell^m_n \) the \( \ell_2 \) space of m-dimensional functions of time, i.e. \( \ell^m_n = \{s : \{0, 1, \cdots \} \rightarrow \mathbb{R}^m | \|s\| < \infty \} \). Define a truncated version of the \( \ell_2 \)-norm \( \|s\|_{\ell_2(t)} \triangleq \sum_{\tau=t}^{t_f} s(\tau)^2 d\tau \) for any function \( s \) of time and \( t_f \geq t_s \geq 0 \) and denote the extended space of \( \ell^m_n \) by \( \ell^m_{2e} = \{s : \{0, 1, \cdots \} \rightarrow \mathbb{R}^m | \|s\|_{\ell_2(t)} < \infty \forall t \in \{0, 1, \cdots \} \} \). The order \( m \) is omitted when there is no confusion.

Definition 1: (Stability) A mapping (or a system) \( Q \) is said to be stable if there exist constants \( \alpha_q \) and \( \beta_q \) such that, for any given input signal \( s \in \ell_{2e} \),

\[
\|Q(s)\|_{\ell_2(T)} \leq \alpha_q \|s\|_{\ell_2(T)} + \beta_q \forall t \in \{0, 1, \cdots \}
\]

(1)

where \( Q(s) \) is the output signal of \( Q \) corresponding to the input signal \( s \). Otherwise, \( Q \) is said to be unstable.

Definition 2: (Unfalsified stability) For a given mapping \( Q \) and a pair of an input signal \( s \) and its corresponding output signal \( Q(s) \), stability of \( Q \) is said to be unfalsified by the pair if there exist constants \( \alpha_q \) and \( \beta_q \) such that the inequality in (1) holds. Otherwise, stability of \( Q \) is said to be falsified by the pair.

An adaptive control system in Fig. 1 is considered a mapping whose input signals are a known reference signal \( w = [r' \ v'] \in \ell_{2e} \) and an unknown disturbance signal \( d \in \ell_{2e} \) and whose output signal is an observed signal \( z = [u' \ y'] \) where \( u \in \ell_{2e} \) and \( y \in \ell_{2e} \) are an input signal and an output signal of an uncertain system \( P \), respectively. In most cases, a zero signal is assigned to \( v \) and the role of the signal \( v \) is explained later. The system \( P : \ell_{2e} \times \ell_{2e} \rightarrow \ell_{2e} \) is an uncertain mapping from \( u \) and \( d \) to \( y \). With an unknown initial condition \( x_0 \) of \( P \) at time 0 and a disturbance signal \( d \), the system \( P \) at time 0 can be represented by a mapping \( P|_{x_0,d} \) that assigns to each input signal the corresponding output signal. Then, the input-output relationship of \( P \) for input signals in \( \mathbb{U}(u, \pi) = \{s_u \in \ell_{2e} \ | \ u \leq s_u(t) \leq \pi \forall t \in \{0, 1, \cdots \}\} \) with given constants \( u \) and \( \pi \) can be expressed by \( \mathbb{Z}(\theta|_{x_0,d}, \mathbb{U}(u, \pi)) = \{[s_u' \ s_y'] \ | \ s_u \in \mathbb{U}(u, \pi), s_y = P|_{x_0,d}(s_u)\} \) whose element is one possible experimental data over a time interval \( \{0, 1, \cdots \} \). The behavior of \( P \) corresponding to input signals in \( \mathbb{U}(u, \pi) \) is of interest since it is guaranteed that \( u \in \mathbb{U}(u, \pi) \) by candidate controllers that are introduced in the following.

We consider models \( P_0 \)'s of \( P \), parametrized in terms of a parameter \( \theta \). The parametrization can be performed over different model structures and the details are omitted for simplicity. Then, controllers are developed via an MPC design scheme combined with the models \( P_0 \)'s. Specifically, the output signal \( y_0 \) of a controller, denoted by \( C_0 \), associated with a model \( P_0 \) is given as

\[
y_0(t) = u^*_f(\theta, t)(0) \forall t \in \{0, 1, \cdots\}
\]

(2)

where \( u^*_f(\theta, t)(0) \) is a solution to the optimization problem

\[
\min_{u_f(\theta) \in \ell_{2e}} \|y_f(\theta) - r_f\|_{\ell_2(0,T)} + \|\Delta u_f(\theta)\|_{\ell_2(0,T-1)}
\]

s.t.

\[
y_f(\theta) \leq \overline{y}_f(\theta) \forall k \in \{0, 1, \cdots, T\}
\]

(3)

\[
y_f(\theta) \leq \underline{y}_f(\theta) \forall k \in \{0, 1, \cdots, T-1\}
\]

(4)

\[
|\Delta u_f(\theta)| \leq \overline{\Delta} u_f(\theta) \forall k \in \{0, 1, \cdots, T-1\}
\]

(5)

with given constants \( \overline{y}_f, \underline{y}_f, \), and \( \overline{\Delta} u_f \). A positive integer \( T \) represents a prediction horizon and \( r_f \in \ell_{2e} \) and \( \Delta u_f \in \ell_{2e} \) are defined by \( r_f(k) = r(t + k) \) for \( k \in \{0, 1, \cdots\} \) and \( \Delta u_f(k) = \{u_f(\theta)(k) - u(\theta)(t+k-1)\} \) respectively, with a given value \( u(-1) \). In the optimization problem above, the constraint in (3) is treated as a soft constraint but the constraints in (4) and (5) are treated as hard constraints, which guarantees \( u \in \mathbb{U}(u, \pi) \). The controller corresponding to \( \theta \) employs the model \( P_0 \) for prediction and a mapping \( P_0|_{\tilde{x}_\theta(t),0} \) represents the input-output relationship of \( P_0 \) with an initial condition \( \tilde{x}_\theta(t) \), which is obtained in the following, and a zero disturbance signal.

The initial condition \( \tilde{x}_\theta(t) \) of \( P_0|_{\tilde{x}_\theta(t),0} \) represents an estimate of the state of the model \( P_0 \) at time \( t \) regarding observed data \( u(0), \cdots, u(t-1) \) and \( y(0), \cdots, y(t-1) \) up until time \( t \) as past input-output data of \( P_0 \). Since there is no data up until time \( t = 0 \), a zero state is assigned to \( \tilde{x}_\theta(0) \).
At a time \( t \geq 1 \), with collected input-output data of \( P[x_0, d] \) in the adaptive control system in Fig. 1 up until time \( t \), the model \( P_\theta \) is fitted to the data and the model’s mismatch to the data is defined by

\[
V_M(\theta, z, t) = \min_{\tilde{x}_\theta, d_\theta} \| \tilde{d}_\theta \|_{[0, t-1]}
\]

s.t. \( y = P_\theta |_{\tilde{x}_\theta, d_\theta}(u) \)

with a constant \( \rho_x \) where \( z = [u' \ y'] \). Denote by \( \tilde{x}_\theta(t) \) and \( d_\theta^*(t) \) a solution to this optimization problem. This solution can be interpreted as a fictitious state of \( P_\theta \) and a fictitious disturbance signal for the model. In other words, the mapping \( P_\theta |_{\tilde{x}_\theta^*(t), d_\theta^*(t)} \) combined with the input signal \( u(0), \ldots, u(t-1) \) exactly reproduces the output signal \( y(t) \), \( \ldots, y(t-1) \). Then, the state of \( P_\theta \) at time \( t \) is determined by \( P_\theta |_{\tilde{x}_\theta^*(t), d_\theta^*(t)} \) and \( u \) and is assigned to the estimate \( \hat{x}_\theta(t) \).

If \( P_\theta \) is a discrete time causal linear time-invariant system, it can be described as

\[
y(t) = G_\theta u(t) + \Psi_\theta \hat{x}_\theta + H_\theta \tilde{d}_\theta(t), \quad t = 0, 1, \ldots \tag{7}
\]

with appropriate matrices \( G_\theta, \Psi_\theta, \) and \( H_\theta \) where \( y(t) = [y(0)' \ y(t)'] \), \( u(t) = [u(0)' \ u(t)'] \), and \( \tilde{d}_\theta(t) = [\tilde{d}_\theta(0)' \ \ldots \ \tilde{d}_\theta(t-1)'] \). Due to the causality, the matrices \( G_\theta \) and \( H_\theta \) are lower triangular and we assume that \( H_\theta \) is invertible. Then, it can be shown that a vector

\[
\begin{bmatrix}
\tilde{x}_\theta(t)

\tilde{d}_\theta(t)
\end{bmatrix} = \begin{bmatrix}
I - (H_\theta^{-1}(\Psi_\theta) + H_\theta^{-1})

\begin{bmatrix}
H_\theta^{-1}(\Psi_\theta) + H_\theta^{-1}

G_\theta
\end{bmatrix}
\end{bmatrix}
\times \begin{bmatrix}
y(t) - G_\theta u(t)
\end{bmatrix}
\tag{8}
\]

where + means the Moore-Penrose pseudo inverse, is an optimal solution to the optimization problem in (6) for a sufficiently large \( \rho_x \). For simplicity, we can fix \( \tilde{x}_\theta(t) = \mathbf{0} \) in the optimization problem in (6) and solve it only for \( \tilde{d}_\theta^*(t) \). Then, the solution in (8) reduces to \( \tilde{d}_\theta^*(t) = H_\theta^{-1}(y(t) - G_\theta u(t)) \) and \( \tilde{d}_\theta^* \) can be generated as a filtered signal of \( u \) and \( y \) [12].

Overall, a controller \( C_\theta \) described in (2) is considered a mapping \( \ell_{2x} \times \ell_{2e} \mapsto \ell_{2e} \) that assigns to each pair of two controller-input signals \( r \) and \( y \) the corresponding controller-output signal \( c_\theta \) and the input-output relationship of \( C_\theta \) can be expressed by

\[
\mathcal{Z}(C_\theta) = \left\{ [s_r' \ \ s_y' \ \ s_c'] | s_r, s_y, s_c \in \ell_{2e}, s_c = C_\theta(s_r, s_y) \right\}.
\]

Among all the parameters, we consider \( N \) number of selective parameters \( \Theta_n, n = 1, \ldots, N \), and parameter set \( \Theta_n, n = 1, \ldots, N \), for their vicinities, i.e. \( \theta_n \in \Theta_n, n = 1, \ldots, N \). These parameters may be chosen for different operating points of the system \( P \) and these choices are made based on any prior knowledge of the system \( P \). It is assumed that \( P_{\theta_n}, n = 1, \ldots, N \), are distinct in terms of input-output relationships. Then, we define a candidate controller set \( \mathcal{C} \) by \( \mathcal{C} = \{ C_{\theta_n}, n \in \{ 1, \ldots, N \} \} \).

A switching algorithm selects a parameter at each selecting time, with a help of the selective candidate controllers \( C_{\theta_n}, n = 1, \ldots, N \), and, then, keeps its corresponding controller-output signal delivered to the loop of the adaptive control system until the next selecting time. Denote by \( \hat{\theta}(t) \) the parameter whose corresponding controller is connected in the loop of the adaptive control system at time \( t \).

When a candidate controller \( C_{\theta^*} \) is selected by the switching algorithm, the input signal \( u \) of \( P \) is given by the output signal of \( C_{\theta^*} \) until the next selecting time. Thus, the output signal of \( C_{\theta^*} \) satisfies

\[ c_\theta(t) = u(t) - u(t) \tag{9} \]

for any time \( t \geq 0 \) satisfying \( \hat{\theta}(t) = \theta \).

### III. A Switching Algorithm

For the purpose of switchings, candidate controllers are assessed at each time \( t \in \{0, 1, \cdots \} \) and a cost function \( V_C(\theta, r, z, t) \) represents the assessment of a candidate controller \( C_\theta \). An example of \( V_C(\theta, r, z, t) \) is provided after fictitious reference signals are introduced in the following.

**Definition 3:** (Fictitious reference signal) Given the system \( P \) with an initial condition \( x_0 \), the disturbance signal \( d \), the candidate controllers \( C_\theta \)'s, and the input-output signal \( z = [u' \ y'] \) of \( P \) in Section II, a signal is said to be a fictitious reference signal for \( C_\theta \) if injection of this signal, as a reference signal, into a closed-loop system \( (P[x_0, d], C_\theta) \) would have produced the same input-output signal \( z \).

**Definition 3** leads us to a set \( \mathcal{W}(\theta, z) = \{ [s_r' \ s_y' \ s_c'] | s_r \in \mathcal{Z}(C_\theta) \} \) of the fictitious reference signals for \( C_\theta \) [13], which implies existence and non-unicity of fictitious reference signals. Alternative types of fictitious reference signals are introduced in [2], [14], and [15] under various assumptions on candidate controllers. Among all the fictitious reference signals, one of the simple yet efficient fictitious reference signals is

\[ \bar{w}_\theta = \begin{bmatrix} r \\ u - c_\theta \end{bmatrix} \in \mathcal{W}(\theta, z) \tag{10} \]

where \( c_\theta \) is the computed output signal of \( C_\theta \) in (2). It is clear, from (9) and (10), that \( \bar{w}_\theta(t) = w(t) \) for any \( t \in \{0, 1, \cdots \} \) satisfying \( \bar{\theta}(t) = \theta \).

As already reflected in the MPC design in (2), our main interest lies on making \( y \) to track \( r \). Thus, it is natural to assess candidate controllers using a cost function

\[ V_C(\theta, r, z, t) = \max_{r \in [0, t]} \| y - r \|_{[0, r]} + \rho \tag{11} \]

with a positive constant \( \rho \) and a particular choice of a fictitious reference signal \( \bar{w}_\theta \) in (10). This function reflects the quality of the tracking performance of \( C_\theta \), based on the data from a thought experiment on a closed-loop system \( (P[x_0, d], C_\theta) \). Clearly, if all the possible fictitious reference signals are considered in \( V_C(\theta, r, z, t) \) in (11), it will bring a better assessment of \( C_\theta \) at the expense of computation speed.

Note that \( V_C(\theta, r, z, t) \) in (11) is not an \( \ell_{2e} \)-gain-related cost function ([2] and [6]) since the input signal \( u \) of \( P \) is not included in it. Thus, the boundedness of \( V_C(\theta, r, z, t) \) does not necessarily imply unfalsified stability of the closed-loop system \( (P[x_0, d], C_\theta) \). However, the signal \( u \) is guaranteed, by (4), to be always bounded and our primary interest lies on the tracking performance of candidate controllers.

**Definition 4:** (Feasibility) Given the system \( P \) with its initial condition \( x_0 \) and the disturbance signal \( d \) in the adaptive control system in Section II, together with a cost function \( V_C(\theta, r, z, t) \) in (11), a controller \( C_{\theta^*} \) is said to
be a feasible controller if there exists a constant \( \alpha_f \) such that, for any given input-output signal \( z = [u', y'] \in Z(P)_{x_0,d}, \mathbb{U}(u, \pi) \) of \( P|_{x_0,d} \),
\[
V_C(\theta, r, z, t) \leq \alpha_f \quad \forall t \in \{0, 1, \cdots \}.
\]

Whether a controller is a feasible controller or not depends on the system \( P \) including its initial condition \( x_0 \) and the disturbance signal \( d \) in the experiment conducted from time 0 to \( \infty \).

Although the main goal of a switching control is set by \( V_C(\theta, r, z, t) \) in (11), it is natural to believe that a good model represents the system \( P \) with high accuracy and, hence, a controller in (2) corresponding to a good model leads to a good performance. Thus, we employ both cost functions \( V_M(\theta, r, z, t) \) in (6), which reflects model mismatch, and \( V_C(\theta, r, z, t) \) in (11), which reflects tracking performance, in a switching algorithm. Moreover, it is possible that there are multiple feasible controllers so that we employ the following switching algorithm, which is an extension of the switching algorithm in [6].

Algorithm 1: (i) At \( t = 0 \) : Use any prior knowledge to select \( \hat{\theta}(0) \) or set \( \hat{\theta}(0) = \theta_1 \). Select a constant \( \alpha_w > 1 \) and a sequence \( \varepsilon_1 \geq \varepsilon_2 \geq \cdots \) of nonnegative constants. Define \( N' = \{1, \cdots, N \} \).

(ii) At \( t = 1, 2, \cdots \) : Obtain
\[
N_C(t) = \left\{ n \in N' \left| V_C(\theta_n, r, z, t) \leq \min_{n \in N} V_C(\theta_n, r, z, t) + \varepsilon_t \right. \right\}
\]
\[
\hat{\Theta}(t) = \left\{ \theta \in \cup_{n \in N_C(t)} \Theta_n \left| V_C(\theta, r, z, t) \leq \min_{n \in N} V_C(\theta_n, r, z, t) + \varepsilon_t \right. \right\},
\]
\[
\hat{\theta}_w(t) = \left\{ \theta \in \hat{\Theta}(t) \left| \max_{r \in [0, t]} \|\tilde{w}_r\|_{[0, t]} + \rho \leq \alpha_w \right. \right\}
\]
with the fictitious reference signal \( \tilde{w}_r \) in (10).

(a) If \( n(t-1) \notin N_C(t) \), set
\[
\hat{\theta}(t) = \arg \min_{\theta \in \hat{\Theta}(t)} V_M(\theta, z, t).
\]
Set \( n(t) \) to any \( n \) satisfying \( \hat{\theta}(t) \in \Theta_n \).

(b) If \( n(t-1) \in N_C(t) \) and \( \hat{\theta}_w(t) \neq \emptyset \), set
\[
\hat{\theta}(t) = \arg \min_{\theta \in \hat{\theta}_w(t)} V_M(\theta, z, t).
\]
Set \( n(t) \) to any \( n \) satisfying \( \hat{\theta}(t) \in \Theta_n \).

(c) If \( n(t-1) \in N_C(t) \) and \( \hat{\theta}_w(t) = \emptyset \), set
\[
\hat{\theta}(t) = \hat{\theta}(n(t-1)).
\]
and \( n(t) = n(t-1) \).

The main extension of this algorithm from [6] is that this algorithm consider, in addition to a finite number of parameters \( \theta_1, \cdots, \theta_N \), any parameter in the parameter sets \( \Theta_1, \cdots, \Theta_N \). Thus, the number of parameters of the models is possibly infinite.

The set \( N_C(t) \) contains all the good parameters among the selective parameters \( \theta_n \)'s in terms of \( V_C(\theta, r, z, t) \) and is not empty for any \( t \in \{1, 2, \cdots \} \). And, the sets \( \hat{\Theta}_C(t) \) and \( \hat{\theta}_w(t) \) contain other qualified parameters in terms of \( V_C(\theta, r, z, t) \) and \( \tilde{w}_r \), respectively, in the vicinities of the good selective parameters.

A switching performed in (12) is similar to a switching in the \( \varepsilon \)-Hysteresis Switching Algorithm [17] in the sense that if the previous selective parameter \( \theta_{n(t-1)} \) shows relative inadequacy in \( V_C(\theta, r, z, t) \) by the amount of \( \varepsilon_t \), another adequate selective parameter is selected. Thus, the number of switchings in (12) is finite. However, the difference is that, instead of choosing the new selective parameter, Algorithm 1 searches, in (12), for a better parameter in the vicinity of the new selective parameter.

Even when the previous selective parameter \( \theta_{n(t-1)} \) shows relative inadequacy in \( V_C(\theta, r, z, t) \), additional switchings in (13) are performed, provided that the condition \( \hat{\theta}_w(t) \neq \emptyset \) is satisfied. This condition implies that there is a parameter \( \theta \) whose corresponding candidate controller has a cost value in \( V_C(\theta, r, z, t) \) within \( \varepsilon_t \) difference of \( \min_{n \in N} V_C(\theta_n, r, z, t) \) and has its fictitious reference signal bounded by the reference signal to the level \( \alpha_w \). If this condition is not satisfied, the algorithm chooses \( \theta_{n(t-1)} \) in (14).

Since the number of switchings performed in (13) may not be finite, there is no guarantee of a finite number of switchings in Algorithm 1. Yet, the tracking performance of the adaptive control system is guaranteed by the following theorem.

Theorem 1: Consider the adaptive control system in Section II, together with a pair of cost functions \( V_M(\theta, z, t) \) in (6) and \( V_C(\theta, r, z, t) \) in (11) and Algorithm 1. Suppose that \( \inf_{t \in \{1,2,\cdots\}} \varepsilon_t > 0 \) and the candidate controller set \( \mathbb{C} \) contains at least one feasible controller (Definition 4). Then, there exist constants \( \alpha_y \) and \( \beta_y \) such that
\[
\|y - r\|_{[0, t]} \leq \alpha_y \|w\|_{[0, t]} + \beta_y \quad \forall t \in \{0, 1, \cdots \}. \tag{15}
\]

Proof: At each time \( t = 1, 2, \cdots \), the set \( \hat{\Theta}(t) \) contains the parameter chosen by Algorithm 1, i.e., \( \hat{\theta}(t) \in \hat{\Theta}(t), t = 1, 2, \cdots \).

(i) In the case where the number of switchings is finite, i.e. \( \hat{\theta}(t) = \hat{\theta}(t_f) \) for \( t = t_f, t_f + 1, \cdots \) where \( t_f \) is the time of the last switching, there exist constants \( \alpha_f, \alpha > 0 \) such that
\[
\|y - r\|_{[0, t]} \leq \alpha_f \quad \text{and} \quad \|\tilde{w}_r\|_{[0, t]} + \rho \leq \alpha_w
\]
\[
\forall t \in \{0, 1, \cdots \} \text{ since } \hat{\theta}(t) \in \hat{\Theta}(t), t = 1, 2, \cdots, \text{ and } \hat{\theta}_w(t) \neq \emptyset \quad \forall t \in \{0, 1, \cdots \}.
\]

Then, it follows that
\[
\|y - r\|_{[0, t]} \leq \|y - r\|_{[0, t]} + \rho \quad \|\tilde{w}_r\|_{[0, t]} + \rho \leq \alpha_f \quad \forall t \in \{0, 1, \cdots \}.
\]

(ii) In the case where the number of switchings is not finite, the number of switchings in (13) is not finite. And, at each switching time \( t \) associated with (13), we have
\[
\|y - r\|_{[0, t]} \leq \|y - r\|_{[0, t]} + \rho \quad \|\tilde{w}_r\|_{[0, t]} + \rho \leq \alpha_f \quad \forall t \in \{0, 1, \cdots \}.
\]

Note that Theorem 1 guarantees the inequality in (15) even when the number of switchings is not finite due to the switchings by (13). This depends on the value \( \alpha_w \) and this constant can be adjusted during the experiment depending on the reference signal and the fictitious reference signals but this adjustment is out of the scope of the paper.
The merit of Algorithm 1 is that the algorithm keeps pursuing the best controller in terms of $V_M(\theta, z, t)$ as far as a certain level of tracking performance of the adaptive control system is preserved. This pursuit is useful especially when there are more than one feasible controllers in the candidate controller set and we are interested in finding a controller with a better performance. Also, Algorithm 1 has decreasing hysteresis levels $\varepsilon_\gamma$'s, which makes the model mismatch the major criterion for early switchings. Thus, a swift search for a good model at the beginning of the algorithm becomes critical and beneficial. A method for this is studied in the next section.

IV. MODIFICATION OF THE MPC FOR DUAL CONTROL

Since the input signals of the system $P$ are limited in the set $U(\gamma, \rho)$, discrete-time, causal, and linear time-invariant (LTI) models may be appropriate for representing $P$. The models $P_{\theta_n}$'s in Section II are regarded as the potential models of $P$ and we modify the MPC to pursue both the tracking performance and the model discrimination at the same time, which is called dual control. Fast model discrimination is crucial for early search for a good model in the switching algorithm in the previous section.

Consider the models $P_{\theta_n}$'s that can be described by (7) and each model is fitted to collected data from the adaptive control system in Section II via the cost function and we perform this model discrimination as in the following.

From (6), it follows that $V_M(\theta_n, z, t) = \|\tilde{d}_{\theta_n}(t)\|_2$ for any $n \in \{1, \cdots, N\}$. Moreover, we have

$$\tilde{d}_{\theta_n}(t) = \mu_{\theta_n, \theta_n}(t) + H^{-1}_{\theta_n} \psi_{\theta_n, \theta_n}(t) \tilde{x}_{\theta_n, \theta_n}(t)$$

for any $n_1, n_2 \in \{1, \cdots, N\}$ where

$$\mu_{\theta_n, \theta_n}(t) = H^{-1}_{\theta_n} (G_{\theta_n} - G_{\theta_n}) u(t).$$

Then, the following lemma, whose proof is omitted, guarantees the model discrimination.

Lemma 1: Suppose that the system $P$ in the adaptive control system in Section II satisfies $V_M(\theta_n, z, t) < \delta$ with a parameter $\theta_n$ and a constant $\delta$. If, for any $n_2 \in \{1, \cdots, N\} \setminus \{n_1\}$,

$$\|\mu_{\varphi, \psi}(t)\|_2 \geq \gamma(\varphi, \psi)$$

for either $(\varphi, \psi) = (\theta_n, \theta_n)$ or $(\theta_n, \theta_n)$ where

$$\gamma(\varphi, \psi) = \rho_x \left( \|H^{-1}_{\varphi} \psi_{\varphi, \psi}\|_2 + \|H^{-1}_{\psi} \psi_{\varphi, \psi}\|_2 \right) + (1 + \|H^{-1}_{\varphi} H_{\psi}\|_2) \delta$$

with $\rho_x$ in (6) and $\|H^{-1}_{\varphi} H_{\psi}\|_2$ is the induced matrix norm of $H^{-1}_{\varphi} H_{\psi}$, then we have

$$V_M(\theta_n, z, t) > \delta \quad \forall n_2 \in \{1, \cdots, N\} \setminus \{n_1\}.$$

Based on this lemma, we design an input signal satisfying

$$\|\mu_{\theta_n, \theta_n}(T_M)\|_2 \geq \delta_{\gamma_{n_1, n_2}}$$

for all $n_1, n_2 \in \{1, \cdots, N\}$ satisfying $n_1 < n_2$ where $T_M$ is the time when we want the model discrimination is complete and $\delta_{\gamma_{n_1, n_2}} = \max\{\gamma(\theta_n, \theta_n), \gamma(\theta_n, \theta_n)\}$. Then, Lemma 1 guarantees that at most one model among $P_{\theta_n}$'s has the cost value of model mismatch less than $\delta$ at time $T_M$. This statement can also be developed in a probabilistic framework.

The choice of $\delta$ relies on any prior knowledge of the system $P$ and can be supported by placing lots of models. Note that, for any given unordered pair $(n_1, n_2)$ or any given two models, there is only one condition imposed by (17). Thus, the total number of conditions in (17) is $M = \binom{N-1}{2}$, which is the total number of the unordered pairs of the models $P_{\theta_n}$'s. Then, using (16), we rewrite the condition in (17) as

$$\min_{m \in \{1, \cdots, M\}} \frac{1}{\gamma_m} \|\bar{C}_m u(T_M)\|_2 \geq 1$$

where $\bar{H}_{\theta_n} (G_{\theta_n} - G_{\theta_n})$ and $\bar{r}_{n_1, n_2}$ are represented by $\bar{C}_m$ and $\bar{r}_m$, respectively.

The condition in (18) is imposed on $u(0), \cdots, u(T_M)$ and is implemented in the MPC in (2) as an additional constraint. Thus, $T_M$ is chosen to be large enough to make the optimization problem in (2) feasible or the condition in (18) is implemented as a soft constraint. After time $T_M$, the candidate controllers return back to their original form and focus on the tracking performance.

The optimization problems in (2) combined with the condition in (18) is a quadratic optimization problem with non-convex quadratic constraints and its computation becomes demanding as $M$ and $T_M$ increase. Thus, we may perform semidefinite relaxation techniques (e.g. [10]) to obtain a semidefinite programming (SDP) problem and, then, perform a random search procedure based on an optimal solution to the SDP problem.

V. EXAMPLES

In this section, simple examples of the switching algorithm in Section III is presented.

Consider the adaptive switching control system in Section II with an uncertain discrete-time unstable system that is described, in terms of its transfer function, by

$$P: Y(z) = \frac{1}{(z-1.1)(z+0.2)} U(z) + D(z).$$

Its initial condition and the disturbance signal $d$ are unknown. For the candidate controllers, we consider two models $P_1: Y(z) = \frac{1}{(z-1.1)(z+0.2)} U(z) + D(z)$ and $P_2: Y(z) = \frac{1}{(z-1.1)(z+0.2)} U(z) + \frac{1}{z^2} D(z)$.

In a MATLAB simulation with a sampling period $T_s = 1$ sec, the reference signal and the disturbance signal are given by $w(t) = [r(t), v(t)] = [1, 0]^T$ and $d(t) = 0.2 \cos(0.6283t)$ for $t = 0, 1, \ldots$, respectively. The initial condition of $P$ is set to $[0.1 \ 0]^T$. Algorithm 1 employs $\varepsilon_1 = \varepsilon_2 = \cdots = 0.1$ and the MPC employs the prediction horizon $T = 10$ and only the input constraints with the amplitude bounds $\bar{u} = 2$ and $\bar{w} = -2$.

The input and the output signals of $P$ are shown in Fig. 2 (a) and the selection of the models by Algorithm 1 is shown in Fig. 2 (b). Even though the model $P_1$ is exactly the same as the system $P$ and Algorithm 1 starts with $P_1$ in the simulation, the switching algorithm switches early to and keeps the model $P_2$ in the closed-loop system. This is because the sinusoidal disturbance signal $d$ is well-described
by the disturbance model in $P_2$. Thus, the model $P_2$ is a
better model for representing the input and the output signals
of $P$ and this can also be explained by the internal model
principal [18]. Therefore, despite the sinusoidal disturbance
signal, the switching algorithm shows a good tracking
performance. Further, parametrized disturbance models can be
considered and be searched for the best model to represent
the unknown frequency of the disturbance signal via the
switching algorithm, which is omitted for simplicity.

Next, the same simulation is performed with different
models $P_1 : Y(z) = \frac{1}{(2z-1)(z+0.3)} U(z) + D(z)$ and $P_2 :$
\[ Y(z) = \frac{1}{(2z-1)(z+0.2)^2} U(z) + \frac{z^2-0.809z-1}{z^2-0.015z-1} D(z), \]
both of which have more model mismatches than the models in
the previous simulation. The input and the output signals of $P$
are shown in Fig. 3 (a) and the selection of the models by
Algorithm 1 is shown in Fig. 3 (b). It is clear that the tracking
performance has been compromised due to the increased
model mismatches. However, Algorithm 1 decides, after a
couple of switchings, that the model $P_1$ is suited better for
the collected data of the system $P$ than the model $P_2$.

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