Localized Distributed State Feedback Control with Communication Delays

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Abstract—This paper introduces the notion of localizable distributed systems. These are systems for which a distributed controller exists that limits the effect of each disturbance to some local subset of the entire plant, akin to spatio-temporal dead-beat control. We characterize distributed systems for which a localizing state-feedback controller exists in terms of the feasibility of a set of linear equations. We then show that when a feasible solution exists, it can be found in a distributed way, and used for the localized synthesis and implementation of controllers that lead to the desired closed loop response. In particular, by allowing controllers to exchange both state and control actions, the information needed by a particular controller is limited to a local subset of the system’s state and control inputs.

I. INTRODUCTION

Distributed control problems arise when several decision makers, or controllers, need to determine their actions based only on a subset of the total information available about the system. Broadly speaking, the types of problems addressed in this field fall under one of three settings: (1) synthesizing completely decentralized stabilizing controllers, (2) explicitly taking lossy communication channels into account and (3) synthesizing optimal distributed controllers. Representative papers addressing the first problem class can be found in the references of [1]. Explicitly dealing with realistic communication networks is traditionally the realm of networked control systems (NCS) theory [2]. The synthesis of optimal distributed controllers subject to information constraints is known to be convex for a broad class of systems that satisfy a quadratic invariance (QI) property [3]–[5]: a survey of recent results in this area, and a more exhaustive list of references, can be found in [6].

The ultimate goal of this line of work is the implementation of controllers for large scale distributed systems in a scalable manner. For systems comprised of thousands or more sub-systems, an appealing way to achieve scalable design and implementation is to localize the design and implementation of, as well as the coordination between, distributed controllers. This intuitive idea has been explored in the literature, with [7], [8] being representative examples.

In this paper, we attempt to formalize this notion of locality by defining a class of localizable systems. In particular, we show that systems for which a state-feedback controller exists that yields localized (to be formally defined in Section III) finite impulse response (FIR) behavior are characterized in terms of the feasibility of a set of linear equations. This set of linear equations has the added property of being verifiable via a distributed test, and any set of feasible solutions from these distributed tests can be used locally to construct feedback controllers that achieve the desired closed loop properties. We additionally show that by allowing these controllers to exchange both state and control actions, their implementation is both distributed and localized as well. Finally, through numerical case studies, we show that this localized scheme has many favorable properties with respect to “traditional” distributed control schemes.

The paper is structured as follows. Section II starts with a simple example to illustrate the main ideas used throughout this paper. Section III introduces a more general system and formally defines the notion of a state-feedback FIR localizable system. In Section IV, we show that state-feedback FIR localizable systems are characterized by a linear programming (LP) based test, and that this test can be verified in a distributed and localized manner. Section V then shows how transmitting both state and control action allows for any feedback controller achieving a localized FIR closed loop to be implemented and synthesized in a localized and distributed way. Section VI demonstrates the effectiveness of our method by simulation. Lastly, Section VII ends with conclusions and offers some future research directions.

II. ILLUSTRATIVE EXAMPLE

In this section, we introduce a simple example that will be used throughout the paper to illustrate the various concepts that we define. Consider a system with dynamics given by

\[ x[k + 1] = Ax[k] + u[k] + w[k] \]

with \( A = (A_{ij}) \) a stable \( 7 \times 7 \) tridiagonal matrix, and \( x = (x_i) \), \( u = (u_i) \) and \( w = (w_i) \) vectors of local state, control, and disturbances, respectively. In particular, we let each local sub-system \( i \) have scalar state \( x_i \), and local controller with scalar input \( u_i \). The topology of this cyber-physical system is shown in Figure 1. We also impose a communication delay of 0.5 between each controller, and assume that information can be forwarded from neighbor to neighbor.

Our goal is to find a dynamic controller \( K(z) \) satisfying these communication delay constraints such that applying the control action \( u = Kx \) results in a localized and FIR closed
loop response. One such solution is given by

\[ K = -\frac{1}{2}A^2(I + \frac{1}{2}A)^{-1} = -A^2(zI + A)^{-1}, \]  

(1)

yielding the closed loop transfer function from \( w \) to \( x \)

\[ R_{sw} = \frac{1}{z}I + \frac{1}{z^2}A. \]  

(2)

Not only is \( R_{sw} \) FIR, but it is also localized (to be formally defined in Section III), as each disturbance only affects a local neighborhood of states. From (1), unless \( A \) is nilpotent, the controller \( K \) is neither localized (i.e. it requires information from all other subsystems) nor FIR - this need to collect information about the full system leads to a non-scalable implementation. Notice that it is in general impossible to simultaneously achieve both a FIR localized \( K \) and \( R_{sw} \) due to the relationship \( R_{sw} = (zI - A - K)^{-1} \).

To circumvent this architectural limitation, we use an implicit implementation of the controller in terms of its Youla parameter \( Q \). Letting \( P_{22} = (zI - A)^{-1} \) and \( Q = K(I - P_{22}K)^{-1} \), we have that \( K = (I + QP_{22})^{-1}Q \).

Multiplying both sides of \( u = Kx \) by \( (I + QP_{22}) \) and rearranging terms, we obtain the implicit representation

\[ u = Qx - QP_{22}u. \]  

(3)

Moreover, we can show that

\[ Q = -\frac{1}{2}A^2 + \frac{1}{z^2}A^3 \]  

\[ QP_{22} = -\frac{3}{z^2}A^2. \]  

(4)

We therefore have that this implicit representation

1) is both FIR and localized, all the while maintaining the same (FIR and localized) closed loop response \( R_{sw} \).

2) leads to the distributed implementation of local controllers \( C_i \): by combining (3) and (4), we see that only sub matrices of \( A \) (i.e. only local models of the dynamics) are needed to implement the local controllers, and that each such controller only needs to collect a finite history of state and control from a local neighborhood of plants.

Specializing these general observations to our particular example, we see that due to the tridiagonal sparsity pattern of \( A \), and (2) that, for example, the effect of a disturbance entering at \( P_1 \) will be limited to that plant and its neighbors, \( P_2 \). Additionally, from equations (3) and (4), \( C_1 \) can compute its control action based on the state measurement \( (x_i)_{i=1}^4 \) and control inputs \( (u_i)_{i=1}^3 \). The synthesis of \( C_1 \) is in fact also independent of the model parameters of \( P_6 \) or \( P_7 \) – this follows from the sparsity pattern of the first row of \( Q \) and \( QP_{22} \) in (4).

By combining the two simple ideas of localizing the effect of disturbances and allowing controllers to exchange their inputs as well as states, we are thus able to achieve many desirable properties in a completely local and distributed way. Of course, the previous example is predicated on there already existing a controller \( K \) that achieves a FIR localized \( R_{sw} \) – we will additionally show that such a \( K \) can be found in a completely local and distributed manner as well, completing our scalable design process.

The rest of this paper will formalize and generalize these ideas.

III. Problem Formulation

A. System Model

We consider a discrete-time distributed system, described by the triple \( (A, B, S) \), with dynamics given by

\[ x[k + 1] = Ax[k] + Bu[k] + w[k], \]  

(5)

and \( S \) a controller information sharing constraint that will be formally defined in the next subsection.

We assume that \( A \) is stable. In addition, we assume that \( B \) has full column rank (and hence has a unique left inverse \( B^\dagger \)) with exactly one non-zero entry per column – i.e. we assume that each control action \( u_i \) is scalar and only directly affects one scalar state \( x_j \). We refer to this as the scalar sub-system plant model, and use \( c_i \) to index the unique location of the non-zero entry of the \( i \)-th column of \( B \), i.e. if \( u_i \) directly affects \( x_j \), then \( c_i = j \). We define the \( i \)-th local controller to be the controller in location \( c_i \), and which generates control action \( u_i \). Similarly we define the \( j \)-th local plant to be the plant with state \( x_j \), and which is affected by disturbance \( w_j \).

Our goal is to find a dynamic state-feedback controller \( K(z) \in S \) (i.e. a distributed controller that respects the information sharing constraints of the system) that yields a
FIR and localized closed loop response
\[ R = (zI - A - BK)^{-1}. \] (6)

We will use \( R \) to denote the closed loop transfer function from \( w \) to \( x \), and thus \( M = KR \) is the closed loop transfer function from \( w \) to \( u \) such that \( u(z) = K(z)x(z) = M(z)w(z) \).

B. Sparsity Patterns

Let \( sp(\cdot) : \mathbb{R}^{m \times n} \to \{0,1\}^{m \times n} \) be the support operator, where \( (sp(M))_{ij} = 1 \) if \( M_{ij} \neq 0 \), and 0 otherwise. We denote by \( S_1 \cup S_2 \) the entry-wise OR operation of the two binary matrices \( S_1, S_2 \in \{0,1\}^{m \times n} \). We say that \( S_1 \subseteq S_2 \) if \( S_1 \cup S_2 = S_2 \). The product \( S_1 = S_2S_3 \), with binary matrices \( S_2 \) and \( S_3 \) of compatible dimension, is given by

\[(S_1)_{ij} = 1 \text{ iff there exists a } k \text{ such that } (S_2)_{ik} = 1 \text{ and } (S_3)_{kj} = 1.\]

From this definition, it follows that \( sp(M_1M_2) \subseteq sp(M_1)sp(M_2) \). For a square binary matrix \( S_0 \), we define \( S_0^{i+1} := S_0^{i}S_0 \) for all positive integers \( i \), and let \( S_0^n = I \). In particular, if \( S_0 \) is the support of the adjacency matrix of a graph, we can define the distance from node \( k \) to \( j \), with respect to the constraint set \( S_0 \), as

\[ \text{dist}_{S_0}(k \to j) := \min \{i \in \mathbb{N} \cup 0 \mid (S_0^i)_{kj} \neq 0\} \] (7)

We can then define the information constraint \( S \) as the space \( S := \sum_{i=0}^{\infty} \frac{1}{2^i} S_i \) where each \( S_i \) is a binary matrix. We abuse notation slightly by writing that a stable transfer function \( K := \sum_{i=0}^{\infty} \frac{1}{2^i} K_i \in S \) if and only if \( sp(K_i) \subseteq S_i \) for all \( i \). Finally, to ease notational burden, we write \( SA \) as a shorthand for \( S sp(A) \).

Example 1: The information sharing constraint of the system in Figure 1 is given by \( S = \sum_{i=0}^{\infty} \frac{1}{2^i} sp(A)^2 \).

As is standard, let \( P_{22} = (zI - A)^{-1}B \) once again with a slight abuse of notation, we say that an information constraint \( S \) is quadratically invariant (QI) under \( P_{22} \) (c.f. [3], [4]) if \( K P_{22} K \in S \) for all \( K \in S \). If \( S \) is QI under \( P_{22} \), then \( K \in S \) if and only if \( Q \in S \), where \( Q = K(I - P_{22} K)^{-1} \). In this paper, we assume \( S \) is QI and impose the additional assumption that \( \frac{1}{2} SA \subseteq S \). The latter constraint states that we require the communication delay to be less than or equal to the plant propagation delay for every connected edge.

As our controller implementation is an implicit one (i.e. each local control action is a function of both the states and control actions of other sub-systems), we need to understand what constraints \( Q P_{22} \) should satisfy to be consistent with the information sharing constraints of the system as well. We claim that \( Q P_{22} \in \frac{1}{2} SB \) is the corresponding information sharing constraint on control inputs \( u \). To see this, note that \( SB \) is the projection of the information sharing constraint \( S \) on to the subspace of plants with controllers. Finally, multiplying by \( \frac{1}{2} \) ensures that the transfer function from \( u \to u \) is strictly causal; i.e. that only previous control actions are used to generate the current control signal.

C. Definition of Localizability

We begin with a definition that compares a (transfer) matrix with how quickly dynamics spread through the physical topology of the plant.

Definition 1: We say that a real matrix \( X \) is \((A,d)\) sparse if

\[ \text{sp}(X) \subseteq \bigcup_{i=0}^{d} \text{sp}(A)^i. \]

A transfer function \( R := \sum_{i=0}^{\infty} \frac{1}{2^i} R_i \), with each \( R_i \) a real matrix, is \((A,d)\) sparse if and only if \( R_i \) is \((A,d)\) sparse for all \( i \).

We now define the following two sets, which we will use to characterize the localized region associated with each plant. Let \( A = sp(A) \),

\[ E_{(j,d)} = \{ s \mid (\bigcup_{i=0}^{d} A^i)_{js} = 1 \} = \{ s \mid \text{dist}_A(s \to j) \leq d \} \]

\[ F_{(j,d)} = \{ s \mid (\bigcup_{i=0}^{d} A^i)_{sj} = 1 \} = \{ s \mid \text{dist}_A(j \to s) \leq d \} \]

In particular, if \( A \) is symmetric (i.e. it corresponds to an undirected graph), then \( E_{(j,d)} = F_{(j,d)} \).

Example 2: The system in Figure 1 has \( E_{(1,2)} = F_{(1,2)} = \{1,2,3\} \).

We may now formally define scalar sub-system plants that are state-feedback FIR localizable.

Definition 2: A scalar sub-system model \((A,B,S)\) is state-feedback \((d,T)\)-FIR localizable if there exists a \( K \in S \) such that the closed loop transfer function \( R \) given by (6) is FIR and \((A,d)\) sparse, with \( R = \sum_{i=1}^{T} \frac{1}{2} R_i \) for some real matrices \( R_i \).

There is a fairly intuitive interpretation of the definition of FIR localizability. If \( R \) is \((A,d)\) sparse, then we know that each local disturbance \( w_j \) will only affect the states \( x_i \) with \( i \in F_{(j,d)} \), and each state \( x_i \) will only be affected by disturbances \( w_j \) with \( j \in E_{(i,d)} \).

Example 3: The system in Figure 1 is \((1,2)\) FIR localizable.

IV. A LP Characterization of State Feedback \((d,T)\)-FIR Localizable Systems

In this section, we present a LP characterization of state-feedback \((d,T)\)-FIR localizable systems. In particular, we show that the feasibility of a set of linear equations is both necessary and sufficient condition to determine whether a system \((A,B,S)\) is state-feedback \((d,T)\)-FIR localizable. We then show that this feasibility test can be performed in a localized and distributed manner.

\footnote{For a large \( d \), it is possible that \( \bigcup_{i=0}^{d} \text{sp}(A)^i = 1 \) – this simply means that the “localized” region is the entire system.}
A. Global Feasibility Test

Fix a scalar-subsystem $j$, and let $x_{j[0]} = 0$, $u_{j[0]} = 0$, and $w_{j[k]} = e_j\delta[k]$. We then seek solutions $w_{j[0]}, \ldots, w_{j[T]}$, $x_{j[1]}, \ldots, x_{j[T+1]}$ to the following set of linear equations:

\begin{align}
& x_{j[0]} = 0, w_{j[0]} = 0, \quad w_{j[k]} = e_j\delta[k] \quad (8a) \\
& x_{j[k+1]} = Ax_{j[k]} + Bu_{j[k]} + w_{j[k]} \quad \text{for} \quad k = 0, \ldots, T \\
& x_{j[T+1]} = 0 \\
& \text{sp}(x_{j[k]}) \subseteq \bigcup_{i=0}^{d} \text{sp}(A_i) \quad \text{for} \quad k = 1, \ldots, T \\
& \text{sp}(w_{j[k]}) \subseteq [S_{k-1}]_j \quad \text{for} \quad k = 1, \ldots, T \quad (8e)
\end{align}

where $[S_{k-1}]_j$ and $\bigcup_{i=0}^{d} \text{sp}(A_i)$ are the $j$-th column of the binary matrices $S_{k-1}$ and $\bigcup_{i=0}^{d} \text{sp}(A_i)$ respectively.

**Theorem 1:** Assume $S$ is QI under $P_{22}$, and that $\frac{1}{2}SA \subseteq S$. The system $(A, B, S)$ is state-feedback $(d, T)$-FIR localizable if and only if the linear equalities $(8a) - (8e)$ are feasible for all sub-systems $j$.

**Proof:** Let $R = \sum_{i=1}^{T} \frac{1}{z} R_i$ be the resulting impulse response of the closed loop system when applying the control actions computed. Then the $j$-th column of each $R_k$ is given by the solution $x_{j[k]}$ to the $j$-th feasibility test. Essentially, equations $(8a) - (8d)$ are a time-domain formulation of the condition that the transfer function $R$ be $(A,d)$ sparse and FIR with time $T$.

Therefore all that remains is to show that feasibility of the LP occurs if and only if there also exists a control law $u = Kx$, with $K \in S$, leading to a closed loop $R$ that is $(d,T)$-FIR localized. The solution to the feasibility test provides us with an $M \in \frac{1}{2}S$ such that $u = Mw$ yields a $(d,T)$-FIR localized $R$. Thus, it is sufficient to show that there exists a bijection between all $K \in S$ and all $M \in \frac{1}{2}S$ s.t. $u = Kx = Mw$.

First, notice that we can rewrite $R$ as

\begin{align}
R &= (zI - A)^{-1} (I + BK) R \\
&= (zI - A)^{-1} + P_{22}KR. \quad (10)
\end{align}

Rearranging (10) and multiplying both sides by $(I - P_{22}K)^{-1}$ we obtain

\begin{align}
R &= (I - P_{22}K)^{-1} (zI - A)^{-1}.
\end{align}

Finally, notice that

\begin{align}
M &= KR \\
&= K(I - P_{22}K)^{-1} (zI - A)^{-1} \\
&= \frac{1}{2}S \quad \text{and} \quad K \in S.
\end{align}

Thus for every $M \in \frac{1}{2}S$, $Q = M(zI - A) \in S$ by our assumption that $\frac{1}{2}SA \subseteq S$. Moreover, by our assumption of QI $Q \in S$ if and only if $K \in S$, leading to the desired implication that $M \in \frac{1}{2}S \implies K \in S$.

To show the reverse implication, assume a $K \in S$, and express (11) as

\begin{align}
M &= Q \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} A^i.
\end{align}

By the QI assumption, we have that $Q \in S$, and therefore through repeated use of the assumption that $\frac{1}{2}SA \subseteq S$, we conclude that $M \in \frac{1}{2}S$, completing the proof. \hfill \blacksquare

B. Local Feasibility Test

We now simplify $(8a)-(8e)$ for a particular disturbance at state $j$. We will show that the feasibility test can be verified in a localized way. In particular, we prove that imposing that all states along the boundary of a local region (defined in terms of $F_{j,d+1}$) remain zero for all time ensures that the disturbance $w_j$ cannot propagate outside of this region, thus automatically satisfying the constraints of the global LP.

In order to state this result, we need to define reduced state and control vectors:

**Definition 3:** The $(j,d)$-reduced state vector of $x$ consists of all states in $F_{j,d+1}$ and is denoted by $x_{j,d}$. Similarly, the $(j,d)$-reduced control vector of $u$ consists of all controllers $i$ such that $c_i \in F_{j,d+1}$, and is denoted by $u_{j,d}$.

We can then define the $(j,d)$-reduced plant model $(A_{j,d}, B_{j,d}, S_{j,d})$ by selecting submatrices of $(A, B, S)$ consisting of the columns and rows associated with $x_{j,d}$ and $u_{j,d}$. In addition, we denote by $w_{j,d}$ the new location of the source of disturbance $j$ within the reduced state $x_{j,d}$.

**Example 4:** The $(7,1)$-reduced state and control for the system in Figure 1 are $x_{7,1} = [x_5, x_6, x_7]^T$ and $u_{7,1} = [u_5, u_6, u_7]^T$, respectively. The new location of the source is $w(7,1) = 3$, as the disturbance entering $P_7$ is the third component in the reduced state. $A_{7,1}$ is the bottom right $3 \times 3$ submatrix of $A$.

We can now formulate a local feasibility test for each subsystem $j$ in terms of these reduced quantifiers:

\begin{align}
& x_{j,d}[0] = 0, u_{j,d}[0] = 0, \quad w_{j,d}[k] = e_{w(j,d)}\delta[k] \quad (12a) \\
& x_{j,d}[k+1] = A_{j,d}x_{j,d}[k] + B_{j,d}u_{j,d}[k] + w_{j,d}[k] \quad \text{for} \quad k = 0, \ldots, T \\
& x_{j,d}[T+1] = 0 \\
& \text{sp}(x_{j,d}[k]) \subseteq \bigcup_{i=0}^{d} \text{sp}(A_{j,d}) \quad \text{for} \quad k = 1, \ldots, T \\
& \text{sp}(w_{j,d}[k]) \subseteq [S_{j,d}]_{k-1} \quad \text{for} \quad k = 1, \ldots, T \quad (12e)
\end{align}

where $[S_{j,d}]_{k-1}w_{j,d}$ and $\bigcup_{i=0}^{d} \text{sp}(A_{j,d})w_{j,d}$ are the $w_{j,d}$-th column of the sparsity patterns $[S_{j,d}]_{k-1}$ and $\bigcup_{i=0}^{d} \text{sp}(A_{j,d})$ respectively.

Although the notation in (12a) - (12e) is complicated, the idea is conceptually straightforward – these constraints are such that no effects from the disturbance “leak” out of $F_{j,d+1}$. For example, (12d) imposes that the boundary $\{s \mid \text{dist}_A(j \rightarrow s) = d + 1\}$ in the full state vector remain zero for all time.

To prove the equivalence between the global feasibility test and local feasibility test, we define the embedding linear operators $E_x(\cdot)$ on $x_{j,d}[k]$ and $E_u(\cdot)$ on $u_{j,d}[k]$, which simply add appropriate zero padding such that $E_x(x_{j,d}[k]) = x^j[k]$ and $E_u(u_{j,d}[k]) = w^j[k]$ – in particular, we have that
Ex(\(e_{w(j,d)}\)) = ej. The following lemma is useful to connect
the two feasibility tests.

**Lemma 1:** Suppose that (12d) and (12e) hold. Then

\[
E_x(A_{j(d)}x_{j(d)}[k]) = AE_x(x_{j(d)}[k]) \tag{13}
\]

\[
E_x(B_{j(d)}u_{j(d)}[k]) = BE_u(u_{j(d)}[k]). \tag{14}
\]

**Proof:** We only prove equality (13), as (14) follows
from a nearly identical argument. As \(x_{j(d)}[k]\) satisfies (12d),
the support of \(x_j'[k] = E_x(x_{j(d)}[k])\) is contained within
\(\mathcal{F}_{j(d)}\). Let \(x_i\) be a state such that \(i \notin \mathcal{F}_{j(d+1)}\) and \(x_l\) a
state such that \(\text{dist}_A(l \rightarrow i) \leq 1\): then \(l \notin \mathcal{F}_{j(d)}\). Let \(\Delta A_1\) be a
matrix of the same dimension as \(A\), but only have one
(possibly) non-zero entry \(-A_{il}\) at \((i, l)\)-th location. Clearly,
\(\Delta A_1 x_j'[k] = 0\) for any time \(k\) as the \(l\)-th entry of \(x\) is always
zero. Therefore, setting the \((i, l)\)-entry of \(A\) to zero does not
change the value of the RHS of (13).

Similarly, let \(x_s\) be a state such that \(\text{dist}_A(i \rightarrow s) \leq 1\).
Letting \(\Delta A_2\) be a matrix of the same dimension as \(A\), but
with only one (possibly) non-zero entry \(-A_{is}\) at \((s, i)\)-th
location, we then also have \(\Delta A_2 x_j'[k] = 0\) for all \(k\). Thus
setting the \((s, i)\)-entry of \(A\) to zero does not change the value
of the RHS of (13).

Repeatedly applying this argument, we can explicitly set
all the elements in the \(i\)-th row/column of \(A\) to zero, without
changing the value of the RHS of (13) – clearly this implies
that the desired equality indeed holds.

**Theorem 2:** \((x_{j(d)}, u_{j(d)})\) is a feasible solution for (12a)-(12e) if and only if
\((E_x(x_{j(d)}), E_u(u_{j(d)}))\) is a feasible solution for (8a)-(8e).

**Proof:** Assume that \((x_{j(d)}, u_{j(d)})\) is a feasible solution
for (12a)-(12e). Applying the \(E_x\) operator to both sides of
(12a)-(12d), \(E_x\) to (12e), and using Lemma 1, it is straightforward
to verify that \((E_x(x_{j(d)}), E_u(u_{j(d)}))\) satisfy
(8a) - (8e) by noting that

\[
E_x \left( \text{sp} \left( x_{j(d)} \right) \right) = \text{sp} \left( E_x(x_{j(d)}) \right)
\]

and

\[
E_u \left( \text{sp} \left( u_{j(d)} \right) \right) = \text{sp} \left( E_u(u_{j(d)}) \right)
\]

Explicitly, to show that (8d) holds, it suffices to note that
this constraint implies that all non-zero states are contained
in \(\mathcal{F}_{j(d)}\). From the zero padding of the \(E_x\) operator, the
states outside of \(\mathcal{F}_{j(d+1)}\) will remain at 0. From (12d),
the boundary states \(s_j \text{dist}_A(j \rightarrow s) = d + 1\) will also be
zero. Therefore, (8d) is satisfied. Similarly, (8e) is satisfied
as (12e) implies that the communication delay constraints
are satisfied within the localized region.

To show the opposite direction, assume that \((x_j', u_j')\)
is a solution to the global feasibility test. It suffices to
show that \((x_j', u_j')\) satisfy the same sparsity constraints as
\((E_x(x_{j(d)}), E_u(u_{j(d)}))\). The sparsity constraint on \(x_j'\)
follows directly from (8d). For \(u_j'\), the sparsity constraint
is not directly stated in (8e). However, combining (8b) and
(8d), we know that the nonzero entries in \(Bu\) are contained
in the set \(\mathcal{F}_{j(d+1)}\). Combining this with the assumed sparsity
pattern on \(B\) and the fact that the active controllers are all
located within \(\mathcal{F}_{j(d+1)}\), we conclude that \(u_j'\) must have the
same sparsity pattern as \(E_u(u_{j(d)}).

**Example 5:** For the first local feasibility test in Figure
1, the constraint on \(x\) force the states \(x_3, \ldots, x_7\) to be
zero for all time. It is clear that the plant model in \(P_4\)
will not affect the solution of the first local feasibility
test. In particular, suppose for simplicity that \(u[k] = 0\): then
the equation \(x[k + 1] = Ax[k]\) can be simplified to
\(x_{(1,1)}[k + 1] = A_{(1,1)} x_{(1,1)}[k]\), where \(A_{(1,1)}\) is the left top
\(3 \times 3\) submatrix of \(A\). After solving for the reduced state
\(x_{(1,1)}\), we can reconstruct a solution \(x\) to the global LP via
the embedding operators, as described in Theorem 2.

Notice that the dimension of \(A_{j(d)}\) only depends on
the size of the set \(\mathcal{F}_{j(d)}\), regardless of the original size of the
plant \(A\). Therefore, the local feasibility test is scalable to
arbitrary large systems.

V. LOCALIZED IMPLEMENTATION AND SYNTHESIS

In this section, we first show how to use the solution of
the local feasibility test to implement the controller in a
localized and distributed fashion. We then show that each
local controller can be synthesized through a local update
based on the local feasibility test’s solution. Finally, we
discuss the possibility of controller redesign and layered
control architecture using this scheme.

A. Localized Synthesis and Implementation

In order to synthesize a \((d, T)\)-localizing feedback controller
\(u = Mw\) in a distributed and local manner, it suffices to
note that the transfer function \(M\) created by setting the
\(j\)-th column of its \(k\)-th spectral component \(M_k\) to be
the solution \(w_j[k] = E_x(u_{j(d)}[k])\) of the \(j\)-th local feasibility
test is one that satisfies the global LP test. From the linearity
of the system, this results in a feedback controller \(u = Mw\)
that achieves a \((d, T)\)-localized closed loop response \(R\).

Taking the z-transform of the dynamics (5), we get \(w =
(zI - A)x - Bu\), allowing us to write

\[
u = M \left[ (zI - A)x - Bu \right] \tag{15}
\]

\[
= Qx - QP_{22}u \tag{16}
\]

where the last equality follows from (11) (recall that \(Q =
M(zI - A)\)).

The next theorem states that (16) provides a means of
localizing the controller implementation if the system is
\((d, T)\)-FIR localized. By a localized implementation, we
mean that in order to compute a particular control action
\(u_j\), only states and inputs within a neighborhood of node \(j\)
need to be collected. Note that we still assume throughout
that \(S\) is IQI under \(P_{22}\), and that \(\frac{1}{2}SA \subseteq S\).

**Theorem 3:** If the system \((A, B, S)\) is state-feedback
\((d, T)\)-FIR localizable, and \(M\) is constructed as previously
described from the solutions \(u_{j(d)}\) to the local feasibility
tests, then \(u = Qx - QP_{22}u\) is an implicit implementation
achieving a localized FIR closed loop \(R\).
In addition, \( Q \in S \) and \( QP_{22} \in \frac{1}{2}SB \), and this implementation is localized in the sense that to compute \( u_i \), the \( i \)-th local controller only needs to collect states \( \{x_j \mid j \in E_{(c_i,d+2)}\} \) and control inputs \( \{ u_k \mid c_k \in E_{(c_i,d+1)}\} \).

**Proof:** By Theorem 1 we know that the global LP test is feasible, and by Theorem 2 this implies that the local feasibility tests also have a solution. From the solution of these tests and (16), it is clear that \( u = Qx - QP_{22}u \) achieves a \((d,T)\)-localized FIR closed loop \( R \). As \( S \) is QF, \( K \in S \) implies \( Q \in S \). Additionally, using \( QP_{22} = MB \) and \( M \in \frac{1}{2}S \), we have \( QP_{22} \in \frac{1}{2}SB \). Thus the communication constraints are satisfied.

Finally, we show that this implementation is localized by examining the sparsity pattern of \( Q \) and \( QP_{22} \). Rearranging (9) as

\[
BKR = (zI - A)R - I
\]

and multiplying both sides by \( B^\dagger \), we obtain

\[
M = KR = B^\dagger[(zI - A)R - I],
\]

allowing us to write

\[
Q = B^\dagger[(zI - A)R - I](zI - A)
\]

\[
QP_{22} = B^\dagger[(zI - A)R - I]B.
\]

Although it may not be obvious, it can be verified that \( Q \) in (19) and \( QP_{22} \) in (20) are proper and strictly proper transfer functions, respectively. The FIR nature of \( Q \) and \( QP_{22} \) follows directly from the fact that \( R \) is FIR. In addition, as \( R \) is \((A,d)\) sparse, applying the multiplication rule of sparsity pattern implies that \( [(zI - A)R - I](zI - A) \) is \((A,d+2)\) sparse. Left-multiplying by \( B^\dagger \) just select some rows of this transfer function, so \( Q \) is localized in the sense that the non-zero entries of \( i \)-th row are contained in the set \( E_{(c_i,d+2)} \). Similarly, we can conclude that \( QP_{22} \) is localized in the sense that the non-zero entries of \( i \)-th row of \( QP_{22} \) are contained in the set \( \{j \mid c_j \in E_{(c_i,d+1)}\} \). Combining these two arguments, we know that the controller implementation is localized as described in the theorem statement.

Theorem 3 has an intuitive interpretation if we rewrite the control rule as

\[
u(z) = M(z)w(z)
\]

\[
w[k] = x[k + 1] - Ax[k] - Bu[k].
\]

By construction, we know that \( M \in \frac{1}{2}S \). The condition \( \frac{1}{2}SA \subseteq S \), which implies that information is shared faster between controllers than disturbances propagate through the plant, ensures that each \( w_i[k-1] \) can be exactly determined once \( x_i[k-1] \) is available.

**Example 6:** To estimate \( w_1[k] \) in Figure 1, \( C_1 \) need to collect \( x_1[k+1], x_1[k], x_2[k], \) and \( u_1[k] \). The information \( x_1[k] \) and \( u_1[k] \) come before \( x_1[k+1] \) due to causality, \( x_2[k] \) is available before \( x_1[k+1] \) due to the condition \( \frac{1}{2}SA \subseteq S \). Therefore, \( w_1[k] \) can be estimated once \( x_1[k+1] \) is available.

**B. A Localized Synthesis Algorithm**

We conclude this section with an algorithm that summarizes our developments, and which provides a scalable, distributed and entirely local means of synthesizing a \((d,T)\)-localizing state-feedback controller.

**Algorithm 1:** Localized Synthesis

Given \((A,B,S)\) and \((d,T)\).

Set feasible = 1;

for each state \( x_j \) do

Perform \( j \)-th local feasibility test with the reduced plant model \((A_{(j,d)},B_{(j,d)},S_{(j,d)})\);

if not feasible then

feasible = 0;

break;

if feasible then

for each \( j \)-th local feasibility test do

Distribute the local control action \( u_{(j,d)} \) to \( i \)-th controller, \( c_i \in \mathcal{F}_{(j,d+1)} \);

for each \( i \)-th local controller do

Synthesize the \( i \)-th row of \( M \) based on the received solutions from within \( E_{(c_i,d+1)} \);

Retrieve the sub matrix of \( A \) associated with the states in \( E_{(c_i,d+2)} \) to synthesize the \( i \)-th row of \( Q \) and \( QP_{22} \);

**Example 7:** Consider \( C_1 \) in Figure 1. During the feasibility stage of the algorithm, each \( j \)-th local feasibility test, for \( j \in E_{(1,2)} = \{1,2,3\} \), is solved to generate the control action that \( C_1 \) should apply in order to localize the closed loop. In the controller synthesis part, we synthesize \( C_1 \)'s control law from these received solutions and the local sub model \( A_{(1,3)} \).

**C. Controller Redesign**

From Theorems 2 and 3 it is clear that each local control law is only a function of a subset of the full model – in particular only the sub model describing the dynamics of states \( x_k \) with

\[
k \in \left\{ \bigcup_{j \in E_{(c_i,d+1)}} \mathcal{F}_{(j,d+1)} \bigcup E_{(c_i,d+2)} \right\}
\]

needs to be taken into account when designing the \( i \)-th controller. In particular, when the topology of plant \( A \) is given by an undirected graph, that is, \( sp(A) \) is symmetric, the \( i \)-th controller only depends on the dynamics of states \( x_k \) such that \( dist_A(k \rightarrow c_i) \leq (2d + 2) \).

**Example 8:** For the case of \( C_1 \), the first three local feasibility tests only depend on the plant model containing \( P_1 \) to \( P_5 \); thus we do not need to take \( P_6 \) or \( P_7 \) into account when synthesizing \( C_1 \). This remains true if these latter two plants change dynamics, or even disconnect from the system.

Similarly, the redesign of a control scheme in light of a local change to dynamics can be done locally. If for...
example, $A_{ba}$ changes, then we only need to resolve the $j$-th local feasibility tests for $j \in \{j|a \in F_{j,d+1}\}$. If any of the feasibility tests become infeasible, then the system is no longer $(d,T)$ FIR localizable. We may then accordingly increase the value of $d$ or $T$ until the LPs become feasible. Once a solution is found, simply follow the second half of Algorithm 1 to locally update the control laws within $F_{j,d+1}$, $j \in \{j|a \in F_{j,d+1}\}$.

Example 9: For the example in Figure 1, if $A_{11}$ changes, then we need to resolve the $j$-th local feasibility test for $j \in \{j|1 \in F_{j,d+1}\} = \{1,2,3\}$. If the system is still $(1,2)$ FIR localizable, then the first local feasibility test distributes its solution to update $C_1$ through $C_3$, the second local feasibility test updates $C_3$ through $C_4$, and the third local feasibility test updates $C_5$ through $C_7$. In particular, we do not need to resynthesize $C_6$ nor $C_7$.

VI. SIMULATIONS

We demonstrate our method by synthesizing a strictly proper localized decentralized controller for the symmetric structure of 118 bus IEEE standard test case power network from The University of Florida Sparse Matrix Collection [9]. The $A$ matrix has dimension 118, and the number of non-zero entries is 476. The sparsity pattern of $A$ is shown in Figure 2.

We randomly generate the entries of $A$, and normalize it such that its maximum eigenvalue has a value of 0.99. We place 61 local controllers in the network, such that the maximum distance (as defined with respect to the plant topology) between nearest neighbor controllers is two. The communication network topology follows that of the physical network, but the speed is assumed to twice as fast, leading

*Fig. 2. Sparsity pattern of $A$.*

*Fig. 3. Sparsity pattern of $Q$.*

*Fig. 4. Sparsity pattern of $QP_{22}$.*

*Fig. 5. The vertical axis represents the value of the states, and the horizontal axis represents time. All states are plotted in the same graph. At $t = 9$, random disturbances are generated into all states. The disturbance is eliminated at $t = 17$.*

"nz = 476"

"nz = 3095"

"nz = 1187"
to an information sharing constraint set:

$$S = \sum_{i=1}^{\infty} \frac{1}{2^i} \text{sp}(B^iA^{2i-1}).$$ (21)

Using the local feasibility test, we find that $(A, B, S)$ is feasible for $(d, T) = (3, 7)$. $Q$ and $QP_{22}$ are then synthesized from the solutions to these feasibility tests and appropriate local sub models. The sparsity pattern for $Q$ and $QP_{22}$ are shown in Figure 3 and 4, clearly demonstrating the localized nature of the controller.

Simulation in time domain is performed. In the first test, we generate 118 disturbances for all states at the same time. Our result shows that multiple disturbances can be eliminated in exactly 7 time steps, as predicted by the local feasibility tests (see Figure 5).

In the second test, we generate 10 disturbances for 10 states and observe the affected region for each disturbance. In Figure 6, we see that the effect of each disturbance is contained to a localized region defined by the sparsity pattern of $A$.

From Figure 3-6, it is clear that we have met our goal of synthesizing a localized controller that achieves a localized $(3,7)$-FIR closed loop response.

VII. Conclusion

In this paper, we introduced the notion of a localizable decentralized system, and gave an LP characterization of such systems. We showed that this LP could be tested in a distributed and local manner, and that its solution could be used for the local synthesis of a state-feedback controller that achieves an FIR localized closed loop response. We also demonstrated that the implicit implementation $u = Qx - QP_{22}u$ also allows this controller to be implemented in a localized way.

There are many fruitful directions for future work. Some include modifying the LP feasibility test to an optimal control synthesis algorithm by minimizing appropriate costs subject to the locality constraints. It is also imperative to extend these ideas to the output feedback setting, and in particular, to investigate the robustness of the resulting controller to modeling and measurement errors.

REFERENCES