Boundary Control of a Vibrating String under Unknown Time-varying Disturbance

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Abstract—In this paper, robust adaptive boundary control for a vibrating string under unknown time-varying disturbance is developed to suppress the string’s vibration. The dynamics of the string is represented by a partial differential equation (PDE) and several ordinary differential equations (ODEs) involving functions of space and time. To deal with the systematic parametric uncertainty and stabilize the string, robust adaptive boundary control is developed at the tip of the string based on the Lyapunov’s direct method. With the proposed boundary control, uniform ultimate boundedness of the closed loop system is achieved. The state of the string system is proven to converge to a small neighborhood of zero by appropriately choosing design parameters. Simulations are presented to illustrate the effectiveness of the proposed control.

I. INTRODUCTION

String-type structures are widely applied in many areas of modern mechanical, civil, and ocean engineering. Examples of practical applications where vibrating strings are exposed to undesirable time-varying disturbances include crane cable used for dynamic positioning of the payload and industry chains used for material transmission. The crane cable used for subsea installation is subjected to the environmental distributed time-varying disturbance due to ocean current, wave, and wind. Since the excessive vibration of the crane cable owing to its light weight and limited support can degrade the performance of the system, the vibration suppression for the string system is well motivated for preventing damage and improving operation performance of the system.

Mathematically, the string with a tip payload can be described by a PDE representing the dynamics of the string. The dynamics of the flexible mechanical system modeled by PDE is difficult to control due to the infinite dimensionality of the system. Modal control method for PDE system is based on truncated finite dimensional modes of the system from element method or assumed modes method, e.g. [1]. For these finite dimensional models, many control techniques developed for ODE systems in [2], [3] can be applied. However, the truncated models are approximated by neglecting high frequency modes, where spillover due to truncation of the model can make the system unstable.

In order to avoid the spillover phenomenon, boundary control is employed based on the original infinite dimensional model. Boundary control has received increasing attention in area of control for flexible structures, for example, [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

In the framework of robust adaptive control, in this paper, we are going to further study the robust adaptive control problem for the string model with system parametric uncertainty and unknown time-varying disturbance. By using the well-understood mathematical tools such as algebraic and integral inequalities, and integration by parts and so on, the stability analysis of the closed-loop system is based on Lyapunov’s direct method without resorting to semigroup theory or functional analysis.

The rest of the paper is organized as follows. The governing equation (PDE) and boundary conditions (ODEs) of the string system are derived in Section II. Robust adaptive boundary control via the Lyapunov’s direct method is proposed in Section III. Simulations are presented to illustrate performance of the proposed control in Section IV. The conclusion of this paper is provided in Section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Dynamic analysis

The dynamics of a vibrating string with system parametric uncertainty under distributed time-varying disturbance \(f(x,t)\) is considered in this paper. For the system shown in Fig. 1, \(w(L,t)\), \(\dot{w}(L,t)\) and \(\ddot{w}(L,t)\) are the displacement, velocity and acceleration of the tip payload respectively, \(u(t)\) is the boundary control force, \(f(x,t)\) denotes the distributed unknown time-varying disturbance along the string, and \(d(t)\) denotes the unknown time-varying boundary disturbance on the tip payload.

The kinetic energy of the string system \(E_k\) can be represented as

\[
E_k = \frac{1}{2} M_s \left[ \frac{\partial w(L,t)}{\partial t} \right]^2 + \frac{1}{2} \rho \int_0^L \left[ \frac{\partial w(x,t)}{\partial t} \right]^2 \, dx, \quad (1)
\]

Fig. 1. A typical string system.
where \( x \) and \( t \) represent the independent spatial and time variables respectively, \( M_s \) denotes the mass of the payload at the right boundary of the string, \( w(x, t) \) is the displacement of the string at the position \( x \) for time \( t \), \( \rho > 0 \) is the uniform mass per unit length of the string, and \( L \) is the length of the string.

The potential energy \( E_p \) due to the tension \( T \) can be obtained from

\[
E_p = \frac{1}{2} T \int_0^L \left[ \frac{\partial w(x,t)}{\partial x} \right]^2 \, dx,
\]

(2)

The virtual work done by unknown time-varying disturbances on the string and the tip payload is given by

\[
\delta W_f = \int_0^L f(x,t) \delta w(x,t) \, dx + d(t) \delta w(L,t),
\]

(3)

We introduce the boundary control \( u \) at the right boundary of the string to produce a transverse force for vibration suppression. The virtual work done by the control can be written as

\[
\delta W_m = u(t) \delta w(L,t),
\]

(4)

and the total virtual work done on the system, i.e., \( W \), is given by

\[
\delta W = \delta W_f + \delta W_m
\]

\[
= \int_0^L f(x,t) \delta w(x,t) \, dx + [d(t) + u(t)] \delta w(L,t).
\]

(5)

Hamilton’s principle permits the derivation of equations of motion from energy quantities in a variational form, and Hamilton’s principle [18] is represented by,

\[
\int_{t_1}^{t_2} \delta (E_k - E_p + W) \, dt = 0,
\]

(6)

where \( t_1 \) and \( t_2 \) are two time instants, \( t_1 < t < t_2 \) is the operating interval and \( \delta \) denotes the variational operator, \( E_k \) and \( E_p \) are the kinetic and potential energies of the system respectively, \( W \) denotes work done by nonconservative force acting on the system, including control force and external disturbances. The principle states that the variation of the kinetic and potential energy plus the variation of work done by loads during any time interval \([t_1, t_2]\) must equal to zero. Integrating Eqs. (1), (2), and (5) by parts respectively and substituting the results into the Hamilton’s principle Eq. (6), we obtain

\[
- \int_{t_1}^{t_2} \int_0^L \left[ \rho \frac{\partial^2 w}{\partial t^2} - T \frac{\partial^2 w}{\partial x^2} - f \right] \delta w \, dx \, dt
\]

\[
- \int_{t_1}^{t_2} \left\{ T \frac{\partial w}{\partial x} \delta (w(x,t)) \right\}_0^L
\]

\[
- \left[ u(t) + d(t) - M_s \frac{\partial^2 w(L,t)}{\partial t^2} \right] \delta w(L,t) \bigg\} \, dt = 0
\]

(7)

As \( \delta w(x,t) \) is assumed to be an nonzero arbitrary variations in \( 0 < x < L \), the expressions under the double integral in Eq. (7) are set equal to zero. Hence, we obtain the governing equation of the string system as

\[
\rho \ddot{w}(x,t) - T \frac{\partial^2 w(x,t)}{\partial x^2} - f(x,t) = 0,
\]

(8)

\( \forall (x,t) \in (0, L) \times [0, \infty) \). Setting the terms with single integrals in Eq. (7) equal to zero, we obtain the boundary conditions of the string system as

\[
w(0,t) = 0,
\]

(9)

\[
Tw'(L,t) = u(t) + d(t) - M_s \ddot{w}(L,t),
\]

(10)

\( \forall t \in [0, \infty) \).

Remark 1: The notations \( w'(x,t) = \frac{\partial w(x,t)}{\partial x} \), \( w''(x,t) = \frac{\partial^2 w(x,t)}{\partial x^2} \), and \( \ddot{w}(x,t) = \frac{\partial^2 w(x,t)}{\partial t^2} \) are used for clarity.

Assumption 1: For the unknown time-varying disturbances \( f(x,t) \) and \( d(t) \), we assume that there exists constants \( f \in R^+ \) and \( d \in R^+ \), such that \( |f(x,t)| \leq f, \forall t \in [0, \infty) \) and \( |d(t)| \leq d, \forall t \in [0, \infty) \).

Remark 2: This is a reasonable assumption as the time-varying disturbances \( f(x,t) \) and \( d(t) \) have finite energy and hence are bounded, i.e., \( f(x,t) \in L_\infty([0, L]) \) and \( d(t) \in L_\infty \). The exact values for \( f(x,t) \) and \( d(t) \) are not required.

B. Preliminaries

For the convenience of stability analysis, we present the following lemmas and properties for the subsequent development.

Lemma 1: [4] Let \( \phi_1(x,t), \phi_2(x,t) \in R \) with \( x \in [0, L] \) and \( t \in [0, \infty) \), the following inequalities hold:

\[
\phi_1 \phi_2 \leq |\phi_1| \phi_2 \leq \phi_1^2 + \phi_2^2, \quad \forall \phi_1, \phi_2 \in R.
\]

(11)

Lemma 2: [4] Let \( \phi_1(x,t), \phi_2(x,t) \in R \) with \( x \in [0, L] \) and \( t \in [0, \infty) \), the following inequalities hold

\[
|\phi_1 \phi_2| = \sqrt{\frac{1}{\delta^2} \phi_1} \sqrt{\phi_2} \leq \frac{1}{\delta} \phi_1^2 + \delta^2 \phi_2^2,
\]

(12)

\( \forall \phi_1, \phi_2 \in R \) and \( \delta > 0 \).

Lemma 3: [19] Let \( A \in R^{n \times n} \) be a real, symmetric, positive-definite matrix; therefore, all the eigenvalues of \( A \) are real and positive. Let \( \lambda_{\min} \) and \( \lambda_{\max} \) denote the minimum and maximum eigenvalues of \( A \), respectively; then for \( \forall x \in R^n \), we have

\[
\lambda_{\min} ||x||^2 \leq x^T Ax \leq \lambda_{\max} ||x||^2,
\]

(13)

where \( || \cdot || \) denotes the standard Euclidean norm.

Lemma 4: [20], [21] Let \( \phi(x,t) \in R \) be a function defined on \( x \in [0, L] \) and \( t \in [0, \infty) \) that satisfies the boundary condition

\[
\phi(0,t) = 0, \quad \forall t \in [0, \infty),
\]

(14)

then the following inequalities hold:

\[
\phi^2 \leq L \int_0^L |\phi|^2 \, dx, \quad \forall x \in [0, L].
\]

(15)

Property 1: [21] If the kinetic energy of the system (8) - (10), given by Eq. (1) is bounded \( \forall (x,t) \in [0, L] \times [0, \infty) \), then \( \dot{w}(x,t), \dot{w}'(x,t) \) and \( \ddot{w}(x,t) \) are bounded \( \forall (x,t) \in [0, L] \times [0, \infty) \).
Property 2: [21]: If the potential energy of the system (8) - (10), given by Eq. (2) is bounded ∀(x, t) ∈ [0, L] × [0, ∞), then w'(x, t) and w''(x, t) are bounded ∀(x, t) ∈ [0, L] × [0, ∞).

III. CONTROL DESIGN

The control objective is to suppress the vibration of the string, i.e., w(x, t), under the unknown time-varying disturbances f(x, t) and d(t). In this section, the Lyapunov’s direct method is used to construct a boundary control u(t) at the right boundary of the string and to analyze the closed-loop stability of the system. Adaptive control is designed to handle the system parametric uncertainties, i.e., T, and M_s are unknown.

To stabilize the system given by governing Eq. (8) and boundary condition Eqs. (9) and (10), we propose the following robust adaptive control:

\[ u = -P\tilde{\Phi} - ku_a - \text{sgn}(u_a)d, \]  
(16)

where vector \( P = [-w'(L, t) \ w'(L, t)] \), parameter estimate vector \( \tilde{\Phi} = [T \ M_s]^T \), \( \text{sgn}(\cdot) \) denotes the signum function, \( k \) is a positive constant and the auxiliary signal \( u_a \) is defined as

\[ u_a = w(L, t) + w'(L, t). \]  
(17)

We define parameter vector \( \Phi \) and parameter error estimate vector \( \tilde{\Phi} \) as

\[ \Phi = [T \ M_s], \]  
(18)

\[ \tilde{\Phi} = \Phi - \hat{\Phi} = [\hat{T} \ \hat{M_s}]^T. \]  
(19)

After differentiating the auxiliary signal Eq. (17), multiplying the resulting equation by \( M_s \), and substituting Eq. (10), we obtain

\[ M_s\dot{u}_a = -T w'(L, t) + d + M_s w''(L, t) + u = P\tilde{\Phi} + d + u. \]  
(20)

Substituting Eq. (16) into Eq. (20), we have

\[ M_s\dot{u}_a = P\tilde{\Phi} - ku_a + d - \text{sgn}(u_a)d. \]  
(21)

The adaptation law is designed as

\[ \dot{\hat{\Phi}} = \Gamma P^T u_a - r\Gamma\tilde{\Phi}, \]  
(22)

where \( \Gamma \in R^{2 \times 2} \) is a diagonal positive-definite matrix and \( r \) is a positive constant. We define the maximum and minimum eigenvalue of matrix \( \Gamma \) as \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) respectively.

Remark 3: All the signals in the boundary control can be measured by sensors or obtained by a backward difference algorithm. \( w(L, t) \) can be sensed by a laser displacement sensor in the right boundary of the string. \( w'(L, t) \) can be measured by an inclinometer.

Remark 4: The control (16) is based on the distributed parameter model Eqs. (8) to (10), and the spillover problems associated with traditional truncated model-based approaches caused by ignoring high-frequency modes in controller and observer design are avoided.

Consider the Lyapunov function candidate

\[ V(t) = V_1(t) + V_2(t) + \Delta(t) + \frac{1}{2}\tilde{\Phi}^T\Gamma^{-1}\tilde{\Phi}, \]  
(23)

where the energy term \( V_1(t) \) and the auxiliary term \( V_2(t) \) and the small crossing term \( \Delta(t) \) are defined as

\[ V_1 = \frac{\beta}{2}\rho \int_0^L [\dot{w}]^2 dx + \frac{\beta}{2} T \int_0^L [w']^2 dx, \]  
(24)

\[ V_2 = \frac{1}{2} M_s u_a^2, \]  
(25)

\[ \Delta = \alpha \rho \int_0^L x w w' dx, \]  
(26)

where \( k \) is a control parameter and \( \alpha \) and \( \beta \) are two positive weighting constants. Note that the terms \( V_1(t) \) and \( V_2(t) \) are positive semi-definite while the term \( \Delta(t) \) is arbitrary.

Lemma 5: The Lyapunov function candidate given by Eq. (23), can be upper and lower bounded as

\[ \lambda_1(V_1 + V_2 + ||\tilde{\Phi}||^2) \leq V \leq \lambda_2(V_1 + V_2 + ||\tilde{\Phi}||^2), \]  
(27)

where \( \lambda_1 \) and \( \lambda_2 \) are two positive constants.

Proof: Substituting of Ineq. (11) into Eq. (26) yields:

\[ |\Delta| \leq \alpha_1 V_1, \]  
(28)

where

\[ \alpha_1 = \frac{2\alpha_2 \rho L}{\min(\beta\rho, \beta T)}. \]  
(29)

Then, we obtain

\[ -\alpha_1 V_1 \leq \Delta \leq \alpha_1 V_1. \]  
(30)

Considering \( \alpha \) is a small positive weighting constant satisfying \( 0 < \alpha < \frac{\min(\beta\rho, \beta T)}{2k_L} \), we can obtain

\[ \alpha_2 = 1 - \alpha_1 = 1 - \frac{2\alpha_1 \rho L}{\min(\beta\rho, \beta T)} > 0, \]  
(31)

\[ \alpha_3 = 1 + \alpha_1 = 1 + \frac{2\alpha_1 \rho L}{\min(\beta\rho, \beta T)} > 1. \]  
(32)

Then, we further have

\[ 0 \leq \alpha_2 V_1 \leq V_1 + \Delta \leq \alpha_3 V_1. \]  
(33)

Given the Lyapunov function candidate Eq. (23), we obtain

\[ 0 \leq \gamma_1(V_1(t) + V_2(t)) \leq V_1(t) + V_2(t) + \Delta(t) \leq \gamma_2(V_1(t) + V_2(t)), \]  
(34)

where \( \gamma_1 = \min(\alpha_2, 1) = \alpha_2 \) and \( \gamma_2 = \max(\alpha_3, 1) = \alpha_3 \) are positive constants. From the properties of matrix \( \Gamma \) and applying Lemma 3, we have

\[ \frac{1}{2\lambda_{\text{max}}} ||\tilde{\Phi}||^2 \leq \frac{1}{2} \tilde{\Phi}^T \Gamma^{-1} \tilde{\Phi} \leq \frac{1}{2\lambda_{\text{min}}} ||\tilde{\Phi}||^2. \]  
(35)

Combining Ineqs. (34) and (35), we have

\[ \lambda_1(V_1 + V_2 + ||\tilde{\Phi}||^2) \leq V \leq \lambda_2(V_1 + V_2 + ||\tilde{\Phi}||^2), \]  
(36)
where $\lambda_1 = \min(\gamma_1, \frac{1}{2\lambda_{\text{max}}})$ and $\lambda_2 = \max(\gamma_2, \frac{1}{2\lambda_{\text{min}}})$ are two positive constants.

Lemma 6: The time derivative of the Lyapunov function candidate Eq. (23) can be upper bounded with
\[
\dot{V}(t) \leq -\lambda V(t) + \psi,
\]
where $\lambda > 0$ and $\psi > 0$.

Proof: Differentiating Eq. (23) with respect to time leads to
\[
\dot{V}(t) = \dot{V}_1 + \dot{V}_2 + \dot{\Delta} + \dot{\Phi}^T\Gamma^{-1}\dot{\Phi}.
\]
The first term of the Eq. (38)
\[
\dot{V}_1 = A_1 + A_2,
\]
where
\[
A_1 = \beta\rho\int_0^L \dot{w}\dot{w}dx,
\]
\[
A_2 = \beta T\int_0^L w'\dot{w}'dx.
\]
Substituting the governing equation (8) into $A_1$, we obtain
\[
A_1 = \beta\int_0^L \dot{w}(Tw'' + f)dx.
\]
Using the boundary conditions and integrating Eq. (41) by part, we obtain
\[
A_2 = \beta T\int_0^L w'd\dot{w},
\]
\[
= \beta T\int_0^L w'(L, t)\dot{w}(L, t) - \beta T\int_0^L \dot{w}w''dx.
\]
Substituting Eqs. (42) and (43) into Eq. (39), we have
\[
\dot{V}_1 = \beta T\int_0^L w'(L, t)\dot{w}(L, t) + \beta\int_0^L f\dot{w}dx.
\]
Substituting the Eq. (17) into Eq. (44), and using Ineq. (12), we obtain
\[
\dot{V}_1 \leq -\frac{\beta T}{2} \left( [\dot{w}(L, t)]^2 + [w'(L, t)]^2 \right) + \frac{\beta T}{2} u_a^2
+ \beta\delta_1\int_0^L [w]^2dx + \frac{\beta}{\delta_1}\int_0^L f^2dx,
\]
where $\delta_1$ is a positive constant. Substituting Eq. (21) into the second term of the Eq. (38), we have
\[
\dot{V}_2 = M_s\dot{u}_u\dot{u}_a = -ku_u^2 + du_u - \text{sgn}(u_u)\ddot{u}_u + P\dot{u}_u.
\]
The third term of the Eq. (38)
\[
\dot{\Delta} = \alpha\int_0^L (x\dot{w}w' + x\dot{w}'w')dx
= \alpha\int_0^L xw'[Tw'' + f]dx + \alpha\rho\int_0^L x\dot{w}w'dx
= B_1 + B_2 + B_3,
\]
where
\[
B_1 = \alpha\int_0^L T\dot{w}w''dx,
B_2 = \alpha\int_0^L f\dot{w}w'dx,
B_3 = \alpha\rho\int_0^L x\dot{w}w'dx.
\]
After integrating Eq. (48) by parts and using the boundary conditions, we obtain
\[
B_1 = \alpha TL[w'(L, t)]^2 - \alpha T\int_0^L ([w']^2 + x\dot{w}'w'')dx.\]
Combining Eq. (48) and Eq. (51), we obtain
\[
B_1 = \frac{\alpha TL}{2}[w'(L, t)]^2 - \frac{\alpha T}{2}\int_0^L [w']^2dx.
\]
Using Ineq. (12), we obtain
\[
B_2 \leq \frac{\alpha L}{\delta_2}\int_0^L f^2dx + \alpha L\delta_2\int_0^L [w']^2dx,
\]
where $\delta_2$ is a positive constants. Integrating Eq. (50) by parts, we obtain
\[
B_3 = \alpha\rho L[w(L, t)]^2 - \frac{\alpha\rho}{2}\int_0^L ([\dot{w}]^2 + x\dot{w}'w')dx.
\]
The last term in Eq. (54) equals $B_3$, and we have
\[
B_3 = \frac{\alpha\rho L}{2}[w(L, t)]^2 - \frac{\alpha\rho}{2}\int_0^L [w]^2dx.
\]
Substituting Eqs. (52), (53) and (55) into Eq. (47) and using the boundary conditions, we obtain
\[
\dot{\Delta} \leq \frac{\alpha TL}{2}[w'(L, t)]^2 - \frac{\alpha T}{2}\int_0^L [w']^2dx + \frac{\alpha}{\delta_2}\int_0^L f^2dx
+ \alpha L\delta_2\int_0^L [w']^2dx + \frac{\alpha\rho L}{2}[w(L, t)]^2 - \frac{\alpha\rho}{2}\int_0^L [w]^2dx.
\]
Substituting Eqs. (45), (46) and (56) into Eq. (23), we obtain
\[
\dot{V} \leq -\left(\frac{\alpha}{2} - \beta\delta_1\right)\int_0^L [\dot{w}]^2dx - \left(\frac{\alpha T}{2} - \alpha L\delta_2\right)\int_0^L [w']^2dx
- \left(k - \beta T\right)u_a^2 - \left(\beta T - \frac{\alpha\rho L}{2}\right)[\dot{w}(L, t)]^2
- \left(\beta T - \frac{\alpha T L}{2}\right)[w'(L, t)]^2 + \left(\frac{\beta}{\delta_1} + \frac{\alpha L}{\delta_2}\right)\int_0^L f^2dx
+ \dot{\Phi}^T\left(P^T u_a + \Gamma^{-1}\dot{\Phi}\right) + \varepsilon,
\]
where the constants $k, \alpha, \beta, \delta_1$ and $\delta_2$ are chosen to satisfy the following conditions:

$$
\alpha < \min\left(\frac{\beta \rho}{2}, \frac{\beta T}{2} - \frac{\alpha \rho L}{2}ight),
\sigma_1 = \frac{\alpha \rho}{2} - \beta \delta_1 > 0,
\sigma_2 = \frac{\alpha T}{2} - \alpha \rho L > 0,
\sigma_3 = k - \beta \lambda T > 0,
\lambda_3 = \min\left(\frac{2 \sigma_1}{\beta}, \frac{2 \sigma_2}{\beta}, \frac{2 \sigma_3}{\beta \lambda T}ight) > 0,
\varepsilon = \left(\frac{\beta}{\delta_1} + \frac{\alpha L}{\rho} \right) \int_0^L f^2 dx \in L_\infty.
$$

Substituting Eq. (22) into Ineq. (57), we have

$$
\dot{V} \leq -\lambda_3 (V_1 + V_2) + r \dot{\Phi}^T \dot{\Phi} + \varepsilon
\leq -\lambda_3 (V_1 + V_2) + \frac{r}{2} ||\dot{\Phi}||^2 + \frac{r}{2} ||\Phi||^2 + \varepsilon
\leq -\lambda_4 (V_1 + V_2 + ||\dot{\Phi}||^2) + \frac{r}{2} ||\Phi||^2 + \varepsilon,
$$

where $\lambda_4 = \min(\lambda_3, \frac{r}{2})$ is a positive constant. Combining Ineqs. (34) and (58), we have

$$
\dot{V} \leq -\lambda V + \psi,
$$

where $\lambda = \lambda_4 / \lambda_2 > 0$ and $\psi = \frac{r}{2} ||\Phi||^2 + \varepsilon > 0$. □

With the above lemmas, we are ready to present the following stability theorem of the closed-loop string system.

**Theorem 1:** For the system dynamics described by (8) and boundary conditions (9), (10), under Assumption 1, and the boundary control Eq. (16), given that the initial conditions are bounded, the closed loop system is uniformly bounded (UB), and the state of the closed loop system $w(x,t)$ will remain in the compact set $\Omega$ defined by

$$
\Omega := \{ w(x,t) \in R \mid |w(x,t)| \leq D, \forall (x,t) \in [0, L] \times [0, \infty) \},
$$

where constant $D = \sqrt{\frac{2L}{\beta T \lambda_1} \left( V(0) + \frac{\psi}{\lambda} \right) }$.

**Proof:** Multiplying Eq. (37) by $e^{\lambda t}$ yields

$$
\frac{\partial}{\partial t} (Ve^{\lambda t}) \leq \psi e^{\lambda t}.
$$

Integrating the above inequality, we obtain

$$
V \leq \left( V(0) - \frac{\psi}{\lambda} \right) e^{-\lambda t} + \frac{\psi}{\lambda} \leq V(0) e^{-\lambda t} + \frac{\psi}{\lambda} \in L_\infty,
$$

which implies $V$ is bounded. Utilizing Ineq. (15) and Eq. (24), we have

$$
\frac{\beta}{2L} Tw^2(x,t) \leq \frac{\beta}{2} \int_0^L |w'(x,t)|^2 dx \leq V_1 + V_2 \leq \frac{1}{\lambda_1} V \in L_\infty.
$$

Appropriately rearranging the terms of the above inequality, we obtain $w(x,t)$ is uniformly bounded as follows

$$
|w(x,t)| \leq \sqrt{\frac{2L}{\beta T \lambda_1} \left( V(0)e^{-\lambda t} + \frac{\psi}{\lambda} \right) } \leq \frac{2L}{T \lambda_1} \left( V(0) + \frac{\psi}{\lambda} \right), \forall x \in [0, L].
$$

**Remark 5:** From Eq. (62), it is shown that the parameter estimation error $\dot{\Phi}$ is bounded $\forall t \in [0, \infty)$. From Eq. (63), we can state that $V_1$ and $V_2$ are bounded $\forall t \in [0, \infty)$. Since $V_1$ and $V_2$ are bounded, $w(x,t), w'(x,t)$ are bounded $\forall (x,t) \in [0, L] \times [0, \infty)$ and $u_\infty$ is bounded $\forall t \in [0, \infty)$. Then, we can obtain that potential energy Eq. (2) is bounded. Using Property 2, we can further obtain that $w''(x,t)$ and $w'''(x,t)$ are bounded. From the boundedness of $u_\infty$ and $w'(x,t)$ in Eq. (17), we can state that $\dot{w}(L,T)$ is bounded. Therefore, we can conclude that the kinetic energy of the system Eq. (1) is also bounded. Using Property 1, we can obtain $\dot{w}(x,t)$ and $\dot{w}'(x,t)$ are also bounded $\forall (x,t) \in [0, L] \times [0, \infty)$. Applying Assumption 1, Eq. (8) and the above statements, we can state that $\dot{w}(x,t)$ is also bounded $\forall (x,t) \in [0, L] \times [0, \infty)$. From the above information, it is shown that the proposed control Eq. (16) ensures all internal system signals including $w(x,t), w'(x,t), \dot{w}(x,t), \dot{w}'(x,t)$ and $\dot{w}''(x,t)$ are uniformly bounded. Since $\dot{\Phi}, \dot{w}'(x,t), \dot{w}'(x,t)$ and $\dot{w}''(x,t)$ are all bounded $\forall (x,t) \in [0, L] \times [0, \infty)$, and we can conclude the boundary control Eq. (16) is also bounded $\forall t \in [0, \infty)$.

**Remark 6:** It is shown that the increase in the control gain $k$ will result in a larger $\sigma_3$, which will lead a greater $\lambda_3$. Then the value of $\lambda$ will increase, which will reduce the size of $\Omega$ and produce a better vibration suppression performance. We can conclude that the bound of the system state $w(x,t)$ can be made arbitrarily small provided that the design control parameters are appropriately selected. However, increasing $k$ will bring a high gain control scheme. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.

**IV. NUMERICAL SIMULATIONS**

To illustrate the design procedure and test the control performance of the proposed method, we consider a string with $L = 1m$, $\rho = 0.1kg/m$, initially at rest $w(x,0) = x$ and $w'(x,0) = 0$, and then is excited by the time-varying disturbances $f(x,t)$ and $d(t)$.

The disturbance $d(t)$ on the tip payload is generated by the following equation.

$$
d(t) = 1 + 0.1 \sin(0.1t) + 0.3 \sin(0.3t) + 0.5 \sin(0.5t).
$$

The distributed disturbance $f(x,t)$ on the string is described as

$$
f(x,t) = [3 + \sin(\pi xt) + \sin(2\pi xt) + \sin(3\pi xt)] \times \frac{x}{1000}.
$$

Displacement of the string for free vibration, i.e., $u(t) = 0$, under disturbance is shown in Fig. 2. Displacement of the string with the proposed robust adaptive control (16), by choosing $k = 10$, $r = 0.001$ and $\Gamma = \text{diag}\{1, 1\}$, is shown in Fig. 3.

Figs. 2 and 3 illustrate that the proposed boundary control is able to stabilize the string at the small neighborhood of its equilibrium position. The corresponding boundary control
closed-looped stability under external disturbance has been proven based on the Lyapunov’s direct method. Numerical simulations have been provided to verify the effectiveness of the proposed boundary control.

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