Stochastic Properties of Switched Riccati Differential Equations

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Abstract—This paper studies switched Riccati differential equations, whose switching is driven by a Poisson-like random signal. First we show that the expected value of the escape time of a switched Riccati differential equation satisfies an integral equation and then give a sufficient condition for the equation to admit a unique solution. Then we study a switched version of so called extended Riccati differential equations, which are obtained by extending the domain of Riccati differential equations to the Grassmannian manifold. We show that the limiting distribution of the random walk given by the switched stochastic equation converges to a unique invariant measure exponentially fast. The theory of products of random matrices is used to derive this result. We do not require Riccati differential equations to be symmetric.

I. INTRODUCTION

This paper studies random walks that follow Riccati differential equations (RDEs) [13], [2]. Freiling [3] gives a recent treatment of RDEs from linear ordinary differential equations is that the solution of an RDE can diverge to infinity in a finite time. This is called finite escape phenomena, and has been studied by various authors [5], [10], [12].

It is well known [13], [2] that the finite escape phenomena can be avoided by extending the domain of RDEs to the Grassmannian manifold. Grassmannians are compactification of the domain of RDE, the vector space of matrices, and an RDE naturally corresponds to a flow on a Grassmannian induced by a one parameter semigroup. This flow is called an extended Riccati differential equation (ERDE).

This paper studies the finite escape phenomena of switched RDEs, where multiple RDEs are switched by a stochastic signal driven by a Poisson process. We show that the expected value of the escape time satisfies a certain integral equation and give a sufficient condition for the equation to admit a unique solution.

Then we study switched ERDEs. Utilizing the theory of the product of random matrices (see, e.g., [11], [4]), we show that, under a certain irreducibility condition, the distribution of the random walk given by an switched ERDE converges to a unique invariant measure exponentially fast. The switching signal is driven by a Poisson-like probability measure so that our random walk cannot be treated by the framework developed by [1], where a diffusion type random process on Grassmannians is studied.

This paper is organized as follows. After preparing necessary mathematical notation and conventions, in Section II we give a brief review on RDEs and Grassmannians. Section III studies the expected value of the first escape time of switched RDEs with a numerical example. Then Section IV studies random walks on Grassmannians given by switched ERDEs.

A. Notations and Conventions

Let \( \mathbb{R} \) denote the field of real numbers. For \( x > 0 \) let \( \log^+(x) := \max(\log(x), 0) \). By \( (\cdot, \cdot) \) we denote the usual inner product in \( \mathbb{R}^d \), which yields the Euclidean norm \( \|v\| := \sqrt{(v, v)} \) for \( v \in \mathbb{R}^d \). We also use another norm \( \|v\|_\infty := \max_{1 \leq j \leq n} |v_j| \).

For a matrix \( M \in \mathbb{R}^{m \times d} \), its maximal singular value is denoted by \( \|M\| \) and its column space is denoted by \( \text{Sp}M \). When \( M \) is square the trace of \( M \) is denoted by \( \text{tr}M \).

For a subspace \( W \) of \( \mathbb{R}^d \) define a subspace \( M(W) \subset \mathbb{R}^m \) by

\[
M(W) := \{Mw : w \in W\}.
\]

Let \( \text{Gl}(d, \mathbb{R}) \) denote the multiplicative group of invertible \( d \times d \) real matrices. Its subgroup consisting of those having determinant 1 is denoted by \( \text{Sl}(d, \mathbb{R}) \). Let \( S \) be a subset of \( \text{Sl}(d, \mathbb{R}) \). \( S \) is said to be strongly irreducible if there does not exist any finite family of proper linear subspaces \( V_1, \ldots, V_m \) of \( \mathbb{R}^d \) such that, for every \( M \in S \),

\[
M(V_1 \cup \cdots \cup V_m) = V_1 \cup \cdots \cup V_m.
\]

Also we say that \( S \) is contracting if there exists a sequence \( \{M_n\}_{n=1}^\infty \subset S \) such that \( M_n/\|M_n\| \) converges to a rank-one matrix as \( n \to \infty \). A measure \( \mu \) on \( \text{Sl}(d, \mathbb{R}) \) is said to have a finite exponential moment if there exists \( \tau > 0 \) such that

\[
\int_{\text{Sl}(d, \mathbb{R})} e^{\tau\|M\|} d\mu(M) \text{ is finite where } \|M\| := \max(\log^+ \|M\|, \log^+ \|M^{-1}\|).
\]

For a measure space \( X \) let \( L^\infty(X)^n \) be the space of \( \mathbb{R}^n \)-valued Lebesgue measurable functions \( f \) on \( X \) such that

\[
\sup_{x \in X} \|f(x)\|_{\infty} < \infty.
\]

\( L^\infty(X)^n \) is a Banach space equipped with the norm \( \|f\| := \sup_{x \in X} \|f(x)\|_{\infty} \). For a Banach space \( X \) let \( \mathcal{L}(X) \) denote the space of continuous linear operators on \( X \). \( \mathcal{L}(X) \) becomes a Banach space with the norm

\[
\|A\| := \sup_{x \neq 0} \frac{|Ax|}{\|x\|}.
\]

For the invertibility of the operators in the space \( \mathcal{L}(X) \) the next sufficient condition is well known.

Lemma 1.1: Let \( X \) be a Banach space and let \( A \in \mathcal{L}(X) \). If \( \|A\| < 1 \) then \( I - A \) is invertible in \( \mathcal{L}(X) \) and the inverse is given by \( \sum_{k=0}^\infty A^k \).
II. Riccati Differential Equations and Grassmannians

This section briefly reviews Grassmannians and their relationship to Riccati differential equations. Let $A \in \mathbb{R}^{d \times d}$ be arbitrary and let $k < d$. Partition $A$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where the sizes of $A_{11}, A_{12}, A_{21},$ and $A_{22}$ are $k \times k, k \times (d-k), (d-k) \times k,$ and $(d-k) \times (d-k)$, respectively. The Riccati differential equation associated with $A$ is defined by

$$\frac{dW}{dt} = A_{21} + A_{22}W - WA_{11} - WA_{12}W. \tag{1}$$

Notice that, since $A$ is arbitrary, this RDE is not required to be symmetric. Let $W(\cdot; W_0)$ denote the solution of this RDE with the initial state $W(0) = W_0$. The first time $t > 0$ when $W(t; W_0)$ does not exist is called the escape time of the RDE (1).

It is well known that RDEs have a close relationship to Grassmannians. In the rest of this section we quote some facts from [13]. Grassmanian $G^k_0(\mathbb{R}^d)$ consists of the set of all $k$-dimensional subspaces of $\mathbb{R}^d$. When $k = 1$ they are called real projective space, denoted by $P(\mathbb{R}^d)$. For a nonzero vector $v \in \mathbb{R}^d$ let us write

$$\psi := \langle v \rangle := \{ xv : x \in \mathbb{R} \} \in P(\mathbb{R}^d).$$

$G^k_0(\mathbb{R}^d)$ acts on $G^k(\mathbb{R}^d)$ since any invertible matrix maps a $k$-dimensional subspace to a $k$-dimensional subspace.

One of the canonical charts of Grassmannians connects RDEs and Grassmannians as follows. Define a mapping $\psi: \mathbb{R}^{(d-k)\times k} \rightarrow G^k(\mathbb{R}^d)$ by

$$\psi(K) = \text{Sp} \begin{bmatrix} I_k \\ K \end{bmatrix}$$

and let $G^k_0(\mathbb{R}^d) \subset G^k(\mathbb{R}^d)$ be the set of $k$-dimensional subspaces in $\mathbb{R}^d$ that are complementary to the subspace $\text{Sp} \begin{bmatrix} 0 \\ I_{d-k} \end{bmatrix}$. Then $\psi$ imbeds $\mathbb{R}^{(d-k)\times k}$ in $G^k(\mathbb{R}^d)$ as the open and dense subset $G^k_0(\mathbb{R}^d)$. In fact the pair $(G^k_0(\mathbb{R}^d), \psi^{-1})$ is one of the standard charts for the manifold $G^k(\mathbb{R}^d)$. Thus $G^k(\mathbb{R}^d)$ can be viewed as a compactification of $G^k_0(\mathbb{R}^d)$. Now one can find (see, e.g., [2]) that

$$\psi(W(t; W_0)) = e^{At} \psi(W_0), \tag{2}$$

whenever $W(t; W_0)$ exists. This means that RDE (1) is the local expression of the differential equation on $G^k(\mathbb{R}^d)$ corresponding to the flow

$$S(t; S_0) := e^{At} S_0 \tag{3}$$

with respect to the chart $(G^k_0(\mathbb{R}^d), \psi^{-1})$. By the extended Riccati differential equation (ERDE), we mean the differential equation on $G^k(\mathbb{R}^d)$ whose flow is given by (3). Therefore, abusing notation we write the solution of RDE (1) as

$$e^{At} W_0 := W(t; W_0).$$

Notice that, by the relationship (2), RDE (1) escapes precisely when $e^{At} \psi(W_0))$ leaves the subset $G^k_0(\mathbb{R}^d)$.

To fix ideas let us see an example. This example will be used later in the numerical example of the next section.

**Example 2.1.** Let $\omega > 0$ and consider the RDE

$$\frac{dw}{dt} = \omega (1 + w^2) \tag{4}$$

with the initial condition $w(0) = 0$. This RDE has the solution $w(t) = \tan(\omega t)$ so that the escape time of the RDE (4) equals $t = \pi/2\omega$.

The ERDE corresponding to the RDE (4) can be found as follows. To the domain $\mathbb{R}$ of the RDE (4) there corresponds the Grassmannian $G^1(\mathbb{R}^2)$, which is the projective space $P(\mathbb{R}^2)$. The canonical chart $\psi$ maps each $t \in \mathbb{R}$ to a straight line passing through the origin and having the slope $t$. In particular the initial state $w(0) = 0$ is mapped to the $x$-axis. With this canonical chart, a flow on $P(\mathbb{R}^2)$ escapes exactly when the flow coincides with the $y$-axis.

Since the RDE (4) is induced by the $2 \times 2$ matrix

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},$$

the corresponding ERDE is the flow

$$e^{At} \psi(W_0) = \left\{ \begin{array}{l} r \cos(\omega t) \\ r \sin(\omega t) \end{array} : r \in \mathbb{R} \right\},$$

which represents a line rotating counterclockwise with the angular speed $\omega$. It is easy to see that, starting from the $x$-axis, the flow escapes at the time $t = \pi/2\omega$, as we saw above.

III. The Escape Time of Switched RDEs

In this section we study the mean escape time of switched RDEs. First define a $\{0, 1\}$-valued random variable $z$ by the stochastic differential equation [7]

$$dz = (1 - 2z) dN, \quad z(0) \in \{0, 1\} \tag{5}$$

where $N$ is the Poisson process of rate $\lambda > 0$. This $z$ is our switching signal. Then let $A, B \in \mathbb{R}^{d \times d}$ and consider the switched RDE on $\mathbb{R}^{(d-k)\times k}$ defined by

$$dW = [z(A_1 + A_2 W - WA_{11} - WA_{12}W) + (1 - z)(B_{21} + B_{22} W - WB_{11} - WB_{12}W)] dt \tag{6}$$

where the matrices $A$ and $B$ are partitioned accordingly.

We are interested in calculating the expected value of the escape time of the switched RDE (6). Let $W_0 \in \mathbb{R}^{(d-k)\times k}$ be arbitrary and let $T_A(W_0)$ ($T_B(W_0)$) be the expected value of the escape time of the switched RDE (6) when $W(0) = W_0$ and $z(0) = 1$ ($z(0) = 0$, respectively). Also let $t_A(W_0)$ and $t_B(W_0)$ be the escape time of the RDEs (1) and

$$\frac{dw}{dt} = B_{21} + B_{22} W - WB_{11} - WB_{12} W$$

when $W(0) = W_0$.

We can derive a relationship between the functions $T_A$ and $T_B$ as follows. Temporarily assume $z(0) = 1$, i.e., the switched RDE starts from the RDE (1). If a switching does not occur after $t = 0$ then the solution eventually escapes
at \( t = t_A(W_0) \) by definition. On the other hand suppose that the first switching occurs at \( t = \tau \), which must be obviously less than \( t_A(W_0) \). At this time the state of the switched RDE equals \( e^{\alpha t_A}W_0 \) so that, by definition, the switched RDE will escape after \( T_B(e^{\alpha t_A}W_0) \) time units in mean. This argument yields the integral equation

\[
T_A(W_0) = \int_0^{t_A(W_0)} f(\tau)[(1 + F(t_A(W_0))) + (1 - F(t_A(W_0)))] d\tau \tag{7}
\]

where \( F \) is the probability distribution function

\[
F(t) := \int_0^t f(\tau) d\tau, \quad t \geq 0.
\]

The integral equation (7) yields the following functional equation

\[
T_A = g_1 + M_1 T_B \tag{8}
\]

where \( g_1 \) and \( M_1 T_B \) are the real valued functions on \( \mathbb{R}^{(d-k)\times k} \) defined by

\[
g_1(W) := \int_0^{t_A(W)} \tau f(\tau) d\tau + t_A(W) (1 - F(t_A(W))) \tag{9}
\]

and

\[
(M_1 T_B)(W) := \int_0^{t_A(W)} f(\tau) T_B(e^{\alpha \tau}W) d\tau. \tag{10}
\]

In the same way we can show that

\[
T_B = g_2 + M_2 T_A \tag{11}
\]

where \( g_2 \) and \( M_2 \) are defined in a similar manner as (9) and (10):

\[
g_2(W) := \int_0^{t_B(W)} \tau f(\tau) d\tau + t_B(W) (1 - F(t_B(W))),
\]

\[
(M_2 T_A)(W) := \int_0^{t_B(W)} f(\tau) T_A(e^{\alpha \tau}W) d\tau.
\]

Summarizing we have shown the following result.

**Theorem 3.1:** The functions \( T_A \) and \( T_B \) satisfy

\[
(I - M) \begin{bmatrix} T_A \\ T_B \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{12}
\]

where

\[
M := \begin{bmatrix} 0 & M_1 \\ M_2 & 0 \end{bmatrix}.
\]

**Proof:** Combine (8) and (11). \hfill \square

The next proposition gives a sufficient condition for the equation (12) to admit a unique solution and also gives a representation of the solution.

**Proposition 3.2:** Assume that there exists a constant \( t_0 > 0 \) such that

\[
t_i(W) \leq t_0 \tag{13}
\]

for every \( i = A, B \) and \( W \in \mathbb{R}^{(d-k)\times k} \). Then the equation (12) has a unique solution in \( L^\infty(\mathbb{R}^{(d-k)\times k}) \). Moreover the solution is given by the power series

\[
\begin{bmatrix} T_A \\ T_B \end{bmatrix} = \sum_{k=0}^{\infty} M^k \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{14}
\]

**Proof:** By Lemma 1.1, it is sufficient to prove the following claims

a) The mapping \( M \) is a continuous linear operator on \( L^\infty(\mathbb{R}^{(d-k)\times k}) \); b) \( \|M\| < 1 \).

The linearity of \( M \) is trivial. Let \( T, T' \in L^\infty(\mathbb{R}^{(d-k)\times k}) \) be arbitrary. Then

\[
\|M \begin{bmatrix} T \\ T' \end{bmatrix}(W)\|_\infty = \sup_{W \in \mathbb{R}^{(d-k)\times k}} \left\| M \begin{bmatrix} T \\ T' \end{bmatrix}(W) \right\|_\infty = \|T\| \sup_{W \in \mathbb{R}^{(d-k)\times k}} \left\| M \begin{bmatrix} T \\ T' \end{bmatrix}(W) \right\|_\infty.
\]

For every \( W \in \mathbb{R}^{(d-k)\times k} \) we have

\[
\left\| (M_1 T')(W) \right\| = \int_0^{t_A(W)} f(\tau) T'(e^{\alpha \tau}W) d\tau \leq \int_0^{t_A(W)} f(\tau) (T'(e^{\alpha \tau}W)) d\tau \leq \sup_{W \in \mathbb{R}^{(d-k)\times k}} \int_0^{t_B(W)} f(\tau) d\tau = \|T'\| F(t_0).
\]

In the same way we can show that

\[
\left\| (M_2 T)(W) \right\| \leq \|T\| F(t_0). \tag{17}
\]

The equation (15) together with (16) and (17) gives that

\[
\left\| M \begin{bmatrix} T \\ T' \end{bmatrix} \right\|_\infty \leq F(t_0) \left\| \begin{bmatrix} T \\ T' \end{bmatrix} \right\|_\infty = F(t_0) \left\| T' \right\|_\infty \tag{18}
\]

which shows that \( M \) is a continuous linear operator on \( L^\infty(\mathbb{R}^{(d-k)\times k}) \). Also the second claim b) follows immediately from (18) since \( F(t_0) < 1 \). This completes the proof. \hfill \square

**Remark 3.3:** The property of the probability distribution function \( F(t) \) that was used in the above proof is only that \( F(t_0) < 1 \) for every \( t_0 < \infty \). Thus the result of this section holds for any probability density function \( F(t) \) with this property. A sufficient condition for the property is that the support of the density function \( f \) is not bounded.

**Example 3.4:** Let \( \omega, \gamma > 0 \) be constants and consider the scalar switched RDE

\[
dw = \left[ z (\omega(1 + w^2)) + (1 - z) (-\gamma(1 + w^2)) \right] dt
\]

where \( z \) follows the stochastic differential equation (5). This switched RDE consists of the two RDEs (4) and

\[
\frac{dw}{dt} = -\gamma(1 + w^2). \tag{19}
\]
As in Example 2.1, we can see that those RDEs are the local expressions for the rotations in $G^1(\mathbb{R}^2)$ with the constant angular speeds $\omega$ and $-\gamma$. Hereafter we identify an element in $G^1(\mathbb{R}^2)$ with its angle $\theta$ measured from the positive $x$-axis. Then it is easy to see that

$$t_A(\theta) = \frac{(\pi/2) - \theta}{\omega},$$

$$t_B(\theta) = \frac{\theta + (\pi/2)}{\gamma}.$$ (20)

In particular, the assumption (13) is satisfied with $t_0 = \max\{\pi/\omega, \pi/\gamma\}$.

Fig. 1 shows the expected value $T_A$ of the escape time as a function of the initial angle $\theta_0 \in [-\pi/2, \pi/2]$ when we fix $\gamma = \lambda = 1$ and move $\omega$ as $\omega = 1$, 10, and 100. This graph is obtained by terminating the power series (14) at the 21st term. We can observe that the escape time decreases as $\omega$ increases. This is because, the more the angular speed $\omega$ is, the faster the line will rotate to pass the critical line of $y$-axis.

Then let us fix $\omega = \gamma = 1$ and move $\lambda$. Fig. 2 shows the expected value $T_A$ of the escape time when $\lambda = 0.01, 1, \text{ and } 10$. It is remarkable that, when $\lambda = 0.01$, the corresponding graph is close to the escape time (20) of the (deterministic) RDE (4). This is because, when the rate $\lambda$ is small, a switching from the first RDE (4) to the second RDE (19) rarely occurs, which implies that the effect of the presence of switching is small.

**Remark 3.5:** One of the difficulty in applying our method lies in computing the escape time of deterministic RDEs. Consider the RDE (1) on $\mathbb{R}^{1 \times 2}$ determined by the matrix

$$A = T^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} T, \quad T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with the initial state

$$W_0 = [1, 1].$$

We can show that the corresponding ERDE is the flow

$$e^{At} (\psi(W_0)) = \left\{ r \begin{bmatrix} 2 \cos t + \sin t - e^t \\ \cos t - 2 \sin t \\ e^t \end{bmatrix} : r \in \mathbb{R} \right\}$$

in $P(\mathbb{R}^3)$. Therefore, under the canonical chart $\psi$, this ERDE escapes when $2 \cos t + \sin t - e^t = 0$. However it is not easy to calculate the zeros of this type of equation effectively.

Though a method to compute a lower bound of escape times is proposed in [8], that method does not necessarily give accurate estimates because it is basically based on Lemma 1.1, which can be sometimes very conservative.

**IV. THE CONVERGENCE OF THE RANDOM WALK BY SWITCHED ERDES**

This section studies stochastic behaviors of switched ERDEs, not switched RDEs. The most remarkable difference from switched RDEs is the absence of finite escape phenomena, as reviewed in Section II.

Define

$$\Omega := \{1, 2, \ldots, N\} \times [0, \infty)$$

and let us introduce in $\Omega$ the product $\sigma$-algebra

$$\mathcal{P}(\{1, 2, \ldots, N\}) \times \mathcal{B},$$

where $\mathcal{P}(\cdot)$ denotes the power set and $\mathcal{B}$ is the smallest $\sigma$-algebra containing the open intervals in $[0, \infty)$. Finally let $\mu$ be the probability measure on $\Omega$ having the Poisson-like probability distribution function

$$p(k,t) = \frac{1}{N} e^{-\lambda t}.$$ (21)

In this section we study the switched ERDE on $P(\mathbb{R}^d) = G^2(\mathbb{R}^1)$ defined by

$$V_{n+1} = e^{A_{\lambda_0} t_0} (V_n), \quad V_0 \in P(\mathbb{R}^d),$$ (22)

where $(k_n, t_n) \in \Omega$ follows the probability measure $\mu$. Notice that this random walk can be expressed with the product of random matrices as

$$V_n = (e^{A_{\lambda_0} t_0} \cdots e^{A_{\lambda_0} t_1}) (V_0).$$
A. Preliminaries

This subsection collects some relevant facts about the convergence of the product of random matrices from [6], [9], [14]. Let \( \mu \) and \( \nu \) be probability measures on \( \text{Sl}(d, \mathbb{R}) \) and \( P(\mathbb{R}^d) \), respectively. Define a measure \( \mu \ast \nu \) on \( P(\mathbb{R}^d) \) by

\[
\int_{P(\mathbb{R}^d)} f d(\mu \ast \nu) := \int_{P(\mathbb{R}^d)} \int_{\text{Sl}(d, \mathbb{R})} f(M \cdot \bar{u}) \mu(M) d\nu(\bar{u})
\]

for every bounded Borel function \( f \) on \( P(\mathbb{R}^d) \). \( \nu \) is said to be \( \mu \)-invariant if

\[
\mu \ast \nu = \nu.
\]

For \( \alpha > 0 \) let \( C(\alpha) \) be the set of continuous real-valued bounded functions on \( P(\mathbb{R}^d) \) such that

\[
\sup_{\theta \neq \varphi} \frac{\| \phi(\bar{u}) - \phi(\bar{v}) \|}{\delta(\bar{u}, \bar{v})^{\alpha}} < \infty,
\]

where \( \delta \) is a metric in \( P(\mathbb{R}^d) \) defined by

\[
\delta(\bar{u}, \bar{v}) := \left| 1 - \left( \frac{\| \bar{u} \|}{\| \bar{u} \|}, \frac{\| \bar{v} \|}{\| \bar{v} \|} \right) \right|^{1/2} = | \sin \theta |
\]

with \( \theta \) being the angle between \( \bar{u} \) and \( \bar{v} \). Then it can be checked that \( C(\alpha) \) becomes a normed space with the norm

\[
\| \phi \|_C := \sup_{\bar{u} \in P(\mathbb{R}^d)} | \phi(\bar{u}) | + \sup_{\theta \neq \varphi} \frac{\| \phi(\bar{u}) - \phi(\bar{v}) \|}{\delta(\bar{u}, \bar{v})^{\alpha}}.
\]

The next proposition lists some facts about the measure induced by the products of random matrices.

**Proposition 4.1:** Let \( \mu \) be a probability measure on \( \text{Sl}(d, \mathbb{R}) \). Suppose that \( \mu \) has a finite exponential moment and that the closed smallest semigroup containing the support of \( \mu \) is strongly irreducible and contracting. Then the following conditions hold.

1) There exists a unique \( \mu \)-invariant measure \( \nu \) on \( P(\mathbb{R}^d) \) [6], [14].

2) There exist constants \( \alpha_0 > 0 \), \( c > 0 \), and \( 0 < \rho < 1 \) such that for \( 0 < \alpha \leq \alpha_0 \), the operators \( P \) and \( Q \) on \( C(\alpha) \) defined by

\[
(P\phi)(\bar{u}) := \int_{\text{Sl}(d, \mathbb{R})} \phi(M\bar{u}) d\mu(M),
\]

\[
(Q\phi)(\bar{u}) := \int_{P(\mathbb{R}^d)} \phi(\bar{v}) d\nu(\bar{v})
\]

are bounded and satisfy

\[
\| P^n - Q \|_C \leq c \rho^n
\]

for every \( n \geq 1 \) [9], [14].

Let \( S_0 := X_0 \cdots X_1 \) where each \( X_k \) follows the measure \( \mu \).

Since it can be shown [14] that

\[
E[\phi(S_n\bar{u})] = (P^n\phi)(\bar{u}),
\]

the bound (24) shows that the probability distribution of \( S_n\bar{u} \) exponentially converges to the invariant measure \( \nu \).

B. Main Result

Let us go back to the switched ERDE (22). Let \( A_k' := A_k - d^{-1}(\text{tr}A_k)I_d \). Since the multiplication by a nonzero constant acting on \( P(\mathbb{R}^d) \) is the identity, the actions of the matrices \( e^{At} \) and \( e^{At'} \) are the same as

\[
e^{At} \bar{u} = e^{At'} e^{-d^{-1}(\text{tr}A_k)t} \bar{u} = e^{At'} \bar{u}.
\]

Therefore, because \( \text{tr}A_k' = 0 \), without loss of generality we can assume that \( \text{tr}A_k = 0 \) for every \( k \). In particular, for every \( (k , t) \in \Omega \) we have

\[
e^{At} \in \text{Sl}(d, \mathbb{R})
\]

because \( e^{At} = e^{At'} = e^{(\text{tr}A_k)t} = e^0 = 1 \).

We place the following assumption on the matrices \( A_k \).

**Assumption 4.2:**

1) There does not exist any finite family of proper and distinct linear subspaces \( V_1, \ldots, V_m \) of \( \mathbb{R}^d \) such that, for every \( k \),

\[
A_k(V_1 \cup \cdots \cup V_m) \subset V_1 \cup \cdots \cup V_m.
\]  (25)

2) There exists \( k \) such that the matrix \( A_k \) has the unique eigenvalue having the largest real part among the eigenvalues of \( A_k \).

The next theorem is the main result of this section, which shows that the distribution of our random walk exponentially converges to a unique invariant measure.

**Theorem 4.3:** There exists a unique \( \mu \)-invariant distribution \( \nu \) on \( P(\mathbb{R}^d) \) for the random walk (22). Moreover there exist constants \( \alpha_0 > 0 \), \( c > 0 \), and \( 0 < \rho < 1 \) such that for \( 0 < \alpha \leq \alpha_0 \), the operators \( P \) and \( Q \) defined on \( C(\alpha) \) by (23) are bounded and satisfy (24) for every \( n \).

For the proof of this theorem we need the next proposition.

**Proposition 4.4:** The smallest closed semigroup generated by the support of \( \mu \) is strongly irreducible and contracting.

**Proof:** Let \( T \) be the smallest closed semigroup. First let us show that \( T \) is strongly irreducible. Suppose that \( T \) is not strongly irreducible. Then there exists a family of proper subspaces \( V_1, \ldots, V_m \) of \( \mathbb{R}^d \) such that, for every \( (k, t) \in \Omega \),

\[
e^{At}(V_1 \cup \cdots \cup V_m) = V_1 \cup \cdots \cup V_m.
\]

Considering the derivative at \( t = 0 \) we can show (25), which is impossible by the assumption. Hence the semigroup is strongly irreducible.

Then let us show that \( T \) is contracting. Let \( k \) be the integer in the second condition 2) of Assumption 4.2. It is sufficient to show that the sequence \( \{e^{At_n} / ||e^{At_n}\|\}_{n \geq 0} \) converges to a rank one matrix. Let \( r \) be the eigenvalue having the maximal real part among the eigenvalues of \( A_k \). Then there exists matrices \( T \) and \( J \) such that \( T \) is invertible, \( A = TJT^{-1} \), and \( J \) is of the form

\[
J = \begin{bmatrix} r & 0 \\ 0 & K \end{bmatrix}
\]

where \( K \) is a \((d-1) \times (d-1)\) matrix whose eigenvalues have real parts less than \( r \). Then it is easy to see that \( e^{At} / e^{At'} = T(e^{It} / e^{It'})T^{-1} \) converges to the rank one matrix
$M := T \text{diag}(1,0,\ldots,0)T^{-1}$ as $t \to \infty$. Therefore, by the continuity of the norm,

$$
\frac{e^{A_k n}}{\|e^{A_k n}\|} = \frac{e^{A_k n}}{\|e^{A_k n}\|} \to M
$$

as $n \to \infty$. This completes the proof.

Now we can prove Theorem 4.3.

**Proof of Theorem 4.3:** By Propositions 4.4 and 4.1, there exists a unique $\mu$-invariant distribution $\nu$ on $P(\mathbb{R}^d)$ for the random walk (22). To prove the latter part it is sufficient to show that the distribution $\mu$ has a finite exponential moment. Notice that $\ell(e^{\langle \lambda \ell \rangle}) \leq \|A_k\| t \leq \max_k\|A_k\| t$. Take sufficiently small positive $\tau$ such that $\tau(\max_k\|A_k\|) < \lambda$. Then

$$
\int_{\text{Sl}(d,\mathbb{R})} e^{\langle \lambda \ell \rangle} d\mu = \sum_{k=1}^{N} \int_{0}^{\infty} e^{\langle \lambda \ell \rangle} \frac{\lambda}{N} e^{-\lambda t} dt \leq \frac{\lambda}{N} \sum_{k=1}^{N} \int_{0}^{\infty} e^{\langle \max_k\|A_k\| \rangle} e^{-\lambda t} dt < \infty.
$$

Therefore $\mu$ has a finite exponential moment and this completes the proof.

**Remark 4.5:** It is only the exponential decay of the probability distribution function (21) that was essentially needed in the proof of Theorem 4.3, which means that the results of this section hold for many other distribution functions.

In the rest of this section we discuss how to check the conditions in Assumption 4.2. The second condition is easy to check. Though checking the first condition 1) is not trivial, when $d = 2$ we can give an easy to check sufficient condition, which will cover many cases.

**Proposition 4.6:** Suppose $d = 2$. Suppose that all the matrices $A_k$ are semi-simple and invertible. If the matrices $A_k$ do not share a common eigenvector then the condition 1) of Assumption 4.2 is satisfied.

The proof of this proposition needs the next lemma.

**Lemma 4.7:** Let $A \in \mathbb{R}^{d \times d}$ be a semi-simple matrix and let $n$ be an arbitrary positive integer. Then the set of eigenvectors of $A$ and $A^n$ are the same.

**Proof:** Since $A$ is semi-simple, $A$ has precisely $d$ distinct eigenvectors. Let $E = \{v_1, \ldots, v_d\}$ be the set of eigenvectors of $A$ corresponding to the set of eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$. Because $A^n v_k = \lambda_k^n v_k$, $v_k$ is an eigenvector of $A^n$ for every $k$. Therefore the set $E$ is contained in the set of eigenvectors of $A^n$. In fact those two sets are equal because a $d \times d$ matrix cannot have more than $d$ eigenvectors.

**Proof of Proposition 4.6:** Suppose that the condition 1) of Assumption 4.2 is false; i.e., there exists a family of proper and distinct subspaces $V_1, \ldots, V_m$ of $\mathbb{R}^2$ such that (25) holds for every $k$. We need to show that the matrices $A_k$ share a common eigenvector. Let us fix $k$. Since $d = 2$, all the proper subspaces $V_1, \ldots, V_m$ must be one-dimensional, and hence are spanned by nonzero distinct vectors $v_1, \ldots, v_m \in \mathbb{R}^2$, respectively. Also since $A_k$ is invertible, the inclusion in (25) must be an equality. Therefore we have

$$
A_k(\langle v_1 \rangle \cup \cdots \cup \langle v_m \rangle) = \langle v_1 \rangle \cup \cdots \cup \langle v_m \rangle.
$$

This means that $A_k$ acts transitively on the set $\{\langle v_1 \rangle, \ldots, \langle v_m \rangle\}$. Hence there exists $n$ such that $A_k^n(\langle v_1 \rangle) = (\langle v_1 \rangle)$, which implies that $v_1$ is an eigenvector of $A_k^n$.

By Lemma 4.7, $v_1$ is an eigenvector of $A_k$. Since $k$ was arbitrary, $v_1$ is a common eigenvector of the matrices $A_k$, $k = 1, \ldots, N$. This completes the proof.

**V. CONCLUSIONS AND DISCUSSION**

This paper studied some of the stochastic properties of switched RDEs and ERDEs. For switched RDEs, we showed that the mean escape time satisfies a certain integral equation and gave a power series representation for them. For switched extended RDEs, exponential convergence of the $n$-step probability distribution to the invariant measure was proved.

The following problems are still open:

1) This paper considered two themes, i.e., the escape times of the solutions of stochastically switched RDEs and the convergence property of the distributions on the stochastically switched ERDEs. However the relationship between them was not clarified.

2) As pointed out in Remark 3.5, our method to find the expected value of the escape time sometimes can fail to be applied by the difficulty of computing the escape time of deterministic RDEs. In such a case it would be needed to rely on numerical methods.

3) Though the existence of the unique invariant distribution was shown in Theorem 4.3, we have not given a method to compute the distribution.

**REFERENCES**


