Stabilization of Linear Input Delayed Dynamics Under Sampling
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Abstract—For continuous-time linear time-invariant systems with an arbitrarily large constant pointwise delay in the inputs, we propose a new construction of exponentially stabilizing sampled control laws. Stability is achieved under an assumption on the size of the largest sampling interval. The proposed design is based on an adaptation of the two main results of the reduction model approach. The stability of the closed loop systems is proved through a Lyapunov functional of a new type.

I. INTRODUCTION

Stabilizing linear dynamical systems with delay in the inputs is one of the fundamental problems of the control theory of systems with delay. It admits several solutions. Amongst them, there is the design of control laws based on the reduction technique, also known as finite spectrum assignment. This technique originates in [15], with the well known contributions that have followed in [23], [14], [12] and [1]. It applies in particular to any linear system of the form:

\[ \dot{x}(t) = Ax(t) + Bu(t - \tau), \tag{1} \]

where \( \tau > 0 \) is a pointwise constant delay, \( x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^p \) is the input. Thus, it provides with exponentially stabilizing feedbacks for (1), even in the difficult case where the system (1) with \( u \) identically equal to zero is exponentially unstable and the delay \( \tau \) is large enough for precluding the stabilization of this system with a state feedback of the form \( u(x(t - \tau)) \).

As explained for instance in [11, Section 2.6] and [25], the reduction technique leads to stabilizing control laws of the type:

\[ u(t) = K \left[ e^{TA}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}Bu(\ell)d\ell \right] \]

for the system (1). Two crucial features of these control laws are that they incorporate a term that depends on the past values of \( u \) and they are continuous functions of \( t \). Therefore they cannot be used when only piecewise constant control laws can be used. This is a drawback in the sense that, in many cases, only sampled control laws can be utilized, most notably in control over networks (see for instance [26], [24]). Overcoming this obstacle does not seem to be an easy task; generally speaking, it is a well-known fact that the problem of controlling a system with delay in the inputs with discontinuous feedbacks is a difficult problem. It is worth noting that only a few contributions are devoted to it [3].

These remarks motivate the direct study of the system defined by

\[ \dot{x}(t) = Ax(t) + Bu(t - \tau), \quad \forall t \in [t_i, t_{i+1}), \]

where \( \tau > 0, x \in \mathbb{R}^n \) is the state and \( u \in \mathbb{R}^p \) is the input and \( t_i \) is an increasing sequence such that there exist two constants \( \delta > 0, \varepsilon > 0 \) such that

\[ t_0 = 0, \varepsilon \leq t_{i+1} - t_i \leq \delta, \quad \forall i \geq 0. \tag{3} \]

For this system, some important results are available in the literature.

The system (2) in the case where \( \delta = \varepsilon \) is stabilized through a feedback of the form \( Kx(t) - \varepsilon \) in [7] and in the general case \( (\delta \neq \varepsilon) \) in [26] and [24]. The results of these contributions are delay dependent, i.e. they rely on an assumption on the size of \( \tau \).

The problem of stabilizing the system (2) in the case where \( \delta = \varepsilon \) and \( \tau \) is arbitrarily large admits at least two solutions. In [2, Chapt. 3], it is explained how it can be solved by constructing the zero-order-hold sampling of the system and adding extra variables. That way, a discrete-time system without delay is obtained, which can be stabilized using classical techniques. Notice that the size of the linear system obtained that way is proportional to \( \frac{\tau}{\varepsilon} \) and therefore it is very large when \( \tau \) is large and \( \delta \) is small. The second solution is proposed in the contribution [9], where the stabilization is achieved through a feedback of the form

\[ u_i = Mx(i\delta - \tau) + \sum_{p=1}^{i-1} N_p u_{i-p}. \]

In the present paper, we propose two new techniques of design of piecewise constant stabilizing control laws for (2). More precisely, we adapt two continuous-time stabilizing strategies developed for linear time invariant systems with a delay in the inputs, namely the classical reduction model approach recalled above and the dynamic extension approach of [22], to the case where the control is implemented through zero order holding devices. We recall that this second approach properly provides with control laws that are robust with respect to implementation errors, while control laws resulting from the classical reduction model approach may be not possess this desirable property [4]. We notice that, alternatively, the techniques of the contribution [18] can be used to perform safe implementation of distributed terms. In both cases, we show that the piecewise constant controllers that we propose result in exponentially stable
systems, provided the largest sampling interval is sufficiently small. To establish the results, we use Lyapunov-Krasovskii-like functionals \([6], [13]\) with the difference that they incorporate the state variable first derivative. For both strategies, we determine explicit bounds on the sampling period under which stability is preserved. Observe also that the technique of proof we propose neither leads to LMs that depend on the delay or the largest sampling period, as in \([5]\) nor relies on the introduction of extra state variables as in \([2, \text{Chapt. 3}]\).

The main advantages of our main results relative to \([7], [24]\) and \([26]\) is that they apply for any delay \(\tau \geq 0\). However, in contrast to \([7], [24]\), the control laws we obtain depend on a dynamic extension. The main differences of the designs we propose and the one of \([9]\) are the following: (i) our approach applies to the case of asynchronous sampling (i.e., \(\delta \neq \varepsilon\)), (ii) no sum of the type \(\sum_{p=1}^{l+1} N_{j_{p}} u_{-p}\) is present in the expression of the control laws we propose. The recent work \([10]\) is closer to the present paper than \([9]\); in \([10]\) asynchronous sampling is considered and a dynamic extension dependent on the input is used.

Regarding the results in \([5]\) and \([19]\), which are concerned with systems under sampled control, they may lead to extensions to systems with input delays satisfying some length restrictions. It is worth mentioning that our results are established under a restriction on the size of \(\delta\) relative to \(A, B\) and \(\tau\). This constraint is needed in the sense that the problem may admit no solution if \(\delta\) is larger than some threshold. However, the upper bound that we determine is probably smaller than the maximal admissible sampling period. This is a consequence of the Lyapunov approach we adopt, which offers some advantages, such as the possibility to estimate the rate of convergence of the solutions, but does not allow to determine the maximal admissible sampling period. Since the purpose of the present paper is to propose a stabilization technique that applies for any delay \(\tau \geq 0\) and not the estimation of the largest possible value for \(\delta\), we did not try to improve the upper bound for \(\delta\) that is deduced from our Lyapunov approach. However, improving this upper bound using, for instance, ideas borrowed from \([19]\) may be the subject of further studies.

The paper is organized as follows. In Section II we adapt to sampling the classical reduction model approach. In Section III, a similar result is established for the technique adopted in \([22]\). Some concluding remarks are given in Section IV.

**Notation and definitions.**

- We denote by \(I\) the identity matrix in \(\mathbb{R}^{n \times n}\).
- We denote \(|·|\) the Euclidean norm of matrices and vectors of any dimension.
- Let \(r\) and \(T\) be two positive integers. We denote \(C_{in} = C([-T, 0], \mathbb{R}^{r})\) the set of all continuous \(\mathbb{R}^{r}\)-valued functions defined on a given interval \([-T, 0]\). We denote \(C^{1} = C^{1}([-T, 0], \mathbb{R}^{r})\) the set of all continuously differentiable \(\mathbb{R}^{r}\)-valued functions defined on a given interval \([-T, 0]\).
- For a continuous function \(\varphi : [-\tau, +\infty) \rightarrow \mathbb{R}^{k}\), for all \(t \geq 0\), the function \(\varphi_{t}\) is defined by \(\varphi_{t}(\theta) = \varphi(t + \theta)\) for all \(\theta \in [-\tau, 0]\).
- The notation will be simplified whenever no confusion can arise from the context.

**II. New Reduction Model Approach**

The main result of this section relies on a dynamic extension and the classical Assumption 1 below. It is satisfied if and only if the pair \((A, B)\) is stabilizable.

**Assumption 1:** There exist a matrix \(K \in \mathbb{R}^{p \times n}\) and a symmetric positive definite matrix \(Q \in \mathbb{R}^{n \times n}\) such that the matrix inequality

\[
H^T Q + QH \leq -I
\]

with

\[
H = A + BK
\]

is satisfied. We proceed to state the following result:

**Theorem 1:** Assume that the system \((2)\) with the sequence \(t_{i}\) satisfying \((3)\), satisfies Assumption 1. Assume that the largest sampling period \(\delta > 0\) in \((3)\) is such that the inequality

\[
\delta \leq \min \left\{ \frac{1}{2\sqrt{6} |Qe^{A\delta}BK| |H|}, \frac{1}{2\sqrt{3} |e^{A\delta}BK|} \right\}
\]

is satisfied. Then the origin of this system is globally exponentially stabilized by the control law

\[
u(t_{i} - \tau) = \alpha(t_{i} - \tau),
\]

with, for all \(t \geq 0\),

\[
\alpha(t) = K \left[ e^{A t} x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)} B\alpha(\ell) d\ell \right]
\]

and \(\alpha(t) = \alpha_{0}(t)\) for all \(t \in [-\tau, 0]\), where \(\alpha_{0} \in C_{in}\) is any function such that

\[
\alpha_{0}(0) = K \left[ e^{A t} x(0) + \int_{-\tau}^{0} e^{-A\ell} B\alpha_{0}(\ell) d\ell \right].
\]

**Proof.** Let \(\phi_{t} \in C_{in}\) denote the initial condition of a solution \(x(t)\) of \((2)\) with the feedback \((7)-(8)-(9)\). One can prove easily that this solution is defined over \([0, +\infty)\).

Now, for the sake of simplicity, we introduce the notation:

\[
z(t) = e^{A t} x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)} B\alpha(\ell) d\ell
\]

for all \(t \geq 0\). We observe for later use that \((8)\) and \((10)\) give

\[
\alpha(t) = K z(t),
\]

for all \(t \geq 0\). Now, elementary calculations give, for all \(t \in [t_{i}, t_{i+1})\),

\[
\dot{z}(t) = e^{A t} Ax(t) + e^{A t} Bu(t_{i} - \tau) + A \int_{t-\tau}^{t} e^{A(t-\ell)} B\alpha(\ell) d\ell + B\alpha(t) - e^{A t} Ba(t - \tau)
\]

\[
= A z(t) + e^{A t} Bu(t_{i} - \tau) + B\alpha(t) - e^{A t} Ba(t - \tau)
\]

\[
= A z(t) + Ba(t)
\]

\[
+ e^{A t} B[u(t_{i} - \tau) - \alpha(t - \tau)].
\]
Using (11), we obtain for all $t \in [t_i, t_{i+1})$,
\[ \dot{z}(t) = A z(t) + B K z(t) + e^{A \tau} B [u(t_i - \tau) - \alpha(t - \tau)], \]  
(13)
where $H$ is the matrix defined in Assumption 1. Now, replacing, for all $i \in \mathbb{N}$, $u(t_i - \tau)$ by its expression given in (7), we obtain, for all $t \in [t_i, t_{i+1})$,
\[ \dot{z}(t) = H z(t) + e^{A \tau} B [\alpha(t_i - \tau) - \alpha(t - \tau)]. \]  
(14)
Using (11), we obtain, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ \dot{z}(t) = H z(t) + M [z(t_i - \tau) - z(t - \tau)], \]  
(15)
with $M = e^{A \tau} B K$. The next step of the proof consists in proving the exponential stability of origin of the system (15). To begin with, observe that, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ \dot{z}(t) = H z(t) - M \int_{t_i - \tau}^{t} \dot{z}(\ell) d\ell \]  
(16)
because $\dot{z}$ is a piecewise continuous function of $t$ over $[0, +\infty)$. Now, we analyze the stability of the system (15) via a Lyapunov approach and the representation (16). Let $V_1 : \mathbb{R}^n \to [0, +\infty)$,
\[ V_1(z) = z^T Q z, \]  
(17)
where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix such that (4) is satisfied. According to (16) and Assumption 1, the derivative of $V_1$ along the trajectories of (15) satisfies, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ \dot{V}_1(t) \leq -\frac{1}{2} |z(t)|^2 - 2 z(t)^T Q \int_{t_i - \tau}^{t} \dot{z}(\ell) d\ell. \]  
(18)
From the triangle inequality, we deduce that
\[ V_1(t) \leq -\frac{1}{2} |z(t)|^2 + 2 |Q M|^2 \int_{t_i - \tau}^{t} \dot{z}(\ell) d\ell. \]  
(19)
Using the fact that, for all $t \in [t_i, t_{i+1})$, $(t - \tau) - (t_i - \tau) \in [0, \delta)$, and the Cauchy-Schwarz inequality, we deduce that
\[ V_1(t) \leq -\frac{1}{2} |z(t)|^2 + 2 |Q M|^2 \delta \int_{t_i - \tau}^{t} |\dot{z}(\ell)|^2 d\ell. \]  
(20)
Since $\dot{z}$ is a piecewise continuous function of $t$ over $[0, +\infty)$, it follows that, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ \dot{V}_1(t) \leq -\frac{1}{2} |z(t)|^2 + 2 |Q M|^2 \delta \int_{t_i - \tau}^{t} |\dot{z}(\ell)|^2 d\ell. \]  
(21)
This inequality leads us to consider the functional $V_2 : C^1 \to [0, +\infty)$,
\[ V_2(\phi) = V_1(\phi(0)) + 3 |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t} \int_{0}^{\delta} |\dot{\phi}(\ell)|^2 d\ell d\ell d\ell. \]  
(22)
Along the trajectories of the system (15), for $t \geq \tau + \delta$,
\[ V_2(z(t)) = V_1(z(t)) + 3 |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t} \int_{0}^{\delta} |\dot{z}(\ell)|^2 d\ell d\ell d\ell. \]  
(23)
and the derivative of $V_2$ along the solutions of the system (15) satisfies, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ V_2(t) \leq -\frac{1}{2} |z(t)|^2 + 2 |Q M|^2 \delta \int_{t - \tau - \delta}^{t - \tau} |\dot{z}(\ell)|^2 d\ell \]  
+ 3 |Q M|^2 \delta^2 \int_{t - \tau - \delta}^{t - \tau} |\dot{z}(\ell)|^2 d\ell \]  
(24)
\[ + 3 |Q M|^2 \delta \int_{t_i - \tau}^{t_i - \tau - \delta} |\dot{z}(\ell)|^2 d\ell \]  
\[ - 3 |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell \]  
\[ \leq -\frac{1}{2} |z(t)|^2 - 1 |Q M|^2 \delta \int_{t_i - \tau}^{t_i - \tau - \delta} |\dot{z}(\ell)|^2 d\ell \]  
\[ + 3 |Q M|^2 \delta^2 |\dot{z}(t)|^2. \]  
(25)
Using (16) again, we deduce that, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ |\dot{z}(t)|^2 \leq 2 |H|^2 |z(t)|^2 + 2 |M|^2 \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell. \]  
(26)
Combining (24) and (26), we obtain, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ V_2(t) \leq -\frac{1}{2} |z(t)|^2 - |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell \]  
\[ + 6 |Q M|^2 \delta^2 |H|^2 |z(t)|^2 \]  
\[ + 6 |Q M|^2 \delta^2 |M|^2 \delta \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell. \]  
(27)
By grouping the terms, we obtain
\[ V_2(t) \leq -\frac{1}{2} |z(t)|^2 - \frac{1}{2} |Q M|^2 \delta^2 |H|^2 |z(t)|^2 \]  
\[ + |Q M|^2 \delta (1 + 6 \delta^2 |M|^2) \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell. \]  
(28)
From (6), we deduce that
\[ V_2(t) \leq -\frac{1}{2} |z(t)|^2 - \frac{1}{2} |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell. \]  
(29)
To determine an estimate of the rate of convergence of the solutions, we consider the functional $V_3 : C^1 \to [0, +\infty)$,
\[ V_3(\phi) = V_2(\phi) + \rho \int_{t_i - \tau - \delta}^{t_i - \tau} \int_{0}^{\delta} |\phi(\ell)|^2 d\ell d\ell d\ell \]  
(30)
where the constant $\rho > 0$ is to be selected later. Then along the trajectories of the system (15), for $t \geq \tau + \delta$,
\[ V_3(z(t)) = V_2(z(t)) + \rho \int_{t_i - \tau - \delta}^{t_i - \tau} \int_{0}^{\delta} |\dot{z}(\ell)|^2 d\ell d\ell d\ell \]  
(31)
and the derivative of $V_3$ along the solutions of the system (15) satisfies, for all $t \geq \tau + \delta$ and $t \in [t_i, t_{i+1})$,
\[ V_3(t) \leq -\frac{1}{2} |z(t)|^2 - \frac{1}{2} |Q M|^2 \delta \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell \]  
\[ + \rho (\tau + \delta) |\dot{z}(t)|^2 - \rho \int_{t_i - \tau - \delta}^{t_i - \tau} |\dot{z}(\ell)|^2 d\ell. \]  
(32)
By virtue of (26), we obtain, for all \( t \geq \tau + \delta \) and \( t \in [t_i, t_{i+1}) \)
\[
\dot{V}_3(t) \leq -\frac{1}{4}|z(t)|^2 - \frac{1}{2}|QM|^2 \delta \int_{t-\delta}^{t-\tau} |\ell|d\ell + 2\rho (\tau + \delta) |H|^2 |z(t)|^2 + 2\rho \delta (\tau + \delta) |M|^2 \int_{t-\delta}^{t-\tau} |\ell|d\ell - \rho \int_{t-\tau}^{t-\delta} |\ell|d\ell.
\]
(33)
Choosing
\[
\rho = \min \left\{ \frac{1}{16 (\tau + \delta)} |H|^2, \frac{|QM|^2}{8 (\tau + \delta)} \right\}
\]
we obtain
\[
\dot{V}_3(t) \leq -\frac{1}{8}|z(t)|^2 - \frac{1}{4}|QM|^2 \delta \int_{t-\delta}^{t-\tau} |\ell|d\ell - \rho \delta \int_{t-\tau}^{t-\delta} |\ell|d\ell.
\]
(34)
Since, for all \( \phi \in C^1 \),
\[
V_3(\phi) = \phi(0)^T Q \phi(0) + 3|QM|^2 \delta \int_{0}^{\tau - \delta} m_0 |\phi(\ell)|^2 d\ell + \rho \int_{0}^{0} |\phi(\ell)|^2 d\ell
\]
\[
\leq |Q||\phi(0)|^2 + [3|QM|^2 \delta + \rho] |\tau + \delta| \int_{0}^{0} |\phi(\ell)|^2 d\ell.
\]
(35)
We deduce that, for \( t \geq \tau + \delta \),
\[
V_3(t) \leq -gV_3(z(t))
\]
(36)
with \( g = \min \left\{ \frac{1}{8 |Q|}, \frac{|QM|^2 \delta + \rho}{|\tau + \delta|} \right\} > 0 \). Since \( V_3 \) is a nonnegative functional, by integrating we deduce that, for all \( t \geq \tau + \delta \),
\[
V_3(z(t)) \leq e^{-g(t-\tau-\delta)}V_3(z(\tau+\delta)).
\]
(37)
Since, for all \( \phi \in C^1 \), \( V_3(\phi) \geq V_3(\phi(0)) \), it follows that, for all \( t \geq \tau + \delta \),
\[
V_1(z(t)) \leq e^{-g(t-\tau-\delta)}V_3(z(\tau+\delta)).
\]
(38)
From the definition of \( V_3 \) and the fact that \( Q \) is a symmetric and positive definite matrix, we deduce that there exists a constant \( c > 0 \) such that, for all \( t \geq \tau + \delta \),
\[
|z(t)| \leq ce^{-\frac{g}{2} t} \sqrt{V_3(z(\tau+\delta))}.
\]
(39)
or, equivalently (see (10)), for all \( r \geq \tau + \delta \),
\[
|x(t)| + \int_{t-\tau}^{t} e^{A(t-\ell-\tau)} B \alpha(\ell)d\ell \leq ce^{-\frac{g}{2} t} \sqrt{V_3(z(\tau+\delta))}.
\]
(40)
It follows that, for all \( t \geq \tau + \delta \),
\[
|x(t)| + \int_{t-\tau}^{t} e^{A(t-\ell-\tau)} B \zeta(\ell)d\ell \leq ce^{-\frac{g}{2} t} \sqrt{V_3(z(\tau+\delta))}.
\]
(41)
We deduce that, for all \( t \geq \tau + \delta \),
\[
|x(t)| \leq ce^{-\frac{g}{2} t} \sqrt{V_3(z(\tau+\delta))} e^{A(t-\tau)} B \zeta(t) \int_{t-\tau}^{t} |\ell|d\ell.
\]
(42)
From (39), we deduce that, for all \( t \geq 2\tau + \delta \),
\[
|x(t)| \leq ce^{-\frac{g}{2} t} \sqrt{V_3(z(\tau+\delta))} + e^{A(t-\tau)} |BK| \tau e^{-\frac{g}{2} (t-\tau)} \sqrt{V_3(z(\tau+\delta))}.
\]
(43)
Thus \( x(t) \) converges exponentially to the origin when the time goes to the infinity with a decay rate larger than \( \frac{\tau}{2} \).

A. Some computational aspects

Assuming that, for all \( t \in [t_i, t_{i+1}) \) only the measurement \( x(t_i) \) is available, then the extension (8) cannot be used because the value of \( x(t) \) with \( t \in (t_i, t_{i+1}) \) is unknown. However, the following arguments can be used.

(i) First, for all \( t \in [t_i, t_{i+1}) \), the equality
\[
x(t) = e^{A(t-t_i)} x(t_i) + B^{t-t_i} u(t_i - \tau)
\]
is satisfied with
\[
B^r := \int_{0}^{\tau} e^{-A^r} B \zeta d\ell.
\]
Therefore, we can still stabilize the system by using (7) and replacing (8) by \( \alpha \) defined, for all \( t \in [t_i, t_{i+1}) \), by
\[
\alpha(t) = K \left[ e^{A(t-t_i)} x(t_i) + e^{A^r} B^{t-t_i} \alpha(t_i - \tau) \right] + K \int_{t_i - \tau}^{t} e^{A^r(t-\ell)} \zeta(\ell)d\ell.
\]
(44)
(ii) Determining a continuous function \( \alpha_0 \) such that (9) is satisfied is an easy task. For instance, let us choose a negative constant \( h \) such that the matrix
\[
J = I - K \int_{t_i - \tau}^{t} e^{-A^r} B \zeta d\ell
\]
is invertible and let
\[
\alpha_0(\ell) = e^{h(t)} J^{-1} K e^{A^r} x(0), \forall \ell \in [-\tau, 0].
\]
Then (9) is satisfied. Indeed, this equality holds if
\[
J^{-1} K e^{A^r} x(0) = K \left[ e^{A^r} x(0) + \int_{t_i}^{t} e^{-A^r} B \zeta d\ell \right].
\]
(45)
This equality holds if
\[
J^{-1} = I + K \int_{t_i - \tau}^{t} e^{-A^r} B \zeta d\ell.
\]
(46)
From the definition of \( J \), we deduce that this is the case.

(iii) From a practical point of view, to obtain a well-defined function \( \alpha \) of class \( C^1 \) over \( (0, +\infty) \), it may be worth solving the equation defined, for all \( t \in [t_i, t_{i+1}) \) by
\[
\dot{\alpha}(t) = K \left[ e^{A^r} x(t) + e^{A^r} B \zeta(t) \right] + K \left[ \int_{t_i}^{t} e^{A^r(t-\ell)} \zeta(\ell)d\ell \right] + K \left[ e^{A(t-t_i)} x(t_i) + \alpha(t_i - \tau) e^{A^r(t-t_i)} \right] + K \left[ \int_{t_i}^{t} e^{A^r(t-\ell)} \zeta(\ell)d\ell \right].
\]
(47)
III. NEW REDUCTION APPROACH BASED ON DYNAMIC EXTENSION

In this section, we adapt to the case of piecewise constant inputs the method of [22], which combines the model reduction approach with a dynamic extension to improve robustness of the implementation of the integral term involved in the design.

The assumption below ensures that the system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\beta(t), \\
\beta(t) &= A_f \beta(t) + B_f x(t),
\end{align*}
\]

(48)
is exponentially stable. This assumption is a classical assumption. When the pair \((A, B)\) is stabilizable, there always exist matrices \(A_f\) and \(B_f\) such that the system (48) is exponentially stable.

**Assumption 2:** The matrices \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times p}\) are such that there are constant matrices \(A_f \in \mathbb{R}^{p \times p}, B_f \in \mathbb{R}^{p \times n}\) and a symmetric positive definite matrix \(Q \in \mathbb{R}^{(n+p) \times (n+p)}\) such that the inequality

\[
H^T Q + QH \leq -I,
\]

(49)
is satisfied.

\[H = \begin{bmatrix} A & B \\ B_f & A_f \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+p)},\]

(50)

We are ready to state the following result:

**Theorem 2:** Assume that the system (2) with the sequence \(t_i\) satisfying (3) satisfies Assumption 2. Assume that the largest sampling period \(\delta > 0\) is such that the inequality

\[
\delta \leq \min \left\{ \frac{1}{2\sqrt{6} |QN||H|}, \frac{1}{2\sqrt{3}|N|} \right\},
\]

(51)

with \(H\) defined in (50) and

\[N = \begin{bmatrix} 0 & e^{A\tau}B \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+p)},\]

(52)
is satisfied. Then the origin of this system is globally exponentially stabilized by the feedback

\[u(t_i - \tau) = \beta(t_i - \tau),\]

(53)

\[
\dot{\beta}(t) = A_f \beta(t) + B_f \left[ e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell \right].
\]

(54)

**Remark.** In contrast to the dynamic extension (8), the \(\beta\)-subsystem does not need to satisfy a requirement similar to (9).

\[\dot{r}(t) = e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell\]

(55)

for all \(t \geq 0\). Then, for all \(t \in [t_i, t_{i+1})\),

\[
\dot{r}(t) = e^{A\tau}x(t) + e^{A\tau}Bu(t_i - \tau) + A \int_{t_i}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell + B\beta(t_i - \tau) - e^{A\tau}B\beta(t_i - \tau)
\]

(56)

\[
\dot{\beta}(t) = A_f \beta(t) + B_f \left[ e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell \right].
\]

(57)

Thus, we have, for all \(t \in [t_i, t_{i+1})\),

\[
\dot{r}(t) = A t + B\beta(t) + e^{A\tau}B[u(t_i - \tau) - \beta(t_i - \tau)],
\]

\[
\dot{\beta}(t) = B_f \left[ e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell \right].
\]

With the notation \(z = \begin{pmatrix} r \\ \beta \end{pmatrix}\) and

\[
R = \begin{bmatrix} e^{A\tau}B \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+p) \times p},
\]

we obtain

\[
\dot{z}(t) = Hz(t) + R[u(t_i - \tau) - \beta(t_i - \tau)],
\]

(58)

where \(H\) is the matrix in (50). With \(u(t_i - \tau)\) defined in (53), the system becomes

\[
\dot{z}(t) = Hz(t) + R[\beta(t_i - \tau) - \beta(t_i - \tau)],
\]

(59)

It follows that, for all \(t \geq \tau + \delta\) and \(t \in [t_i, t_{i+1})\),

\[
\dot{z}(t) = Hz(t) + N[z(t_i - \tau) - z(t_i - \tau)],
\]

(60)

with \(N\) defined in (52). Next, arguing as we did in Section II from (15), we can conclude.

**Remark.** In the context of sampled-data, one can solve the same stabilization problem by considering, instead of the dynamic extension (54) the system defined by

\[
\dot{\beta}(t) = A_f \beta(t) + B_f \left[ e^{A\tau}x(t_i) + \int_{t_i-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell \right] + B_f \left[ \int_{t_i-\tau}^{t} e^{A(t-\ell)}B\beta(\ell) d\ell \right],
\]

(61)

for all \(t \in [t_i, t_{i+1})\).

IV. CONCLUSION

By adapting two reduction model approaches, we solved the problem of stabilizing through piecewise constant control linear time-invariant systems with an arbitrary pointwise constant delay in the inputs. Much remains to be done. In particular, extensions to systems with several delays, time-varying delay, or distributed delays are expected as well as the treatment of nonlinear dynamics by taking advantage of the techniques of [16] and [20], [21], [8]. Moreover, we conjecture that robustness properties such as the input-to-state stability property with respect to additive terms can be established by combining the ideas of the construction of a Lyapunov-Krasovskii functional in [17] with the construction of the functional \(V_3\) proposed in the present paper.
REFERENCES