Abstract—The $\ell_2$-induced norm evaluation problem in finite-horizon for switched linear systems is considered in this paper. The contribution of the work is two-fold. First, two complete solutions are provided for the finite-horizon problem. The first solution is obtained based on a time-varying system approach, while the second one tries to characterize the worst-case switching law for the general case, which is state dependent. Second, optimization problems are formulated to identify the set of possible worst-case switching laws, based on which efficient algorithms are derived to solve the finite-horizon $\ell_2$-induced norm problem for general switched linear systems. The results obtained are demonstrated by a numerical example.

I. INTRODUCTION

Significant attention has been paid to quantify the robustness of dynamical systems. The main objective is to develop conditions so that the robustness measures can be evaluated. This makes great sense because modeling errors are inevitable in most design problems and almost all physical systems are subject to uncertainties and unknown disturbances.

Among others, the $\ell_2$-induced norm has been long regarded as an efficient measure of robustness in discrete time. For Linear Time-Invariant (LTI) systems, it is well known that the calculation of this norm can be done by iteratively testing the existence of stabilizing solutions to an upper-bound-dependent algebraic Riccati equation [1]. However, for switched linear systems, it was listed as one of the open problems in systems and control theory [2]. Despite of the remarkable advances in the study of switched linear systems [3], [4], [5], [6], results on this topic did not begin to appear until recently [7], [8], [9]. The potential difficulties involved in this problem include: 1) Solutions to the infinite-dimensional time-varying difference Riccati equations need to be considered; 2) Since the optimization parameters include both real-valued input signals and integer-valued switching laws, the optimization problem is nonconvex. For further details of the difficulties in this type of problems for switched linear systems, see also [10], [11]. For the results in the continuous-time context, see [12], [13], [14], [15], [16].

In [7], complete solutions to the infinite-horizon problem were characterized based on the finite-memory properties of solutions to the infinite-dimensional Riccati inequalities, and the $\ell_2$-induced norm could be approximately calculated by testing a sufficiently large number of linear matrix inequality (LMI) conditions. The generalized $\ell_2$-induced norm problem was considered in [8], and an algorithm was proposed to calculate the generalized $\ell_2$-induced norm based on the state-dependent generating functions; the computational complexity of which depended on the gridding resolution of the unit sphere. In [9], numerically solvable sufficient LMI conditions were provided to calculate the upper bound of the dwell time to ensure a specified $\ell_2$-induced norm.

An interesting topic in the study of the $\ell_2$-induced norm problems is to consider the finite-horizon truncations. The consequent benefits include: 1) The obtained results are numerically tractable; 2) Conditions on the upper bounds of the ‘truncated’ $\ell_2$-induced norm are necessary conditions for the upper bounds of the infinite-horizon norm [18]; 3) The problem of bounding the finite horizon $\ell_2$-induced norm of unstable systems can be dealt with, which is of practical interest for many problems [19]. In this paper, the finite-horizon $\ell_2$-induced norm evaluation problem is further considered; and both theoretical and computational properties of the solutions to this problem are explored. First, two complete solutions are provided for the finite horizon problem. The first solution is obtained by treating the switched system subject to a fixed switching law as a time-varying system and enforcing the Riccati equation conditions to all switching laws; while the second one is obtained by exploiting the worst-case switching law conditions for general switched linear systems, which is generally state-dependent. The difference between the proposed finite-horizon solutions and the infinite-horizon complete solutions in [7] is that the proposed results can be combined to develop numerically simpler algorithms, due to the benefits of the finite-horizon formulation and the underlying variational approach. Second, based on the insights of the obtained complete solutions, two algorithms are proposed to calculate the $\ell_2$-induced norm. The main step is to remove the non-worst-case switching laws, so that the Riccati equation conditions are only tested over a subset of the switching laws considered. Numerical results show that the potential worst-case-switching-law set is usually a small subset of the set of arbitrary switching signals, and that the number of elements in the set does not increase with the increase of the time horizon, therefore the computational complexity can be efficiently reduced.

The rest of the paper is organized as follows. Section II provides the problem formulation and some preliminary results. Complete solutions to the finite-horizon $\ell_2$-induced
norm problem are provided in Section III. In Section IV, the algorithms to compute the $\ell_2$-induced norm are presented. Section V presents a numerical example to illustrate the efficiency of the algorithms, followed by some concluding remarks in Section VI.

The following notation will be used. $[0, N]_{\text{DT}}$ denotes the discrete time interval $\{0, 1, 2, \ldots, N\}$, and $I$ denotes the identity matrix with context-dependent size. If $P$ and $Q$ are symmetric matrices, then $P \succ (\succeq) Q$ means $P - Q$ is positive definite (positive semi-definite).

II. Preliminaries

In this paper, a discrete-time switched linear system of the following form is considered,

$$x(k + 1) = A_v x(k) + B_v u(k), \quad y(k) = C_v x(k) + D_v u(k),$$

(1)

where $v := v(k)$ is the switching signal and takes values in $\{1, 2, \ldots, l\}$, $l$ being the number of modes. For mode $i$, the system matrices are denoted as $\{A_i, B_i, C_i, D_i\}$. It is assumed that $v$ belongs to $\mathcal{V}$, the set of arbitrary switching signals.

Define the $\ell_2$-norm of a discrete-time signal $u(\cdot)$ on $[0, N - 1]_{\text{DT}}$ as

$$||u||_{2,N} := \sqrt{\sum_{k=0}^{N-1} u^T(k) u(k)},$$

and define $\ell_2,N := \{ u : ||u||_{2,N} < \infty \}$. The $\ell_2$-induced norm of (1) over $[0, N - 1]_{\text{DT}}$ is defined by

$$\gamma^*(N) := \inf \{ \gamma \geq 0 : ||y||_{2,N} \leq \gamma ||u||_{2,N}, \forall u \in \ell_2,N, \forall v \in \mathcal{V} \},$$

where $y$ is the zero state output response of (1) with respect to input $u$ and switching signal $v$. Similarly, the $\ell_2$-induced norm of mode $i$ in (1) is defined by

$$\gamma^*_i(N) := \inf \{ \gamma \geq 0 : ||y||_{2,N} \leq \gamma ||u||_{2,N}, \forall u \in \ell_2,N \},$$

where $y$ is the zero state output response of mode $i$ with respect to $u$. Note that $\gamma^*(N) \geq \gamma^*_i(N)$ holds for all $i \in \{1, 2, \ldots, l\}$. When $N \to \infty$, $||\cdot||_{2,N}$ becomes $||\cdot||_2$, and $\ell_2,N$ becomes the usual $\ell_2$.

It is well known that the algebraic Riccati equations play an important role in the study of $H_2$ and $H_\infty$ norms of LTI systems. To analyze the $\ell_2$-induced norm of switched linear systems, the following Riccati difference equation is considered:

$$P(k) = S_\gamma(P(k + 1), v(k))$$

(2)

with terminal condition $P(N) = 0$, $v$ being the switching signal and

$$S_\gamma(P, v) := A^T_v PA_v + C^T_v C_v + (A^T_v PB_v + C^T_v D_v)\frac{[\gamma^2 I - D^T_v D_v - B^T_v PB_v]^{-1}}{(B^T_v PA_v + D^T_v C_v)}.$$

(3)

Definition 1: The Riccati equation in (2) is said to have a solution $P_v(\cdot)$ over $[s, N]_{\text{DT}} \subset [0, N]_{\text{DT}}$ if $\gamma^2 I - D^T_v D_v - B^T_v P_v(k)B_v > 0$ holds for all $k \in [s + 1, N]_{\text{DT}}$.

III. Complete Solutions to the finite-horizon $\ell_2$-induced norm evaluation problem

Although for certain systems, based on Proposition 1, it is possible to iteratively narrow down the interval in which the $\ell_2$-induced norm lies, the norm cannot be evaluated for arbitrary accuracy. In this section, complete solutions for general switched linear systems are provided. The first solution is obtained by enforcing the Riccati equation conditions to all switching laws; while the second one requires the existence of the common solutions to a set of Riccati difference inequalities, which reveals that the worst-case switching law is generally state-dependent.

Theorem 1: Fix $N \in \mathbb{Z}^+$ and $\gamma > 0$. If for any switching law $v \in \mathcal{V}$, the corresponding Time-Varying Riccati Difference Equation (TVRDE) $P(k) = S_\gamma(P(k + 1), v)$ with $P(N) = 0$ has a solution over $[0, N]_{\text{DT}}$, then $\gamma^*(N) \leq \gamma$; otherwise, $\gamma^*(N) \geq \gamma$.

Proof: Fix $v \in \mathcal{V}$, and consider the problem of maximizing

$$J(u) := \sum_{k=0}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t)$$

along the solution of (1), with $x(0) = x_0$. Suppose the corresponding TVRDE has a solution over $[s, N]_{\text{DT}}$. Using the convention that $\sum_{i=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) = 0$, for $k \in [s, N]_{\text{DT}}$, define $V(x, k) := x^T P(k)x$, and

$$m(k) := V(x(k), k) + \sum_{t=s}^{k-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t).$$

(4)

Then for $k \in [s, N - 1]_{\text{DT}},$

$$m(k + 1) - m(k) = V(x(k + 1), k + 1) - V(x(k), k) + y^T(k)y(k) - \gamma^2 u^T(k)u(k) = -[u(k) - u^*(k)]^T R[u(k) - u^*(k)] \leq 0,$$

(5)

where $R = \gamma^2 I - D^T_v D_v - B^T_v P(k + 1)B_v > 0$, and

$$u^*(k) := [\gamma^2 I - D^T_v D_v - B^T_v P(k + 1)B_v]^{-1} (B^T_v P(k + 1)A_v + C^T_v D_v)x(k).$$

Therefore $m(N) \leq m(s)$ and the equality holds for $u(k) = u^*(k)$. Since $P(N) = 0$, we have

$$V(x(k), k) = m(k) - m(N) + \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t).$$

(6)

By combining the monotonicity of $m(k)$ and Eq. (6),

$$V(x_0, k) = x_0^T P_v(k)x_0 = \sup_{u \in \ell_2,N - k} \{ \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) : u \leq \ell_2,N - k, x(k) = x_0 \}.$$

(7)
for all $k \in [s, N]_{\text{DT}}$

If for all $v \in V$, the TVDRE has a solution over $[0, N]_{\text{DT}}$, then taking $s=0$ and $k=N$ in (4), and using the fact that $x(0)=0$ and $P_s(N,0)=0$, we get

$$m(N) = V(x(N),N) + \sum_{t=0}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) = J(u) \leq m(0) = J(u^*) = 0.$$ 

Since the above relationship holds for all $v \in V$, we have $\gamma^*(N) \leq \gamma$.

If there exists $v \in V$, such that $\gamma^2 I - D^T_z D_s - B^T_z P(s)B_z > 0$ no longer holds for some $s \in [0,N]_{\text{DT}}$, there exists a finite nonzero $u(s-1)$ such that

$$u^T(s-1)B^T_z P(s)B_z u(s-1) - u^T(s-1)D^T_z D_s u(s-1) - \gamma^2 u^T(s-1)u(s-1) \geq 0.$$ 

Letting $u(k)=0$ for $k=0,1,...,s-2$, we have $x(k)=0$, $y(k)=0$, for $k=0,1,...,s-2$, $x(s)= B_z u(s-1)$, and $y(s-1) = D_s u(s-1)$. Combining the above discussion with Eq. (7), this shows that there exist switching signal $v$ and nonzero input $u$ such that $J(u) \geq 0$. As a result, $\gamma^*(N) \geq \gamma$.

Remark 1: According to the above result, the $\ell_2$-induced norm could be estimated iteratively for any precision by solving the TVRDE along all possible switching sequences. Unlike Proposition 1, this result can be applied to general switched linear systems, but the cost is that much more computational effort is needed. In addition, this result can be generalized to the infinite-horizon case based on the results for time-varying systems in [17]. The resulting conditions will become infinite dimensional and can only be approximately tested by making the horizon large enough. However, for periodic switched linear systems, this problem is solvable by checking the existence of stabilizing solutions for a finite number of Riccati difference equations, using the discrete-time lifting approach [20].

Remark 2: In Proposition 1, the worst-case switching law and input can be explicitly constructed. As will be shown later, for general $x(0) \neq 0$, it is also possible to construct the worst-case switching law and input that maximize $J(u,v)$, by making $v^*$ state-dependent, i.e., $v^* = v^*(x)$. However, since the worst-case input is in the state-feedback form, the worst case switching law cannot be identified when $x(0)=0$, which is the case in the $\ell_2$-induced norm estimation problem. As a result, all possible switching laws need to be tested.

The above condition requires the TVRDE induced by each switching law to have a solution over $[0, N]_{\text{DT}}$: such solutions are normally different from each other for different switching laws. The following result presents another complete solution by requiring the existence of a common state and time dependent solution to a set of Riccati difference inequalities.

**Theorem 2:** Fix $N \in \mathbb{Z}^+$ and $\gamma > 0$, if there exists a matrix function $P(x,k) = P^T(x,k)$: $\mathbb{R}^{n \times n}$ that satisfies

$$P(x(k),k) \geq C^T_z C_z + A^T_z P(x(k+1),k+1)A_z + [A^T_z P(x(k+1),k+1)B_z + C^T_z D_z] \cdot [\gamma^2 I - D^T_z D_s - B^T_z P(x(k+1),k+1)B_z]^{-1} \cdot [B^T_z P(x(k+1),k+1)A_z + D^T_z C_z],$$

for $k \in [0, N-1]_{\text{DT}}$ and $z \in \{1,2,...,l\}$, then $\gamma^*(N) \leq \gamma$; otherwise, $\gamma^*(N) \geq \gamma$.

**Proof:** 1) The first part of the result can be proved by defining $V(x(k),k) := x(k)^T P(x(k),k)x(k)$ and following a similar argument as in the proof of Theorem 1 in [9], which leads to $\gamma^*(N) \leq \gamma$.

2) On the other hand, suppose there does not exist a matrix function that satisfies Eq. (8) but $\gamma^*(N) < \gamma$. From the proof of Theorem 1, we know that for all $k \in [0, N-1]_{\text{DT}}$, $z \in \{1,2,...,l\}$, $x_0 \in \mathbb{R}^n$, and any given switching law $v \in V$ on $[k, N-1]_{\text{DT}}$,

$$x_0^T P_v(k) x_0 = \sup \{ \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) : u \in \ell_{2,N-k}, x(k)=x_0 \},$$

and $R := \gamma^2 I - D^T_z D_s - B^T_z P_v(k+1)B_z > 0$, where $P_v(k)$ is the solution to the TVRDE induced by $v$ subject to $P_v(N)=0$. For $x(k)=x_0$, let us define

$$V(x(k),k) := x(k)^T P(x(k),k)x(k) = \max \{ x(k)^T P_v(k)x(k) \} = \max \sup \{ \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) : u \in \ell_{2,N-k}, x(k)=x_0 \}. $$

For $k \in [0, N-1]_{\text{DT}}$, we have

$$V(x(k),k) = x(k)^T P(x(k),k)x(k) \leq \max \sup \{ \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) \} = \max \sup \{ \sum_{t=k}^{N-1} y^T(t)y(t) - \gamma^2 u^T(t)u(t) \}.$$

(10)

where $u^*(k) = [\gamma^2 I - D^T_z D_s - B^T_z P_v(k+1)B_z]^{-1} (B^T_z P_v(k+1)A_z + D^T_z C_z) x(k)$, and the inequality is obtained by fixing $v(k) = \gamma, z \in \{1,2,...,l\}$ and choosing the remaining portion of the switching signal as an arbitrary $v \in V$, which is independent of $x(k+1)$. Since
the above relationship holds for all \( P_k(k+1), v \in V \), and
\[
P(x(k+1), k+1) = \arg \max_{P_k(k+1)} \{x^T(k+1)P_k(k+1)x(k+1)\},
\]
we have
\[
x^T(k)P(x(k), k)x(k) \\
\geq x^T(k)\{ A^T_2 P(x(k+1), k + 1)A_z + C^T_z C_z \\
+ [\gamma^2 I - D^T_z D_z - B^T_z P(x(k+1), k + 1)B_z]^{-1} \}
[ B^T_z P(x(k+1), k + 1)A_z + D^T_z C_z \} x(k).
\]
Since \( x(k) = x_0 \) is arbitrarily chosen, we have constructed a solution for (8), which is contrary to the assumption that no matrix function \( P(x, k) \) satisfies Eq. (8), and thus the conclusion follows.

**Remark 3:** Although the conditions in this result are not straightforward to test, it provides insights into the \( \ell_2 \)-induced norm problem: 1) Eqs. (11) and (12) show that the worst-case switching law is state-dependent for general switched linear systems, while the corresponding input is still in the same state-feedback form; 2) to compute the \( \ell_2 \)-induced norm, it suffices to test the conditions on the set of worst-case switching laws, which is a subset of the whole set of switching signals. Based on this point, computational friendly algorithms are proposed in the following section by computing the solutions to the TVRDEs corresponding to the potential worst-case switching laws, over which the conditions in Theorem 1 are tested. In addition, based on the results in Claim 5 of [17], it is also possible to generalize this result to the infinite-horizon case, which is the discrete-time counterpart to the results obtained for continuous-time switched linear systems in [16], [15]. Furthermore, for first-order switched linear systems, simple algebraic conditions can be proposed for the infinite-horizon problem. The details are not provided here since they are not in the scope of the paper.

### IV. EFFICIENT ALGORITHMS FOR THE FINITE HORIZON PROBLEM

According to Theorem 1, to estimate the \( \ell_2 \)-induced norm, TVRDEs need to be solved for all possible switching sequences, which is computationally unacceptable, even when the horizon and the number of modes are not very large. In this section, efficient algorithms are developed so that this problem can be solved at a reasonable cost. The basic idea is to remove the redundant solutions that can never be the ‘worst-case’ ones, so that the Riccati equation conditions are only tested on a subset of the considered switching laws. Since the notations of the potential worst-case switching laws are not computationally essential, they are not explicitly denoted in this section, instead the solutions to the corresponding TVRDEs are employed to represent them implicitly. The following definitions are similar to those given in [21], where the LQR problem for switched linear systems is studied.

**Definition 2:** Consider a set of positive semidefinite matrices \( P := \{P_1, P_2, ..., P_M\} \). A matrix \( P_i \) is said to be redundant if for all \( x \in \mathbb{R}^n \), there exists a \( P_j \in \mathcal{P}\setminus\{P_i\} \) such that \( x^TP_jx \leq x^TP_ix \).

**Definition 3:** Let \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) be two sets of positive semidefinite matrices. The set \( \mathcal{P} \) is said to be equivalent to \( \hat{\mathcal{P}} \), if for all \( x \in \mathbb{R}^n \), \( \max_{P \in \mathcal{P}} x^TPx = \max_{\hat{P} \in \hat{\mathcal{P}}} x^T\hat{P}x \). Furthermore, \( \hat{\mathcal{P}} \) is said to be the minimum equivalent set of \( \mathcal{P} \) if it is the equivalent set of \( \mathcal{P} \) with the fewest elements.

The following lemma provides a necessary and sufficient condition for a solution to be redundant, which results in a non-convex optimization problem.

**Lemma 1:** Consider a set of positive semidefinite matrices \( \mathcal{P} := \{P_1, P_2, ..., P_M\} \). \( P_i \) is redundant if and only if the optimization problem
\[
\begin{align*}
\max_x & \quad x^TP_ix \quad \text{s.t.} \quad x^TP_jx \leq 1, \forall j \in \{1, 2, ..., M\}\setminus\{i\}.
\end{align*}
\]
is feasible and the optimal value is no greater than 1.

**Proof:** 1) Necessity: Suppose \( P_i \) is redundant. For all \( x \in \mathbb{R}^n \), there exists \( P_j \) such that \( x^TP_jx \geq x^TP_ix \). If there exists a feasible \( y \in \mathbb{R}^n \) such that \( y^TP_jy \geq 1 \), then there exists \( P_j \) with \( j \neq i \) such that \( y^TP_jy \geq y^TP_y > 1 \), which is contrary to the feasibility assumption of \( y \). Therefore the optimization problem (14) cannot have an optimal value greater than 1.

2) Sufficiency: Consider an arbitrary nonzero \( x \in \mathbb{R}^n \). Define
\[
\pi := \max\{x^TP_1x, x^TP_2x, ..., x^TP_{1-x}, x^TP_{1-x}, ..., x^TP_{Mx}\}.
\]
Define \( y := x/\sqrt{\pi} \), and we have \( y^TP_jy \leq 1 \) with \( j \neq i \) and at least one constraint active. Since the optimal solution to problem (14) is no greater than 1, there exists at least one \( k \neq i \) such that \( y^TP_1y \leq y^TP_ky = 1 \). As a result, for all \( x \in \mathbb{R}^n \), there exist a \( j \neq i \) such that \( x^TP_jx \leq x^TP_ix \), and thus \( P_i \) is redundant.

Based on this result, all redundant solutions at each iteration can be removed by solving a set of optimization problems. Due to the nonconvex nature of the problems, Genetic Algorithms (GAs) can be used to find an almost-optimal solution. The algorithm is summarized as follows.

**Algorithm 1:**
1. Give an upper bound \( \bar{\gamma} \), a lower bound \( \gamma \), the final time \( N \), and a precision parameter \( \epsilon \).
2. Let \( \gamma = (\bar{\gamma} + \gamma)/2 \), \( \mathcal{P}_N = \{0\} \), and \( k = N - 1 \).
   a. Calculate \( \hat{\mathcal{P}}_k \) by evaluating the one-step solution to the TVRDE for all possible switching signals subject to all \( P_{k+1} \in \mathcal{P}_{k+1} \).
   b. If there exist \( P_{k+1} \) and \( z \in \{1, 2, ..., l\} \) such that \( \gamma^2 I - D^T_z D_z - B^T_z P_{k+1}B_z > 0 \) no longer holds, then \( \gamma = \gamma \), and go to 2).
   c. Find the minimum equivalent set \( \hat{\mathcal{P}}_k \) of \( \mathcal{P}_k \) by solving problem (14) for each element in \( \mathcal{P}_k \).
   d. \( \mathcal{P}_k = \hat{\mathcal{P}}_k \).
According to Lemma 2 using LMI tools.

3) If $\bar{\gamma} - \gamma \geq \epsilon$, repeat step 2.
4) $\gamma^*(N) = (\bar{\gamma} + \gamma)/2$.
5) End.

The above algorithm has the potential to remove all possible redundant solutions at each iteration, although the GAs employed to solve the non-convex optimization problems can be time-consuming. On the other hand, following a similar idea as in [21], a sufficient condition can be proposed to test the redundancy of the solutions, which can be implemented using standard LMI tools [22].

Lemma 2: Consider a set of positive semidefinite matrices $\mathcal{P} := \{P_1, P_2, ..., P_M\}$. $P_i$ is redundant if there exist non-negative $\alpha_1, \alpha_2, ..., \alpha_M$ that satisfy $\alpha_i = 1$, $\sum_{k=1, k\neq i}^{M} \alpha_k = 1$ and $P_i - \sum_{k=1, k\neq i}^{M} \alpha_k P_k \preceq 0$.

Proof: The proof is straightforward and thus is omitted here.

Based on the above result, a similar algorithm can be derived. The difference between the new algorithm and Algorithm 1 only lies in Step 2.c, which is described as follows:

Algorithm 2: 2.c) Test the redundancy of each matrices in $\mathcal{P}_{k+1}$ according to Lemma 2 using LMI tools.

Remark 4: In spite of the difficulties to provide an upper bound for the number of elements in $\mathcal{P}_k$, extensive simulation results show that the size of $\mathcal{P}_k$ in the above algorithms does not increase exponentially with the increase of $k$. Another observation is that the size of $\mathcal{P}_k$ in Algorithm 2 is usually larger than that of Algorithm 1, since Lemma 2 is only a sufficient condition.

V. NUMERICAL EXAMPLES

Example 1: Consider the second-order discrete-time switched linear system in the form of system (1) with $l = 3$,

$$
\begin{align*}
A_1 &= \begin{bmatrix} 0.25 & 0.13 \\ 0.15 & 0.4 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1.5 & 1.0 \\ 1.5 & 1.0 \end{bmatrix}, & D_1 &= \begin{bmatrix} 0.1 & 0.02 \\ 0.0 & 0.1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} 0.2 & 0.12 \\ 0.15 & 0.45 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1.0 & 0.7 \\ 1.0 & 0.7 \end{bmatrix}, & D_2 &= \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.3 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} 0.2 & 0.1 \\ 0.15 & 0.4 \end{bmatrix}, & B_3 &= \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}, \\
C_3 &= \begin{bmatrix} 1.2 & 0.2 \\ 0.6 & 0.1 \end{bmatrix}, & D_3 &= \begin{bmatrix} 0.0 & 0.1 \\ 0.2 & 0.0 \end{bmatrix}.
\end{align*}
$$

First, the evaluation of the $\ell_2$-induced norm over finite horizon $N = 10$ using different algorithms is demonstrated. Three algorithms are compared, including Algorithm 1 and Algorithm 2, and the one testing all possible switching laws, which will be named Algorithm 3 hereafter. All these algorithms yield the same $\ell_2$-induced norm, which equals to 11.1537, while the $\ell_2$-induced norms over $[0, 9]_{DT}$ of the modes equal to 11.1527, 6.109, 4.3039, respectively. However, the computational effort required is dramatically different. To show this, the numbers of solutions to the Riccati equation tested at each iteration of the solution procedure for each algorithm are shown in Fig. 1, where it is apparent that the number of solutions grows exponentially for Algorithm 3, while those of Algorithms 1 and 2 remain small within the whole time horizon. As has been mentioned in Remark 4, the number of solutions at each iteration in Algorithm 1 is no greater than that of Algorithm 2. Also notice that though Algorithm 1 is derived based on a stronger result, it requires greater computational effort as a genetic algorithm is employed to solve the non-convex optimization problem in (14). On the other hand, $\gamma^*(N)$ as a function of $N$ is plotted in Fig. 2. It is interesting to note that $\gamma^*(N)$ increases with $N$ and some convergence phenomenon is observed as $N$ becomes large.

VI. CONCLUSIONS

In this paper, new results on the $\ell_2$-norm evaluation problem are obtained. For the finite-horizon problem, complete solutions are provided, and efficient algorithms are proposed to reduce the computational complexity by removing the redundant solutions that can never be the ‘worst-case’ ones. Further steps of the research include the exact characterization of the set of the worst-case switching signals and the study of the asymptotic properties of the finite-horizon $\ell_2$-induced norm. Another topic for future work is to explore the transient properties of the Riccati equations, so that further non-conservative algebraic conditions can be found for high-
order switched linear systems.

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