Equivalent LMI Constraints: Applications to Discrete-time MJLS and Switched Systems*

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Abstract—Discrete-time Markov jump linear systems (MJLS) and switched linear systems (SLS) stability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance conditions are very similar. Starting from the fact that MJLS second moment stability can be checked through four different linear matrix inequalities (LMIs), we show how one LMI condition can be obtained from the other. Then, we show the stability of SLS may also be checked through equivalent matrix inequalities and apply the same steps for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances, obtaining new conditions for both MJLS and SLS. Special attention is given to the case where the transition probabilities are independent of the mode, which is equivalent to consider the Metzler matrices on switched linear systems framework to have identical columns.

I. INTRODUCTION

The increasing interest on dynamic systems that present sudden changes on their structures or parameters modeled as Markovian processes become decisive for the development of the so called Markov Jump Linear Systems (MJLS) in both continuous and discrete-time domains. An important assumption to consider for MJLS design is if the Markov chain state, often called mode, is available or not at every instant of time. Based on that information the design is said to be either mode-dependent or mode-independent, respectively, see [5], [15] and [23]. One of the first works in the literature dealing with this class of models was presented in [19].

Many procedures have been developed in order to extend the concepts of the deterministic systems to this special class, namely stability concepts and testable conditions [4], [18]; optimal state feedback control [17]; state feedback $\mathcal{H}_2$ optimization via LMIs [11]; state feedback $\mathcal{H}_\infty$ optimization and robustness via LMIs [6], [16]; state feedback $\mathcal{H}_\infty$ design via Riccati equations [1] and $\mathcal{H}_\infty$ filtering [2].

On the other hand, Switched Linear Systems (SLS) also have attracted special attention. They are characterized by having a switching rule which selects, at each instant of time, a particular subsystem among a certain number of available ones. Hence, the switching function can be characterized as a control variable to be designed in order to achieve global asymptotic stability as well as adequate performance. The papers [10] and [21], and the books [20] and [24] are surveys that treat these topics with deepness and particular attention to the most effective design techniques. More specifically, in the context of discrete-time SLS, the literature to date provides few stability conditions obtained from the use of different techniques [7], [13], [25] and [26], as for instance, multiple Lyapunov functions or piecewise quadratic Lyapunov functions. Roughly speaking, although different Lyapunov functions are used, the obtained stability conditions are similar. The main difference among the mentioned approaches consists on the possibility of solving the stability conditions taking into account some supplementary exigences, as for instance, robustness against parameter uncertainty, which naturally leads to an $\mathcal{H}_\infty$ control design problem and optimal guaranteed $\mathcal{H}_2$ performance.

This paper aims at exploiting the similarities on the stability and performance conditions related to these two important classes of discrete-time dynamic systems. Indeed, this arises whenever a given transition probability matrix $\Lambda \in \mathbb{R}^{N \times N}$ in the framework of MJLS is interpreted as the transpose of a Metzler matrix variable $\Pi \in \mathbb{R}^{N \times N}$ in the framework of SLS. They encompass the ones available in the literature to date for the classes of systems under consideration. More specifically, we are able to get several equivalent stability and performance conditions that are simpler to solve.

The notation used throughout is standard. For real matrices or vectors $(\cdot)$ indicates transpose. For square matrices $\text{Tr}(X)$ denotes the trace function of $X$ and, for symmetric matrices, the symbol $(\cdot)$ denotes generically each of its symmetric blocks. The set of natural numbers is denoted by $\mathbb{N}$ while $\mathbb{K} = \{1, \ldots, N\}$. The symbol $\sigma(\cdot)$ denotes mathematical expectation of $\mathbb{E}(\cdot)$. For any stochastic signal $\xi(k)$, defined in the discrete-time domain $k \in \mathbb{N}$, the quantity $\|\xi\|_2^2 := \sum_{k=0}^{\infty} \mathbb{E}(\xi(k)^T \xi(k))$ is its squared norm. We use the same notation if the signal $\xi(k)$ is deterministic, that is $\|\xi\|_2^2 := \sum_{k=0}^{\infty} \xi(k)^T \xi(k)$. The set of signals $\xi(k) \in \mathbb{R}^p$, $k \in \mathbb{N}$ such that $\|\xi\|_2^2 < \infty$ is denoted $\mathcal{L}^p_2$.

II. PROBLEM FORMULATION

Consider the following discrete-time system

$$
\mathcal{G} : \begin{cases}
\vspace{1mm}
\begin{array}{l}
x(k+1) = A_{\sigma(k)} x(k) + H_{\sigma(k)} w(k) \\
\vspace{1mm}
z(k) = E_{\sigma(k)} x(k) + G_{\sigma(k)} w(k)
\end{array}
\end{cases}
$$

(1)

where $x(k) \in \mathbb{R}^n$ is the state variable, $w(k) \in \mathbb{R}^p$ is the external disturbance and $z(k) \in \mathbb{R}^r$ is the output for all $k \in \mathbb{N}$. It evolves from the initial condition $x(0) = x_0$ and $\sigma(0) = \theta \in \mathbb{K}$. The scalar variable $\sigma(k)$ takes a value in the set $\mathbb{K}$ for all $k \in \mathbb{N}$ and has a different meaning depending on the framework, either Markov Jump Linear Systems (MJLS)
of Switched Linear Systems (SLS). In a MJLS, \( \sigma(k) \) is a random variable governed by a Markov chain. The chain is represented by a matrix \( \Lambda \in \mathbb{R}^{N \times N} \) whose elements are the transition probabilities, that is
\[
\text{Prob}(\sigma(k + 1) = j | \sigma(k) = i) := \lambda_{ij}
\]
where \( \lambda_{ij} \geq 0 \) and \( \sum_{j \in \mathbb{K}} \lambda_{ij} = 1 \). Under the SLS framework, \( \sigma(k) \in \mathbb{K} \) indicates a switching function defined in every instant of time \( k \in \mathbb{N} \). The signal \( \sigma(k) \in \mathbb{K} \) can be seen either as a control input or as a disturbance. In this paper, we will consider the control case. To ease the presentation, we define \( A_{\sigma(k)} := A_i, E_{\sigma(k)} := E_i, H_{\sigma(k)} := H_i \) and \( G_{\sigma(k)} := G_i \) whenever \( \sigma(k) = i \in \mathbb{K} \). Without loss of generality, we will assume \( E_i'G_i = 0 \) for all \( i \in \mathbb{K} \).

### III. STABILITY

In the context of MJLS, there are several equivalent forms to define stability of the system (1) with \( w(k) = 0 \) and arbitrary initial condition \( x(0) = x_0 \in \mathbb{R}^n \), like mean-square stability, stochastic or exponential mean-square stability. It has been shown in [18] that those definitions are actually equivalent for a MJLS, being referred to as second-moment stability, or simply stability for conciseness. In [17], it was shown that system (1) with \( w(k) = 0 \) is stable if and only if there exist symmetric matrices \( P_i > 0 \) for all \( i \in \mathbb{K} \) satisfying
\[
A_i' \left( \sum_{j \in \mathbb{K}} \lambda_{ij} P_j \right) A_j - P_j < 0, \quad i \in \mathbb{K}
\]

In [4], condition (3) was also demonstrated and three other Lyapunov-like necessary and sufficient conditions were shown equivalent
\[
\sum_{i \in \mathbb{K}} \lambda_{ij} A_i W_{ij} A_j' - W_j < 0, \quad j \in \mathbb{K}
\]
\[
\sum_{i \in \mathbb{K}} \lambda_{ij} A_i V_{ij} A_j' - V_j < 0, \quad j \in \mathbb{K}
\]
\[
\sum_{i \in \mathbb{K}} \lambda_{ij} A_i' Q_j A_j - Q_i < 0, \quad i \in \mathbb{K}
\]
where \( W_i \in \mathbb{R}^{n \times n}, \ V_i \in \mathbb{R}^{n \times n} \) and \( Q_i \in \mathbb{R}^{n \times n} \) for \( i \in \mathbb{K} \) are positive definite matrices. The method used by [4] to demonstrate those conditions involves Kronecker products and concepts of stochastic processes. We aim to show that those equivalences can be demonstrated using only LMI manipulations, what could be useful to determine alternative LMI conditions for SLS as well, like we will show later. Using duality we can verify that LMI (3) is equivalent to (5) and (4) is equivalent to (6). It is interesting to see that, if we change \( \lambda_{ij} \) by \( \lambda_{ji} \) and \( A_i \) by \( A_i' \) we can change from one condition to its dual. We still need to show that (3) is equivalent to (6), which is done in the next theorem.

**Theorem 1:** The following assertions are equivalent:

1) There exist symmetric matrices \( P_i > 0 \) for all \( i \in \mathbb{K} \) satisfying (3).

2) There exist symmetric matrices \( Q_i > 0 \) for all \( i \in \mathbb{K} \) satisfying (6).

**Proof:** Assume that (3) has a solution. Multiplying it by \( \lambda_{ji} \geq 0 \), summing up for \( i \in \mathbb{K} \) and defining \( Q_i = \sum_{j \in \mathbb{N} \setminus \mathbb{K}} \lambda_{ji} P_j > 0 \), the constraint (6) is also satisfied. Conversely, assume that \( Q_i > 0 \) satisfies (6), define \( P_i = A_i' Q_i A_i + \epsilon I > 0 \) for all \( i \in \mathbb{K} \) and \( \epsilon > 0 \). Hence,
\[
\sum_{j \in \mathbb{K}} \lambda_{ij} Q_j A_j - Q_i + \epsilon I < 0
\]
for all \( i \in \mathbb{K} \) and for \( \epsilon > 0 \) small enough. Therefore for each \( i \in \mathbb{K} \) we can write
\[
A_i' \left( \sum_{j \in \mathbb{K}} \lambda_{ij} P_j \right) A_j - P_i \leq A_i' Q_i A_i - P_i = -\epsilon I
\]
and the proof is concluded.

An interesting corollary can be established for a particular class of MJLS where \( \lambda_{ij} = \lambda_j \) for all \( (i, j) \in \mathbb{K} \times \mathbb{K} \), that is, when the matrix \( \Lambda \) has identical rows [4]. For this particular class of dynamic systems, the number of stability constraints to be tested is drastically reduced to only one.

**Corollary 1:** The following assertions are equivalent:

1) There exist symmetric matrices \( P_i > 0 \) for all \( i \in \mathbb{K} \) such that
\[
A_i' \left( \sum_{j \in \mathbb{K}} \lambda_{ij} P_j \right) A_j - P_j < 0, \quad i \in \mathbb{K}
\]

2) There exists a symmetric matrix \( Q > 0 \) such that
\[
\sum_{j \in \mathbb{K}} \lambda_{ij} A_j' Q A_j - Q_j < 0
\]

**Proof:** The proof follows the same pattern as that of Theorem 1 together with the fact that for the class under consideration the matrix \( Q_i = \sum_{j \in \mathbb{K}} \lambda_{ij} P_j \) is invariant with respect to the index \( i \in \mathbb{K} \).

The main importance of the previous corollary is to show that, for a class of MJLS, the condition to check stability can be written in terms of a single LMI and a single Lyapunov variable \( Q > 0 \), instead of \( N \) LMIs and \( N \) variables as in (9). Since that is a necessary and sufficient condition for the stability of that class of MJLS and the variable \( Q > 0 \) is not dependent on the mode \( i \in \mathbb{K} \), it can be used to design mode-independent linear filters or state-feedback controllers.

According to what was shown in [13], a stability condition for SLS depends on the existence of a solution, given by a set of symmetric matrices \( P_i > 0 \) for all \( i \in \mathbb{K} \) and a Metzler matrix \( \Pi \in \mathbb{R}^{N \times N} \), to the constraints named Lyapunov-Metzler inequalities. The class of Metzler matrices \( \mathcal{M} \) is defined by matrices with elements \( \pi_{ij} \geq 0 \) and \( \sum_{i \in \mathbb{K}} \pi_{ij} = 1 \), for all \( i, j \in \mathbb{K} \). At this point, it is important to emphasize the similarities between the elements of the Metzler matrix \( \Pi \) in the context of SLS with the transition probabilities matrix \( \Lambda \) in the MJLS case. As a matter of fact, any Metzler matrix of the class \( \mathcal{M} \) can be interpreted as the transpose of a particular transition probabilities matrix, that is \( \Lambda = \Pi' \). Hence, a MJLS is a stochastic system where \( \sigma(k) \) is defined by a Markov chain process. On the other hand, a SLS is a deterministic one where \( \sigma(k) \) is defined by an appropriated switching function. The main result for stability of discrete-time switched linear systems is described in the following theorem.
Theorem 2: Assume there exist symmetric matrices $P_i > 0$ for all $i \in K$ and $\Pi \in \mathcal{M}$ such that the Lyapunov-Metzler inequalities
\[
A_i^T \left( \sum_{j \in K} \pi_{i,j} P_j \right) A_i - P_i < 0
\] (11)
are valid for all $i \in K$. The state-feedback switching control
\[
\sigma(k) = \arg \min_{x(k) \in K} x(k)^T P x(k)
\] (12)
makes the equilibrium point $x = 0$ from (1), with $w(k) = 0$, globally asymptotically stable.

Proof: See [13].

The fundamental difference between the transition probabilities $\lambda_{ij}$ for the MJLS and the elements of the Metzler matrix $\Pi$ for SLS is that, in the first case, they are given parameters from the system under study while, in the second one, the elements $\pi_{ij}$ for all $i, j \in K$ are variables to be determined. As a consequence, the corresponding stability conditions are expressed through bilinear matrix inequalities. Unfortunately, the Lyapunov-Metzler inequalities (11) do not determine a convex set, because of the products between variables, preventing the possibility to solve the problem using convex programming tools such as LMIs.

Some of the results presented so far for the MJLS framework can be useful for SLS analysis and design. For example, to the best of the authors knowledge, there is no stability condition equivalent to (11) for SLS. With the previous discussion, it is straightforward to determine alternative Lyapunov-Metzler inequalities like (4)-(6). For example, (4) or (6) can be more practical for the design of a stabilizing state feedback controller, since the Lyapunov variables $W_i > 0$ or $Q_i > 0$ are only involved in products with system matrices of the same index $i \in K$. Another interesting point is that, if the search for a Metzler matrix is restricted to matrices with identical columns, the results from Corollary 1 can be applied to SLS, reducing the number of constraints and variables of the stability condition.

IV. $\mathcal{H}_2$ AND $\mathcal{H}_\infty$ PERFORMANCES

Given the stability criteria for MJLS and SLS, the main goal of this section is to present performance indexes in order to design optimal filters and controllers. For MJLS, we will use the definition of $\mathcal{H}_2$ norm from [11].

Definition 1: The $\mathcal{H}_2$-norm of a stable system $\mathcal{G}$ from the input $w$ to the output $z$ is
\[
\|\mathcal{G}\|_2^2 := \sum_{i=1}^{p} \sum_{s=1}^{p} \mu_i \|z^d_i\|_2^2
\] (13)
where $\mu_i = \text{Prob}(\theta = i)$ and $z^d_i$ represents the output $z(k)$ obtained from the input $w(k) = e_s \delta(k)$, where $e_s \in \mathbb{R}^p$ is the $s$-th column of the $p \times p$ identity matrix, $\delta(k)$ is the discrete impulse, $x(0) = 0$ and $\theta = i \in K$.

As expected, when the system is constituted by only one mode, that is $N = 1$, the previous definition is equivalent to the $\mathcal{H}_2$ norm of the corresponding linear time-invariant system (LTI). The $\mathcal{H}_2$ norm of a MJLS can be calculated [11] as the solution of the following convex problem
\[
\|\mathcal{G}\|_2^2 = \inf_{P_i > 0, i \in K} \sum_{i \in K} \mu_i \text{Tr} \left( H_i^T \left( \sum_{j \in K} \lambda_{ij} P_j \right) H_i + G_i'^T G_i \right)
\] (14)
subject to
\[
A_i^T \left( \sum_{j \in K} \lambda_{ij} P_j \right) A_i - P_i + E_i' E_i < 0
\] (15)
for all $i \in K$. If the distribution of the initial mode $\mu_i$ is unknown, it is possible to work with the worst case norm, as suggested in [12] and [15]. For further comparisons, assuming $\sigma(0) = \theta \in K$ is known, the initial probability $\mu_i, i \in K$ is set accordingly and provides
\[
\|\mathcal{G}\|_2^2 = \inf_{P_i > 0, i \in K} \text{Tr} \left( H_0'^T \left( \sum_{j \in K} \lambda_{ij} P_j \right) H_0 + G_0' G_0 \right)
\geq \inf_{P_i > 0, i \in K} \text{min} \text{Tr} \left( H_0'^T P_i H_0 + G_0' G_0 \right)
\] (16)
where the inequality follows from the fact that the minimum is always smaller or equal to any value of the convex combination.

It is possible to argue that the quantity defined by (13) cannot be named as $\mathcal{H}_2$ norm since this concept is connected to a transfer function and is, therefore, only valid for LTI systems. This is the reason why, in the switched systems framework, instead of norm, it is more usual to refer to it as an $\mathcal{H}_2$ functional cost [14].

Definition 2: The $\mathcal{H}_2$-functional cost of a switched stable system (1) from the input $w$ to the output $z$, for a given switching function $\sigma(k) \in K$, is
\[
J_2(\sigma) := \sum_{i=1}^{p} \|z^i\|_2^2
\] (17)
where $z^i$ represents the output $z(k)$ obtained from the input $w(k) = e_s \delta(k)$, where $e_s \in \mathbb{R}^p$ is the $s$-th column of the $p \times p$ identity matrix, $\delta(k)$ is the discrete impulse and $x(0) = 0$.

In the absence of switching, that is, if $\sigma(k) = i$ for all $k \in \mathbb{N}$, the value of this functional cost is equal to the square of the $\mathcal{H}_2$ norm of the subsystem $\mathcal{G}_i = (A_i, H_i, E_i, G_i)$ from input $w$ to output $z$. Since the switching function is time-variant, the exact calculation of (17) is not simple. However, it is possible to work with an upper bound by using the same stability results that were mentioned before. In [14] it is shown that
\[
J_2(\sigma) < \inf_{P_i > 0, i \in K} \text{min} \text{Tr} \left( H_0'^T P_i H_0 + G_0' G_0 \right)
\] (18)
subject to
\[
A_i^T \left( \sum_{j \in K} \pi_{ij} P_j \right) A_i - P_i + E_i' E_i < 0
\] (19)
for all $i \in K$, where $P_i > 0$ and $\pi_{ij}$ are the elements of a Metzler matrix $\Pi \in \mathcal{M}$. In such a case, the switching rule that allows the calculated upper bound to be attained is given by (12). It is important to notice, that although they refer
to different systems and that, in the MJLS the parameters \( \lambda_{ij} \) represent given probabilities and for switched systems \( \pi_{ii} \) are variables to be determined, the inequalities (15) and (19) are similar to each other. Moreover, for \( \Lambda = \Pi' \) and the same initial condition \( \sigma(0) = \theta \in K \) from the previous relations (16) and (18) we have \( \|G\|_\infty^2 > J_2(\sigma) \). The next theorem presents two equivalent inequalities.

**Theorem 3:** The following assertions are equivalent:

1) There exist symmetric matrices \( P_i > 0 \) for all \( i \in K \) such that (25) are valid.

2) There exists a symmetric matrix \( Q \) such that

\[
\sum_{j \in K} \lambda_{ij} (A_i'Q_j A_j + E_i' E_j) - Q_i < 0, \quad i \in K
\]  

**Proof:** If (15) are valid, define \( Q_i = \sum_{j \in K} \lambda_{ij} P_j > 0 \). On the other hand, if (19) are valid, define \( P_i = A_i'Q_j A_j + E_i' E_j + \epsilon I > 0 \) with \( \epsilon > 0 \) small enough and follow the same steps of the proof of Theorem 1.

This is the first time that an LMI condition like (20) is presented for the calculation of the \( \mathcal{H}_2 \) norm of MJLS or for the \( \mathcal{H}_2 \)-functional cost \( J_2(\sigma) \) of a switched system. Again, it is interesting to notice that the Lyapunov variables \( Q_i \) for all \( i \in K \) are only involved in products with system matrices \( A_i \), of the same index \( i \in K \), what can simplify the design of controllers and filters. For the particular case where \( \lambda_{ij} = \lambda_j \) for all \( i, j \in K \) (or \( \pi_{ii} = \pi_j \) for all \( i, j \in K \) for a switched linear system), it is possible to show that the \( N \) inequalities (20) become a single one.

**Corollary 2:** The following assertions are equivalent:

1) There exists symmetric matrices \( P_i > 0 \) for all \( i \in K \) such that

\[
A_i' \left( \sum_{j \in K} \lambda_{ij} P_j \right) A_i - P_i + E_i' E_i < 0, \quad i \in K
\]  

2) There exists a symmetric matrix \( Q > 0 \) such that

\[
\sum_{j \in K} \lambda_{ij} (A_i'Q_j A_j + E_i' E_j) - Q < 0
\]  

**Proof:** Follows immediately from Theorem 3.

Again, this fact can be exploited in the framework of SLS. Since there is no direct method to solve the BMI (19), it is usual to perform a search on a particular structure of \( \Pi \in \mathcal{H} \) and solve the remaining LMIs for the other variables. Considering the class of Metzler matrices with identical columns, the LMI to be solved at each step is unique, with a single variable. Moreover, by dealing with a single LMI for each choice of a Metzler matrix, it may be easier to study other properties of a given switched system, like consistency [8], for example. In the framework of MJLS, the fact that (22) is a necessary and sufficient condition can help obtaining the \( \mathcal{H}_2 \) optimal mode-independent state feedback controller, since the variable \( Q > 0 \) does not depend on the mode \( i \in K \).

That design will be subject to future work. We now move our attention to the \( \mathcal{H}_\infty \) norm of the Markov jump linear system \( G \) with state space realization given in (1). The formal definition of this important concept [3] is as follows.

**Definition 3:** The \( \mathcal{H}_\infty \)-norm of a stable system \( G \) from the input \( w \) to the output \( z \) is

\[
\|G\|_\infty := \sup_{0 \neq w \in L_2^w, \theta \in K} \frac{\|z\|_2}{\|w\|_2}
\]  

As before, it is interesting to observe that in the deterministic case characterized by \( N = 1 \) the previous definition reduces to the square of the usual \( \mathcal{H}_\infty \)-norm of the linear time invariant system \( G \). The \( \mathcal{H}_\infty \)-norm of the MJLS (1) can be calculated, as indicated in [22], from the optimal solution of the following convex programming problem

\[
\|G\|_\infty = \inf_{\gamma > 0} \gamma
\]  

subject to

\[
\begin{bmatrix}
A_i & H_i \\
E_i & G_i
\end{bmatrix}' \begin{bmatrix}
\sum_{j \in K} \lambda_{ij} P_j & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
A_i & H_i \\
E_i & G_i
\end{bmatrix} - \begin{bmatrix}
P_i & 0 \\
0 & \gamma I
\end{bmatrix} < 0, \quad i \in K
\]  

where the minimization is done with respect to the positive definite matrix variables \( P_i > 0 \) for all \( i \in K \) and the scalar \( \gamma > 0 \). Since in the context of MJLS the probabilities \( \lambda_{ij} \) are fixed, this problem can be faced with any LMI solver.

In the framework of switched linear systems, the \( \mathcal{H}_\infty \)-functional cost [9] is defined as a limit to the influence of the worst case disturbance \( w \) on the output \( z \) of the system.

**Definition 4:** The \( \mathcal{H}_\infty \)-functional cost of a switched stable system (1) from the input \( w \) to the output \( z \), for a given switching function \( \sigma(k) \in K \) is

\[
J_\infty(\sigma) := \sup_{0 \neq w \in L_2^w} \frac{\|z\|_2}{\|w\|_2}
\]  

Notice that this definition reduces to the square of the \( \mathcal{H}_\infty \)-norm of the system \( G = (A_i, E_i, H_i, G_i) \) when the commutation rule is fixed at \( \sigma(k) = i \in K \) for all \( k \in \mathbb{N} \). On the other hand, when the switching function \( \sigma(\cdot) \) is time variant, (26) becomes very hard to calculate and, for this reason, the idea is to determine an upper bound

\[
\sup_{0 \neq w \in L_2^w} \frac{\|z\|_2}{\|w\|_2} < \gamma
\]  

In [9], it is shown that a condition for a given \( \gamma > 0 \) to satisfy (27) is that (25) is satisfied, where the variables are symmetric matrices \( P_i > 0 \) and \( \lambda_{ij} = \pi_{ij} \), where \( \pi_{ij} \) are elements of a Metzler matrix \( \Pi \in \mathcal{H} \). The switching function will be (12). Like we did for the stability and \( \mathcal{H}_2 \) norm conditions, we will show there are inequalities equivalent to (25). They will be nonlinear, but they can be linearized with the help of additional variables and Schur complement.

**Theorem 4:** Let \( \gamma > G_i' G_i, \forall i \in K \) be given. Let \( \Gamma_i = H_i(\gamma I - G_i' G_i)^{-1} H_i' \) to ease the notation. The following assertions are equivalent:

1) There exist symmetric matrices \( P_i > 0 \) for all \( i \in K \) such that (25) are valid.
2) There exist symmetric matrices \( Q_i > 0 \) such that \( Q_i^{-1} > \Gamma_i \) and
\[
\sum_{j \in K} \lambda_{ij} A_j^T (Q_j^{-1} - \Gamma_j)^{-1} A_j + E_j^T E_j - Q_i < 0, \quad i \in K
\] (28)

Proof: Applying Schur complement to (25) we get
\[
\begin{bmatrix}
P_i - E_i^T E_i & \cdot \\
A_i & (\sum_{j \in K} \lambda_{ij} P_j)^{-1} - \Gamma_i
\end{bmatrix} > 0, \quad i \in K
\] (29)

If (29) is feasible, then defining \( Q_i = \sum_{j \in K} \lambda_{ij} P_j > 0 \) the second diagonal block of (29) imposes \( Q_i^{-1} > \Gamma_i \), \( i \in K \). Performing the Schur complement with respect to the second diagonal block of (29), multiplying the result by \( \lambda_{ij} \geq 0 \) and summing up for all \( i \in K \) we obtain (28). Conversely, assume (28) hold, define the matrices
\[
P_i = A_i^T (Q_i^{-1} - \Gamma_i)^{-1} A_i + E_i^T E_i + \epsilon I, \quad i \in K
\] (30)
for all \( i \in K \) that are positive definite for all \( \epsilon > 0 \). We have
\[
\left( \sum_{j \in K} \lambda_{ij} P_j \right) = \sum_{j \in K} \lambda_{ij} \left[ \left( Q_j^{-1} - \Gamma_j \right)^{-1} A_j + E_j^T E_j + \epsilon I \right] < Q_i, \quad i \in K
\] (31)
whenever \( \epsilon > 0 \) is taken small enough. Since the two matrices on the left side of (31) are positive definite we also have that \( (\sum_{j \in K} \lambda_{ij} P_j)^{-1} > Q_i^{-1} \) for all \( i \in K \), which yields
\[
\left( \sum_{j \in K} \lambda_{ij} P_j \right)^{-1} - \Gamma_i > Q_i^{-1} - \Gamma_i
\] (32)

hence taking again the inverses on each side of (32) yields
\[
A_i^T \left( \sum_{j \in K} \lambda_{ij} P_j \right)^{-1} - \Gamma_i^{-1} A_i < A_i^T \left( Q_i^{-1} - \Gamma_i \right)^{-1} A_i
\]
\[
< P_i - E_i^T E_i - \epsilon I
\]
\[
< P_i - E_i^T E_i
\] (33)

which, from the fact that this inequality is valid for all \( i \in K \), implies (29) is valid and the claim follows.  

Differently from the case of stability and \( H_\infty \) norm, the alternative inequality constraints are not linear. Nonetheless, it is possible to write (28) as LMIs, after applying the Schur complement and by using additional variables \( R_i > 0 \) for all \( i \in K \). The result is
\[
\sum_{j \in K} \lambda_{ij} R_j - Q_i < 0, \quad i \in K
\] (34)

\[
\begin{bmatrix}
A_j & H_j \\
E_j & G_j
\end{bmatrix}^T \begin{bmatrix}
Q_j & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
A_j & H_j \\
E_j & G_j
\end{bmatrix} - \begin{bmatrix}
R_j & 0 \\
0 & \gamma I
\end{bmatrix} < 0, \quad j \in K
\] (35)

One advantage of (34)-(35) over (25) is that the system matrices \( \mathscr{G}_i = (A_i, E_i, H_i, G_i) \) are involved in products only with variables \( Q_i \) of the same index \( i \in K \). Once again, we believe that this can simplify the filter and control design procedures. An important corollary can be obtained for the particular case where \( \lambda_{ij} = \lambda_j \) for all \( i, j \in K \).

**Corollary 3:** Assume that \( \lambda_{ij} = \lambda_j \) for all \( i, j \in K \). The following assertions are equivalent:

1) There exist symmetric matrices \( P_i > 0 \) for all \( i \in K \) such that (25) is satisfied.

2) There exists a symmetric matrix \( Q > 0 \) such that \( Q^{-1} > \Gamma_i \) for all \( i \in K \) and
\[
\sum_{i \in K} \lambda_i \left( A_i^T \left( Q^{-1} - \Gamma_i \right)^{-1} A_i + E_i^T E_i \right) - Q < 0
\] (36)

Proof: It follows the proof of Theorem 4.

The inequality (36) is not linear, as opposed to (21) or (10). Nonetheless, it is still possible to write it in LMI terms by using additional variables \( R_i > 0 \) for all \( i \in K \) as
\[
\sum_{i \in K} \lambda_i R_i - Q < 0
\] (37)

\[
\begin{bmatrix}
A_i & H_i \\
E_i & G_i
\end{bmatrix}^T \begin{bmatrix}
Q & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
A_i & H_i \\
E_i & G_i
\end{bmatrix} - \begin{bmatrix}
R_i & 0 \\
0 & \gamma I
\end{bmatrix} < 0, \quad i \in K
\] (38)

At this point, we recall the results from [23] in its Theorem 4, where it is claimed that it is possible to calculate the \( H_\infty \) norm of a MJLS, under the assumption that \( \lambda_{ij} = \lambda_j \) for all \( i, j \in K \times K \) using a single LMI. The LMI constraint given in Theorem 4 of [23] is actually a necessary condition from (37) and (38), but not a sufficient one. Indeed, the following example is used to illustrate this fact. It consists of a MJLS system with state space realization (1) with two modes defined by matrices
\[
A_1 = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix}, \quad H_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]
and \( G_i = 0 \) for \( i = \{1, 2\} \). Furthermore, we consider the transition probability matrix as
\[
\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 - \lambda \\ \lambda & 1 - \lambda \end{bmatrix}
\] (39)

where \( \lambda \in [0, 1] \). For each value of \( \lambda \) in this interval we have solved four different problems which consist:

(a) \( \inf_{\gamma > 0} \{\gamma\} \) subject to the constraints (25);

(b) \( \inf_{Q, R} \{\gamma\} \) subject to the constraints (34)-(35);

(c) \( \inf_{Q, R} \{\gamma\} \) subject to the constraints (37)-(38);

(d) \( \inf_{Q, R, \gamma} \{\gamma\} \) subject to the constraint given in Theorem 4 of [23] 1.

Figure 1 shows in solid line the value of the \( H_\infty \) norm (equal to \( \sqrt{\mathbf{P}} \)) for cases a), b) and c) against \( \lambda \in [0, 1] \). As expected, the first three curves coincide which confirms the previous results reported in this paper. In dashed line it is shown the result produced by the fourth problem. It is apparent that it produces a value that is always smaller (in the open interval) than the true \( H_\infty \) norm which enables us to conclude that the result from [23] in its Theorem 4 is a necessary but, in general, not a sufficient condition. We believe, and this will be subject of future work, that there is an extra assumption to allow the LMI from [23] to be a necessary and sufficient condition for bounded \( H_\infty \) norm.

1 The constraint has been obviously modified to cope with \( H_\infty \) norm less than \( \gamma > 0 \) instead of 1.
In the SLS framework, the fact that (36) is a single inequality with only one matrix variable $Q > 0$ and additional variables $\pi_j, j \in \mathbb{K}$ is useful to test system (1) for consistency [8]. In the context of MJLS, the design of optimal $H_\infty$ mode-independent state feedback controllers or filters can be considered, since the Lyapunov variable $Q > 0$ multiplying the system matrices is independent of the mode $i \in \mathbb{K}$.

V. CONCLUSION

There are several similarities between discrete-time MJLS and SLS. This similarities became evident when [13] has proposed a stability condition and a stabilizing switching rule design for SLS. However, some aspects already known for the MJLS did not have a counterpart in terms of switching systems, for example, it is known [4] that the stability from a discrete-time MJLS can be tested through four equivalent Lyapunov-like inequalities, but there was only one Lyapunov-Metzler inequality for SLS. We have shown an alternative method to determine the equivalence between the four MJLS Lyapunov inequalities and used it to show that the stability for SLS can be checked through different Lyapunov-Metzler inequalities as well. The same method was used to MJLS and SLS $H_2$ and $H_\infty$ conditions and, to the best of the authors knowledge, we have shown different matrix constraints for those cases that can be useful for the design of filters and controllers. Special attention was given to the particular case where the rows of the transition probabilities are identical for a discrete-time MJLS, or the columns of the Metzler matrix are identical for SLS.

In the future, we want to show what assumption should be taken in order to allow the constraint first shown in [23], which is different from the one we have demonstrated here, to be useful for design. To this end we need to demonstrate the validity of our conjecture. In the switched systems framework, we would like to use the particular case of Metzler matrices with identical columns to improve the linear search that is needed in order to solve the constraints using available LMI packages. We also think it is possible to address switched system properties like consistency with the help of the results reported here.

REFERENCES