Dual algebraic framework for discrete-time nonlinear control systems

Zbigniew Bartosiewicz, Ülle Kotta, Claude H. Moog, Tanel Mullari, Ewa Pawłuszewicz

Abstract—An algebraic framework for discrete-time nonlinear control systems is introduced, based on the forward and backward shifts of the vector fields, dual to that based on differential 1-forms. As an application, the accessibility criterion of a control system in terms of vector fields is given and compared to those obtained under more restrictive assumptions.

I. INTRODUCTION

The algebraic approach based on the differential 1-forms over a suitable field of meromorphic functions provides simple tools for addressing various problems in nonlinear control [1], [2]. The methods based on 1-forms and the related methods based on the theory of skew polynomial rings [3], [4], [5], [6] are complementary to differential geometric methods [7], [8], characterized by strong similarity to their linear counterparts.

The goal of this paper is to develop, for discrete-time nonlinear control systems, dual algebraic approach based on the vector fields over the field of meromorphic functions. This requires to define the forward and backward shift operators acting on the vector fields in a manner that agrees with the geometric meaning of shifting (moving) the vector field along the (discrete) trajectory of the control system [15]. As for the forward shift of the vector field, this concept has been used in [2], though defined algebraically as the solution of system of equations constructed from the property that the forward shift of the scalar product of the 1-form and the vector field as a function is equal to the scalar product of the shifted 1-form and the shifted vector field. However, the paper [2] did not pursue this approach further, the forward shift was only used in the proof of a specific theorem. It has to be proved that the concept introduced in [2] agrees with the (geometric) interpretation of the forward shift and moreover, one has to introduce also the notion of the backward shift. Moreover, in [10], [11] a so-called $\text{Ad}$ operator has been introduced in the discrete-time context which, as shown in [13], can be understood as the backward shift operator, acting on the vector field. Alternatively, in [15] the same object has been independently reintroduced and denoted as $\Theta$ operator. In the discrete-time setting the backward shift may be interpreted as the analogue of the Lie derivative of the vector field along the (discrete) trajectory of the control system.

We will demonstrate the usefulness of the new algebraic set-up on a fundamental problem of accessibility, and will show that the dual solution in terms of the linear space of the vector fields is expressed in terms of the annihilator of that given in terms of 1-forms. Finally, we compare the criterion with those, obtained earlier in [14], [15] and [16] under more restrictive assumption.

II. PRELIMINARIES

Consider the discrete-time nonlinear control system

$$x^\sigma(t) = \Phi(x(t), u(t)), \quad (1)$$

where by $x^\sigma(t) := x(t+1)$ is denoted the forward shift of $x(t)$, $t \in \mathbb{Z}$ and the variables $x \in \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}^m$ and the state transition map $\Phi : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{X}$ is supposed to be analytic and generically submersive$^1$, i.e. the condition below holds everywhere except on a set of measure zero:

$$\text{rank} \left[ \frac{\partial \Phi(x,u)}{\partial (x,u)} \right] = n. \quad (2)$$

Denote by $\mathcal{K}$ the set of meromorphic functions of a finite number of the independent variables from the infinite set $\mathcal{K}$, [2]

$$\mathcal{C} = \left\{ x, u^{(k)}, z^{(-l)}, k \geq 0, l > 0 \right\}. \quad (3)$$

Let us extend the map $\Phi : (x,u) \mapsto y$ to the map $\Phi : (x,u) \mapsto (y,z)$, where

$$z = \chi(x,u), \quad (3)$$

$z \in \mathbb{R}^m$. The forward shift operator $\sigma_\Phi : \mathcal{K} \rightarrow \mathcal{K}$ is defined as follows. If $\varphi \in \mathcal{K}$ depends on $x, u, \ldots, u^{(k)}, z^{(-l)}, \ldots, z^{(-l+1)}$, then

$$\varphi^{\sigma_\Phi}(x,u,...,u^{(k+1)},z^{(-1)},...,z^{(-l+1)}) := \varphi(\Phi(x,u),u^{(1)},...,u^{(k+1)},z,...,z^{(-l+1)}),$$

where $z = \chi(x,u)$. The subscript $\Phi$ points to the fact that the shift operator $\sigma_\Phi$ is defined by equations (1) and (3). Different control systems and/or different map extensions define different shift operators. If $\varphi$ depends only on $x$, then $\sigma_\Phi(\varphi)$ depends on $x$ and $u$: $\varphi^{\sigma_\Phi}(x,u) = \varphi(\Phi(x,u))$. If $(x(t), u(t)), t \in \mathbb{Z}$, satisfies (1), then

$$\varphi^{\sigma_\Phi}(x(t), u(t)) := \varphi(\Phi(x(t), u(t))) = \varphi(x^\sigma(t)). \quad (4)$$

$^1$The assumption (2) is not restrictive, since it is a necessary condition for system accessibility.
which gives the relation between the two shifts. If \( \varphi(x) = x \), then we get \( u^\sigma(x, u) = \Phi(x, u) \). Similarly one obtains \( u^\sigma(u^{(1)}) = u^{(1)} \). In order to define the inverse of \( \sigma_\Phi \), i.e. the backward shift operator, let us assume that \( \Phi \) has a global analytic inverse, defined on its image \( \Phi(X \times U) \). Then \( \Phi^{-1} : (y, z) \mapsto (x, u) \) exists:

\[
\begin{pmatrix}
x \\
u
\end{pmatrix}
= \Phi^{-1}(y, z).
\]

(5)

Now the inverse of \( \sigma_\Phi(x, u) \) exists. It is the backward shift \( \rho_\Phi(x, u) \) defined by

\[
\begin{align*}
\rho_\Phi^\sigma(x, u, \ldots, u^{(k-1)}, z^{(-1)}, \ldots, z^{(-l-1)}) = \\
\varphi(\Phi^{-1}(x, z^{(-1)}), u, \ldots, u^{(k-1)}, z^{(-2)}, \ldots, z^{(-l-1)}),
\end{align*}
\]

(6)

where \( \varphi \in K \) depends on \( x, u, \ldots, u^{(k)}, z^{(-1)}, \ldots, z^{(-l)} \). In particular, \( \rho_\Phi^\sigma = z^{(-1)} \) as well as

\[
\begin{pmatrix}
x \\
u
\end{pmatrix}
= \Phi^{-1}(x, z^{(-1)}).
\]

(7)

Since \( \sigma_\Phi \) is an automorphism of \( K \), the pair \( (K, \sigma_\Phi) \) is an inversive difference field [12]. Denote by \( \varphi^{(l)} \) the \( l \)-th order forward shift of a function \( \varphi \) constructed recursively by \( \varphi^{(0)} = \varphi, \varphi^{(l+1)} = (\varphi^{(l)})^\sigma_\Phi \). Analogously, \( \varphi^{(l-1)} = (\varphi^{(-l)})^\rho_\Phi \). Obviously, \( \sigma_\Phi \rho_\Phi = \rho_\Phi \sigma_\Phi = \text{id} \).

Remark 1: In [13], the operator \( \epsilon : X \times U_0 \rightarrow X \) was introduced as

\[
\sum_{i=1}^n \frac{\partial}{\partial x_i} x_i + \sum_{j=1}^m \frac{\partial}{\partial u_j} u_j.
\]

In a similar manner one can also define the \textit{backward shift} of a vector field (8) as

\[
\begin{align*}
\epsilon^\sigma &= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^m b_j \frac{\partial}{\partial u_j} + \sum_{j=1}^m c_j \frac{\partial}{\partial z_j},
\end{align*}
\]

(11)

where \( a_i, b_j, k \) and \( c_j, l \) are computed from the scalar products \( \langle \omega, \Xi \rangle^\sigma = \langle \omega, \Xi \rangle^\rho, \) yielding

\[
\begin{align*}
a_i &= \langle dx_i, \Xi^\sigma \rangle = \langle dx_i^\sigma, \Xi \rangle = \langle d\Phi_i, \Xi \rangle^\rho, \\
b_j,k &= \langle du_j, \Xi^\sigma \rangle = \langle du_j^{(k)}, \Xi \rangle - \langle du_j, \Xi \rangle^\rho, \\
c_j,0 &= \langle dz_j, \Xi \rangle \Xi^\sigma = \langle dz_j, \Xi \rangle^\rho, \\
c_j,l &= \langle dz_j^{(l)}, \Xi \rangle \Xi^\sigma = \langle dz_j^{(l)}, \Xi \rangle^\rho, \quad l > 0.
\end{align*}
\]

(12)

A. \textit{Projection of a vector field}

It is obvious from (10), that the backward shift \( \Xi^\sigma \) of a vector field \( \Xi \) in the space \( X \times U_0 \), having the form \( \Xi = \sum_{i=1}^n \Xi_i \partial/\partial x_i + \sum_{j=1}^m \Xi_j \partial/\partial u_j \), has also the components in direction of the basis vector fields \( \partial/\partial u_j^{(k)}, \) i.e. \( b_j \neq 0 \), and moreover, its \( k \)-th order forward shift has the components in directions of \( \partial/\partial u_j^{(k)} \). In a similar manner, by (12), the \( k \)-th order backward shift of \( \Xi \) has the components in directions of \( \partial/\partial z_j^{(-l)} \). In the study below we sometimes have to stay in the space \( X \times U_0 \), therefore we will operate with the projections of \( \Xi \in E^* \) into \( X \times U_0 \).

\textbf{Definition 1}: The projection of the vector field (8) into \( X \times U_0 \), denoted as \( \Xi^\sigma \), is the vector field

\[
\Xi^\sigma = \sum_{i=1}^n \Xi_i \frac{\partial}{\partial x_i} + \Xi_j \frac{\partial}{\partial u_j}.
\]

(13)

\textbf{Remark 1}: In [13], the operator \( \Theta_\Phi : X \times U_0 \rightarrow X \) was introduced as

\[
\Theta_\Phi \Xi = \sum_{i=1}^n \left( \frac{\partial \Phi_i}{\partial x} \Xi_x + \frac{\partial \Phi_i}{\partial u} \Xi_u \right)^\rho \frac{\partial}{\partial x_i}.
\]
The expression in the parentheses is actually \( \langle \Phi_i, \Xi \rangle \) and therefore, due to (11), (12) and (13), \( \Theta_{\Phi} \Xi = \Xi^{\sigma_{\Phi}} \).

**Example 1:** Consider the control system

\[
x_1'(t) = x_1(t) + \mu x_1(t)x_2(t), \quad x_2'(t) = x_2(t) + \mu u(t),
\]

where \( \mu \in \mathbb{R} \) and calculate the forward and backward shifts of the vector field \( \Xi = \partial / \partial x_2 \) for three choices of \( z \).

Take first \( z = u \), and compute

\[
\begin{align*}
    d x_1^{\sigma_{\Phi}} &= (1 + \mu x_1) dx_1 + \mu x_1 dx_2, \\
    d x_2^{\sigma_{\Phi}} &= dx_2 + \mu du, \quad dz = du.
\end{align*}
\]

Applying the backward shift to the inverse map \( \tilde{\Phi}^{-1}(x^\sigma, z) \) yields

\[
\begin{align*}
    x_1^{\sigma_{\Phi}} &= \frac{x_1}{1 + \mu x_2 - \mu^2 z(-1)}, \\
    x_2^{\sigma_{\Phi}} &= x_2 - \mu z(-1), \quad u^{\sigma_{\Phi}} = z(-1),
\end{align*}
\]

whose differentials are

\[
\begin{align*}
    dx_1^{\sigma_{\Phi}} &= \frac{dx_1}{1 + \mu x_2 - \mu^2 z(-1)} - \frac{x_1 \mu (dx_2 + \mu dz(-1))}{1 + \mu x_2 - \mu^2 z(-1)}, \\
    dx_2^{\sigma_{\Phi}} &= dx_2 - \mu dz(-1), \quad du^{\sigma_{\Phi}} = dz(-1).
\end{align*}
\]

Calculate the coefficients of \( \Xi^{\sigma_{\Phi}} \) according to (10), using the backward shifts of the differentials from (15):\footnote{Recall that \( z = u \).}

\[
\begin{align*}
    \tilde{a}_1 &= \left( \frac{\mu x_1}{1 + \mu x_2 - \mu^2 z(-1)} \right)^{\sigma_{\Phi}} = -\frac{\mu x_1}{1 + \mu x_2}, \\
    \tilde{a}_2 &= 1, \quad \tilde{b}_k = c_l = 0,
\end{align*}
\]

yielding

\[
\Xi^{\sigma_{\Phi}} = -\frac{\mu x_1}{1 + \mu x_2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.
\]

In a similar manner we get from (12) and (14),

\[
\Xi^{\rho_{\Phi}} = \frac{\mu x_1}{1 + \mu x_2 - \mu^2 z(-1)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.
\]

As the second possible choice, take \( \tilde{z} = x_1 \). Then \( dx_1^{\sigma_{\Phi}} \) and \( dx_2^{\sigma_{\Phi}} \) as in (14), but \( d \tilde{z} = dx_1 \) and

\[
\begin{align*}
    x_1^{\sigma_{\Phi}} &= \tilde{z}(-1), \\
    x_2^{\sigma_{\Phi}} &= \frac{x_1 - \tilde{z}(-1)}{\mu \tilde{z}(-1)}, \\
    u^{\sigma_{\Phi}} &= \frac{x_2 - \tilde{z}(-1)}{\mu}.
\end{align*}
\]

From (16) we get

\[
\begin{align*}
    dx_1^{\sigma_{\Phi}} &= d \tilde{z}(-1), \quad dx_2^{\sigma_{\Phi}} = \frac{dx_1}{\mu \tilde{z}(-1)} - \frac{x_1 d \tilde{z}(-1)}{\mu \tilde{z}(-1)^2}, \\
    du^{\sigma_{\Phi}} &= -\frac{dx_1}{\mu^2 \tilde{z}(-1)} + \frac{dx_2}{\mu} + \frac{x_1 d \tilde{z}(-1)}{\mu^2 \tilde{z}(-1)^2}.
\end{align*}
\]

Calculate again \( \Xi^{\sigma_{\Phi}} \), now according to (10) and (17):

\[
\begin{align*}
    \tilde{a}_1 &= \tilde{a}_2 = c_l = 0, \quad l > 0, \\
    \tilde{b}_k &= \left( du^{(k-1)} \frac{\partial}{\partial x_2} \right)^{\sigma_{\Phi}} = \frac{\delta_{0,k}}{\mu}, \quad k \geq 0,
\end{align*}
\]

where \( \delta_{0,k} \) stands for Kronecker symbol, yielding \( \Xi^{\sigma_{\Phi}} = (1/\mu) (\partial / \partial u) \). Next compute \( \Xi^{\rho_{\Phi}} \) according to (12):

\[
\begin{align*}
    a_1 &= \mu x_1^{\rho_{\Phi}} = \mu \tilde{z}(-1), \quad a_2 = 1, \\
    b_k &= c_l = 0, \quad k \geq 0, \quad l > 0,
\end{align*}
\]

to obtain

\[
\Xi^{\rho_{\Phi}} = \mu \tilde{z}(-1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.
\]

As the last choice, set \( \tilde{z} = x_2 \), yielding \( dx_1^{\sigma_{\Phi}} \) and \( dx_2^{\sigma_{\Phi}} \) are as in (14), but \( d \tilde{z} = dx_2 \) and

\[
\begin{align*}
    x_1^{\sigma_{\Phi}} &= \frac{x_1}{1 + \mu \tilde{z}(-1)}, \quad x_2^{\sigma_{\Phi}} = \tilde{z}(-1), \quad u^{\sigma_{\Phi}} = \frac{x_2 - \tilde{z}(-1)}{\mu}.
\end{align*}
\]

From above

\[
\begin{align*}
    dx_1^{\sigma_{\Phi}} &= \frac{dx_1}{1 + \mu \tilde{z}(-1)} - \frac{-\mu x_1 d \tilde{z}(-1)}{(1 + \mu \tilde{z}(-1))^2}, \\
    dx_2^{\sigma_{\Phi}} &= d \tilde{z}(-1), \quad du^{\sigma_{\Phi}} = -\frac{dx_1}{\mu} \left( dx_2 - d \tilde{z}(-1) \right).
\end{align*}
\]

Calculate the components of \( \Xi^{\sigma_{\Phi}} \) according to (10):

\[
\begin{align*}
    \tilde{a}_1 &= \tilde{a}_2 = c_l = 0, \quad l > 0, \\
    \tilde{b}_k &= \left( du^{(k-1)} \Xi^{\sigma_{\Phi}} \right) = \left( du^{(k-1)} \frac{\partial}{\partial x_2} \right)^{\rho_{\Phi}} = \frac{\delta_{0,k}}{\mu},
\end{align*}
\]

where \( k \geq 0 \), yielding \( \Xi^{\sigma_{\Phi}} = (1/\mu) (\partial / \partial u) \). Finally, compute \( \Xi^{\rho_{\Phi}} \) according to (12):

\[
\begin{align*}
    a_1 &= \mu x_1^{\rho_{\Phi}} = \frac{\mu x_1}{1 + \mu \tilde{z}(-1)}, \quad a_2 = 1, \quad b_k = 0, \quad c_l = \delta_{1,l},
\end{align*}
\]

for \( k \geq 0, \quad l > 0 \), yielding

\[
\Xi^{\rho_{\Phi}} = \frac{\mu x_1}{1 + \mu \tilde{z}(-1)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \tilde{z}(-1)}.
\]

### III. ACCESSIBILITY

As an application of the introduced algebraic framework we study the accessibility property of system (1) in terms of vector fields, applicable to a larger subclass of systems than those in [14], [15]. Note first that the criteria in [14], [15] are valid only under the reversibility assumption

\[
\text{rank} \left( \frac{\partial \tilde{\Phi} (x, u)}{\partial x} \right) = n,
\]

being more restrictive than (2). For the criterion given in [16], (19) is not required to hold.

Denote, according to [14] by \( A_k(x) \) the set of points reachable from \( x \in \mathbb{R}^n \) in \( k \) (forward) steps, using an arbitrary sequence of inputs and by \( A(x) = \bigcup_{k\geq0} A_k(x) \) the set of points reachable from \( x \) in any number of forward steps, and call the system (1) *accessible from* \( x \) if its reachable set \( A(x) \) has a nonempty interior. A generic notion of accessibility has been derived from the following pointwise definition.

**Definition 2:** [17] System (1) is said to be accessible if its reachable set \( A(x) \) has a nonempty interior in \( \mathbb{R}^n \) for almost all \( x \in \mathbb{R}^n \).
Definition 2 is shown in [2] to be equivalent to Definition
3 below. A non-constant function \( \varphi \in K \) is said to be
an autonomous element of (1), if there exists an integer
\( k \geq 1 \) and a non-constant meromorphic function \( \psi \) such that
\( \psi \left( \varphi, \varphi^{(1)}, ..., \varphi^{(k)} \right) \equiv 0, \partial \psi / \partial \varphi^{(k)} \neq 0. \) [2].

**Definition 3:** [2] System (1) is said to be accessible if it has no non-constant autonomous elements.

**Theorem 1:** [2] System (1) is accessible iff \( \mathcal{H}_\infty \equiv 0 \), where

\[
\mathcal{H}_\infty = \text{span}_K \left\{ \omega \in \mathcal{H}_\infty \mid \omega^{(1)} \in \mathcal{H}_\infty \right\}
\]

is the subspace of 1-forms in \( \text{span}_K \{ dx \} \) being invariant with respect to the forward shift operator\(^3\).

The non-accessibility codistribution \( \mathcal{H}_\infty \) is proved to be in-
tegrable and the autonomous elements being its independent integrals. Because \( \mathcal{H}_\infty \subset \text{span}_K \{ dx \} \), each \( \omega \in \mathcal{H}_\infty \)
can be written in the form \( \omega(x) = \sum_{i=1}^n \omega_i(x) dx_i \). The accessibility criterion of Theorem 1 is easy to check, but unfortunately, it does not allow to find the (possible) singular points, where the system may lose its accessibility property.

Define the vector space \( \mathcal{D}_\infty \subset X \times U_0 \), spanned by the projections of backward shifts of the vector fields \( \partial / \partial u \):

\[
\mathcal{D}_\infty = \text{span} \left\{ \left( \frac{\partial}{\partial u} \right)^{(-k)\pi}, \ k \geq 0 \right\} .
\]

Obviously, for each\(^4\) \( \Xi \in \mathcal{D}_\infty \)

\[
\Xi^{\bar{\sigma} \bar{\pi}} \in \mathcal{D}_\infty .
\]

Denote by \( \mathcal{D}_\infty^{\perp} \) the annihilator of \( \mathcal{D}_\infty \).

**Theorem 2:** \( \mathcal{D}_\infty^{\perp} = \mathcal{H}_\infty \).

**Proof:** To prove the lemma one has to show that:

(i) all 1-forms \( \omega \in \mathcal{D}_\infty^{\perp} \) belong to \( \text{span}_K \{ dx \} \) and

(ii) \( \mathcal{D}_\infty^{\perp} \) is closed under the forward shift: \( \omega^{\sigma \pi} \in \mathcal{D}_\infty^{\perp} \) for each \( \omega \in \mathcal{D}_\infty^{\perp} \).

The first assertion is easy to prove. Since \( \partial / \partial u \in \mathcal{D}_\infty \), the 1-forms, annihilating \( \mathcal{D}_\infty \), cannot depend on \( du \).

To prove (ii), recall that from \( \omega \in \text{span}_K \{ dx \} \) follows

\[
\langle \omega, \Xi^{\bar{\sigma} \bar{\pi}} \rangle = \langle \omega, \Xi^{\bar{\sigma} \bar{\pi}} \rangle .
\]

Due to (21), this scalar product is identically zero, as well as its forward shift \( \langle \omega, \Xi^{\bar{\sigma} \bar{\pi}} \rangle^{\bar{\sigma} \bar{\pi}} \).

Now use the relationship \( \langle \omega, \Xi^{\bar{\sigma} \bar{\pi}} \rangle^{\bar{\sigma} \bar{\pi}} = \langle \omega^{\sigma \pi}, \Xi \rangle \), which leads to

\[
\langle \omega^{\sigma \pi}, \Xi \rangle \equiv 0 .
\]

This completes the proof. \( \blacksquare \)

**Corollary 1:** The system (1) is accessible iff \( \dim \mathcal{D}_\infty = n + m \).

**Proof:** Follows directly from the dual accessibility condition \( \dim \mathcal{H}_\infty = 0 \). \( \blacksquare \)

**Corollary 2:** For non-accessible system (1), the au-
tonomous element \( \varphi \) is the invariant of all vector fields

\[
\left( \frac{\partial}{\partial u} \right)^{(-k)\pi}, \ k \geq 0, \ i.e.
\]

\[
\left\{ d\varphi, \left( \frac{\partial}{\partial u} \right)^{(-k)\pi} \right\} \equiv 0 .
\]

**Proof:** Follows directly from Lemma 2, integrability of \( \mathcal{H}_\infty \) and that autonomous element is the integral of \( \mathcal{H}_\infty \). \( \blacksquare \)

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**IV. Comparison of Different Criteria**

Our goal is to find explicit relationships of accessibility criterion in Corollary 1 with those given in [14], [15], [16], obtained earlier in terms of vector fields for discrete-time nonlinear systems.

**A. Jukubczyn-Sontag criterion [14]**

In [14], one defines the map \( \Phi_u(x) = \Phi(x, u) \) parametrized by fixed \( u \in \mathbb{R}^m \), where \( \Phi_u : X \rightarrow X \). The vector fields \( X_{u;j}^+ \) and the operators \( \text{Ad}_u, \text{Ad}^{-1}_u : X \rightarrow X \) are introduced as follows:

\[
X_{u;j}^+ = (T \Phi_u(x))^{-1} \frac{\partial \Phi_u(x)}{\partial u_j} .
\]

and

\[
\text{Ad}_u \Xi(x) = (T \Phi_u(x))^{-1} \cdot \Xi(\Phi_u(x)) ,
\]

where by \( T \Phi_u(x) \) is defined the Jacobi matrix of \( \Phi_u(x) \) and

\[
(T \Phi_u(x))^{-1} = (T \Phi_u^{-1}) \cdot \Phi_u(x) .
\]

Moreover, \( \text{Ad}_u \Xi = \Xi, \text{Ad}^{-1}_u \Xi = \text{Ad}_u(\text{Ad}_u \Xi) , \text{Ad}^{-1}_u \Xi = \text{Ad}_u^{-1}(\text{Ad}_u \Xi) .
\]

**Theorem 3:** [14] System (1) is accessible iff

\[
\dim \text{span} \left\{ \text{Ad}_u \Xi_{u;j}^+, \ k \geq 0 \right\} = n .
\]

To compare the result in Theorem 3 with that of Corollary 1 consider the vector fields \( X_{u;j}^+ \in X \subset \mathcal{E}^* \) in a larger space, but with nonzero coefficients corresponding to only \( \partial / \partial x \), and recall that the kernel of Jacobi matrix \( T \Phi \) may be interpreted as a \( m \)-dimensional involutive distribution

\[
\text{Ker} T \Phi = \left\{ K \mid T \Phi \cdot K \equiv 0 \right\} \subset X \times U_0 .
\]

Recall also that for an involutive distribution there exists a commutative basis (the Lie brackets of the basis vectors are zero).

**Proposition 1:** For the vector fields \( X_{u;j}^+ \in X \subset \mathcal{E}^* \) the following holds

\[
X_{u;j}^+ = \frac{\partial}{\partial u_j} - K_j, \ j = 1, ..., m ,
\]

where \( K_j \) are the suitably chosen basis vector fields of \( \text{Ker} T \Phi \) satisfying the condition \( [K_j, K_p] \equiv 0 \), for each \( j, p = 1, ..., m \).

**Proof:** If rank \( (\partial \Phi_u / \partial x) = n \), one can define the extended map \( \Phi_u(x) \) as \( x^{(1)} = \Phi_u(x) \), \( u \). Note that the variables \( z, u \) must be the canonical parameters of the basis vector fields \( K_j \in \text{Ker} T \Phi, j = 1, ..., m \). Under assumption (19), one may take \( z = u \), yielding \( \langle du_i, K_j \rangle = \delta_{ij} \). The latter holds if the vector fields \( K_j \) have the form

\[
K_j = K_j^0 \frac{\partial}{\partial z} + \frac{\partial u_i}{\partial u_j} \]

yielding \( [\partial \Phi_u / \partial x] K_j + \langle \partial \Phi_u / \partial u_j \rangle \equiv 0 \), from which we get

\[
K_j = - \langle \partial \Phi_u / \partial x \rangle^{-1} \cdot \langle \partial \Phi_u / \partial u_j \rangle .
\]

Replacing \( K_j \) into...
(25), subtracting the result from ∂/∂u as in (24) and comparing with (22) completes the proof.

**Proposition 2:** For the vector field $\text{Ad}_u^{-1}\Xi \in \mathbf{X} \subset \mathcal{E}$, the following holds:

$$\text{Ad}_u^{-1}\Xi = \Xi^{ρeπ}. \quad (26)$$

*Proof:* Follows directly from formulae (11) and (12) for computation $\Xi^{ρ}$ and the definition (13) of the projection.

**Theorem 4:** Under assumption (19), the condition (23) and that in Corollary 1 are equivalent.

*Proof:* According to (24) and (26), the condition (23) may be rewritten as

$$\dim \text{span} \left\{ \left( \partial/\partial u - K \right)^{(−k)π}, k > 0 \right\} = n.$$  

Since $K \in \text{Ker} T\Phi$, $K^{(−1)π} = 0$, and

$$\left( \partial/\partial u - K \right)^{(−1)π} = \left( \partial/\partial u \right)^{(−1)π},$$

yielding

$$\left( \partial/\partial u - K \right)^{(−k)π} = \left( \partial/\partial u \right)^{(−k)π}, \quad k > 0. \quad (27)$$

Then due to definition (20) and (27),

$$\mathcal{D}_∞ = \text{span} \{ \text{Ad}_u^{-k}X_u^+, k ≥ 0 \} \subset \text{span} \{ \partial/\partial u \}$$

and from (23) Corollary 1 follows.

**B. Rieger-Schlacher criterion [15]**

In ([15]), the set of vector fields has been defined:

$$g_0 = \frac{\partial\Phi}{\partial u} \bigg|_{x^{(−1)}, u^{(−1)}} \frac{∂}{∂x}, \quad (28)$$

$$g_{k+1} = \left( \frac{∂\Phi}{\partial x} g_k \right) \bigg|_{x^{(−1)}, u^{(−1)}} \frac{∂}{∂x}, \quad k ≥ 0,$$

where $x^{(−1)}$ is calculated by (7) under the assumption that $z = u$.

**Theorem 5:** [15] The system (1) is accessible iff

$$\dim \text{span} \{ g_k, k ≥ 0 \} = n. \quad (29)$$

**Lemma 1:** In case $z = u$, the vector fields

$$g_k = \left( \frac{∂}{∂u} \right)^{(−k)π}, \quad k ≥ 0.$$  

*Proof:* Taking into account that $z = u$,

$$(φ(x, u))^{ρe} = φ \left( Φ^{−1} \left( x^{(−1)}, u^{(−1)} \right) \right),$$

see (6). Calculate the backward shifts of $∂/∂u$ according to (11) and (12), and their projections according to (13):

$$\left( \frac{∂}{∂u} \right)^{(−1)π} = \sum_{i=1}^{n} \left( \frac{dΦ_i}{∂u} \right)^{ρe} \frac{∂}{∂x_i} = \sum_{i=1}^{n} \left( \frac{∂\Phi_i}{∂u} \right) \bigg|_{x^{(−1)}, u^{(−1)}} \frac{∂}{∂x_i}, \quad (30)$$

which is the vector field $g_0(x)$ in (28). Applying the backward shift and the projection to an arbitrary vector field $\Xi(x, u) = \sum_{i=1}^{n} ξ_i(x, u) \partial/∂x_i$ yields

$$\Xi^{(−1)π} = \sum_{i=1}^{n} \left( ξ_i \right)^{ρe} \frac{∂}{∂x_i} = \sum_{i=1}^{n} \left( \frac{∂\Phi_i}{∂x} \right) \bigg|_{x^{(−1)}, u^{(−1)}} \frac{∂}{∂x_i}. \quad (31)$$

Taking in (31) $\Xi = g_k$ we get, by (28), that $g_{k+1}$ is the projection of the backward shift of $g_k$, which proves the lemma.

Consequently, under assumption (19), the distribution (29) agrees with that in Corollary 1.

**C. Verriest-Grey condition [16]**

**Theorem 6:** [16] The system (1) is reachable iff for an index $l ≤ n − 1$

$$\text{rank} \left[ g^{(l)} \ (it_l^1g)^{(l−1)} \ ... \ (it_l^ng)^{(0)} \right] = n, \quad (32)$$

where

$$g = \left( \frac{∂\Phi}{∂u} \right), \ (it_l^1g)^{(l)} = g^{(l)}, \quad (33)$$

$$\text{rank} \left[ g^{(l)} \ (it_l^1g)^{(l−1)} \ ... \ (it_l^ng)^{(l−k+1)} g^{(l−k)} \right] = \text{rank} D ≥ k. \quad (34)$$

According to Definition 2, if (32) holds generically, the system is accessible. We will show below how the criterion in Theorem 6 is related to that given in Corollary 1.

The accessibility distribution $\mathcal{D}_∞$ (see (20)) defines the $((n + m) \times ml)$ matrix

$$D = \left[ \left( \frac{∂}{∂u} \right)^{(−1)π}, \ ... , \left( \frac{∂}{∂u} \right)^{(−l)π} \right].$$

If the condition in Corollary 1 is satisfied, the span of vector fields $(∂/∂u)^{(−k)π}, \ k = 1, \ ... , l$, has dimension $n$ and therefore, $\text{rank} D = n$ and vice versa. From direct calculations one can easily see, that the $m$ lower rows of $D$ equal to zero. We will prove now, that the $n$ upper elements in each column of the $k$th block $(∂/∂u)^{(−k)π}$ of matrix $D$ are obtained by the application of $(l + 1)$th order backward shift of the $k$th block $(it_l^{k−1}(l−k+1))$ of the matrix (32).

Note first, that according to (11), (12) and (13)

$$\left( \frac{∂}{∂u} \right)^{(−1)π} = \left( \frac{∂\Phi}{∂u} \right)^{(−1)π} \frac{∂}{∂x}. \quad (34)$$

One can show, step by step, that for $k > 1$

$$\left( \frac{∂}{∂u} \right)^{(−k)π} = \left( \frac{∂\Phi}{∂x} \right)^{(−1)π} \ ... \ \left( \frac{∂}{∂u} \right)^{(−k+1)} \left( \frac{∂\Phi}{∂x} \right)^{(−k)} \frac{∂}{∂x}. \quad (35)$$

Really, by Remark 1, for a vector field $\Xi \in \mathbf{X} \times \mathbf{L}_0$, to find $\Xi^{ρe}$ one has to multiply $\Xi$ by the Jacobi matrix $T\Phi$ and apply the backward shift operator $ρ$ to the coefficients of the result, see (14). We start with $(∂/∂u)^{(−1)π}$ as in (34) and calculate recursively $(∂/∂u)^{(−k)π}$, multiplying $(∂/∂u)^{(−k)π}$ by $T\Phi$ and applying the backward shift operator for all $k = 1, \ ... , l$. 1850
Comparing (33) and (35), equality of the respective blocks is demonstrated. Therefore, if (32) holds, also rank $D = n$, i.e. the span of vector fields $(\partial/\partial u)^{(k)}$, $k \geq 1$, is $n$-dimensional and consequently $D_\infty$ has dimension $n + m$.

Note that (32) suggests an easier criterion for checking the accessibility property than those in Corollary 1 and Theorems 3 and 5 (since there is no need to calculate the backward shifts of functions), but it does not allow to find the autonomous element of the non-accessible system. The reason is, that if we consider the columns of the matrix $D$ as the vector fields, the autonomous elements $\varphi$ are their common invariants, but for the columns of the matrix in (32) as the vector fields, this is not the case.

**Example 2:** Consider the rotation of the 2-dimensional plane around the origin, where the forward shift of a point with coordinates $(x_1, x_2)$ means the rotation by an angle $u$. The extended map $\Phi$ reads as $x_1^{(1)} = x_1 \cos u - x_2 \sin u$, $x_2^{(1)} = x_1 \sin u + x_2 \cos u$, $z = u$.

The accessibility distribution of the system,

$$D_\infty = \text{span} \left\{ \frac{\partial}{\partial u}, -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} \right\},$$

has the dimension $2 = n + m - 1$ and therefore the system is not accessible. One possible choice for autonomous element is $\varphi(x_1, x_2) = 1/2 (x_1^2 + x_2^2)$, that satisfies the condition $\langle d\varphi, \Xi \rangle = 0$ for all $\Xi \in D_\infty$.

We demonstrate that the condition (32) allows to check accessibility property, but does not allow to find the autonomous element. Because the state dimension is $2$, we calculate the reachability matrix (32) for $l = 2$. Its first column is

$$\begin{pmatrix} i_0 \phi^g & i_1 \phi^g \end{pmatrix}^{(2)} = \begin{pmatrix} \varphi^0 & \varphi^1 \end{pmatrix}^{(2)} = \begin{pmatrix} -\beta, \alpha \end{pmatrix}^T =$$

$$= \begin{pmatrix} -\alpha \cos u^{(1)} - \beta \sin u^{(1)} \sin u^{(2)} - \\ \alpha \sin u^{(1)} \cos u^{(2)} - \\ \alpha \sin u^{(1)} + \beta \cos u^{(1)} \sin u^{(2)} \end{pmatrix}^T,$$

where $\alpha = x_1 \cos u - x_2 \sin u$, $\beta = x_1 \sin u + x_2 \cos u$, and the second column

$$\begin{pmatrix} i_1 \phi^g & i_2 \phi^g \end{pmatrix}^{(1)} = \begin{pmatrix} \varphi^0 & \varphi^1 \end{pmatrix}^{(1)} = (i_0 \phi^g)^{(2)}.$$

Consequently, the rank of matrix (32) equals to 1 and the system is not accessible. However, considering its columns as the vector fields, one can easily see that the autonomous element $\varphi$ is not their invariant.

**V. Conclusions**

An algebraic framework for discrete-time nonlinear control systems has been introduced, based on the vector fields. The new framework is dual to the one that is based on the differential 1-forms [2]. The forward and backward shifts of vector fields have been defined. As an application, the accessibility criterion of a control system is expressed in terms of new tools and shown to be equivalent to the earlier conditions. The new framework has some contact points with the Lie-Bäcklund approach for continuous-time case [18]. The forward shift of the vector field corresponds to the total time derivative, and in both cases the vector fields are defined on the infinite spaces. However, like in the case of 1-forms, nothing corresponds to the backward shift.

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