Stability Analysis of Input and Output Finite Level Quantized Discrete-Time Linear Control Systems

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Abstract—This paper analyzes the stability of input and output quantized discrete-time linear control systems considering static finite-level logarithmic quantizers. The sector bound approach together with a relaxed stability notion are applied to derive LMI based conditions for estimating a set of initial conditions and its attractor assuming that the controller and quantizers are known a priori. These conditions ensure that all state trajectories belonging to the first set will enter the attractor in a finite time and remains inside it. A numerical example is presented to illustrate the application of the derived results.

I. INTRODUCTION

Networked control systems (NCS) have recently attracted a huge interest of the control community motivated by the fact that new technologies on sensing, computing and wireless communication technologies are bringing about a new range of emerging control applications [1], [2]. Since in many situations quantization errors are unavoidable, their effects cannot be neglected at the cost of an inadequate closed-loop performance and even the lost of stability. As a result, the study of feedback systems subject to quantization errors has been attracting increasing interest [3]–[5].

Early results on quantized feedback systems were concentrated on evaluating and mitigating the quantization errors on digital implementation of feedback systems [6]–[8]. In the last ten years, NCS became the most common feedback control systems subject to quantization errors because of limited bandwidth in communications links. Notice that in NCS the control loop elements (plant, controller, actuator and sensor) exchange information through digital communication networks. Therefore, a natural issue arising in NCS is the amount of information needed to be sent via communication link such that a certain closed-loop performance is achieved. Very recently, increasing attention has been focused on the topic of quantization errors in NCS as, for instance, the references [3]–[5], [9]–[11] to cite a few. In [4], it has been shown that for a quadratically stabilizable system a logarithmic quantizer (i.e., the quantization levels are linear in a logarithmic scale) is the optimal solution in terms of coarse quantization density, and for that an ideal quantizer, namely a quantizer with an infinite number of quantization levels, is required. In this line, Fu and Xie in [5] have introduced the sector bound approach for quantized feedback systems giving simple formulae to the stabilization problem considering state and output feedback controllers. Since then, the sector bound approach has been applied to solve a wide diversity of problems such as dynamic and static finite level quantization in [12], [13], quantization dependent stabilization [14], quantized robust control [15] and state estimation [16].

Notice that the latter works assume the presence of a single quantizer in the feedback loop either in the input channel or in the output channel. However, since in NCS the information (control signal and measurements) is generally exchanged through a shared communication channel with limited bandwidth, it is natural to suppose that both control and measurement signals are quantized [17]. To date, very few works have addressed stability and stabilization problems for input and output quantized feedback systems excepting, for instance, [11], [17]–[19]. In [11], the authors have extended the sector bound approach of [5] to cope with input and output quantization for single input-single output (SISO) discrete-time linear output feedback systems assuming that the input and output quantizers have an infinite number of levels. To the authors’ knowledge, the study of stability properties of input and output finite-level quantized linear control systems has not yet been addressed in the literature.

This paper deals with local stability analysis of SISO discrete-time linear feedback systems with static finite-level logarithmic quantizers in the input and output. Firstly, we present the main result derived in [11] on quadratic stability of input and output quantized SISO discrete-time linear feedback systems considering ideal static logarithmic quantizers. Then, based on the notion of wide quadratic stability introduced in [13], we propose an LMI approach to derive a set of admissible initial states and an associated attractor set in the neighborhood of the origin such that the state trajectories starting in the first set will enter the attractor in a finite time and will remain in the latter set. An academic example is considered to illustrate the potentials of the proposed approach.

Notation: For a real matrix \( S, S' \) denotes its transpose and \( S > 0 \) (\( S \geq 0 \)) means that \( S \) is symmetric and positive definite (nonnegative definite). For a symmetric block matrix, \( \bullet \) stands for the transpose of the blocks outside the main diagonal block. For two sets \( A \) and \( B \) with \( B \subset A \), \( A \setminus B \) stands for \( A \) excluded \( B \).
II. PROBLEM STATEMENT

Consider the input and output quantized feedback system in Fig. 1 where the system to be controlled is represented by the following state-space model:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k)
\end{align*}
\]  

(1)

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n}, C \in \mathbb{R}^{1 \times n}, x \in \mathbb{R}^{n} \) is the state, \( u \in \mathbb{R} \) is the control input and \( y \in \mathbb{R} \) is the measurement.

The above system is supposed to be controlled by an output feedback controller with a state-space model as follows:

\[
\begin{align*}
    \xi(k+1) &= A_c \xi(k) + B_c Q_1(Cx(k)) \\
    w(k) &= C_c \xi(k)
\end{align*}
\]

(2)

where \( \xi \in \mathbb{R}^{n_c} \) is the controller state, \( v \in \mathbb{R} \) is the controller input, \( w \in \mathbb{R} \) is the controller output, and \( A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c} \) and \( C_c \in \mathbb{R}^{1 \times n_c} \) are given matrices.

The input to the controller and to the system are given by the following relations:

\[
\begin{align*}
    v(k) &= Q_1(y(k)) \\
    u(k) &= Q_2(w(k))
\end{align*}
\]

(3)

where \( Q_1(\cdot) \) and \( Q_2(\cdot) \) are static finite-level logarithmic quantizers as described in the sequel.

\[\text{System} \xrightarrow{\text{Quantizer 2}} \text{Dynamic Controller} \xrightarrow{\text{Quantizer 1}} y(k) \quad \text{Input} \]  

Fig. 1. Feedback control system with input and output quantization.

It is assumed that the quantizers \( Q_1(\cdot) \) and \( Q_2(\cdot) \) have a logarithmic law with quantization levels given by the set:

\[
\mathcal{V}_i = \{ \pm m_{i,j} : m_{i,j} = \rho_i^j \mu_i, \ j = 0, 1, 2, \ldots, N_i-1 \} \cup \{0\},
\]

(4)

where \( N_i \) is the number of positive quantization levels and \( \mu_i > 0 \) is the largest admissible level. Note that a small (large) \( \rho_i \) implies coarse (dense) quantization. As an abuse of terminology, \( \rho_i \) will be referred to as quantization density.

This paper is concerned with investigating the stability of the closed-loop system of (1)-(3), where \( Q_1(\cdot) \) and \( Q_2(\cdot) \) are logarithmic quantizers with finite alphabets obeying the following constructive law:

\[
Q_i(v) = \begin{cases} 
    \mu_i, & \text{if } v > \frac{\mu_i}{(1+\delta_i)}, \ \mu_i > 0 \\
    \rho_i^j \mu_i, & \text{if } \frac{\rho_i^j \mu_i}{(1+\delta_i)} < v \leq \frac{\rho_i^{j+1} \mu_i}{(1+\delta_i)}, \\
    0, & \text{if } 0 \leq v \leq \frac{\rho_i^{N_i-1} \mu_i}{(1+\delta_i)} \\
    -Q(-v), & \text{if } v < 0
\end{cases}
\]

(5)

where

\[
\delta_i = 1 - \frac{\rho_i}{1 + \rho_i}.
\]

(6)

It is assumed that the information between the system output and the controller and between the controller and the system input is transmitted via a shared communication link with limited bandwidth. In addition, the input and output quantizers are independent with possibly different quantization densities, which is a natural setting in this scenario.

III. PRELIMINARIES

In this section, we recall a result derived in [11] on quadratic stability of input and output quantized feedback linear systems with ideal static logarithmic quantizers. To this end, consider the feedback system of (1)-(3) and assume that \( Q_1(\cdot) \) and \( Q_2(\cdot) \) are ideal static logarithmic quantizers. An ideal static logarithmic quantizer \( \breve{Q}(\cdot) \) is defined as follows:

\[
\breve{Q}(v) = \begin{cases} 
    \rho_i^j \mu_i, & \text{if } \frac{\rho_i^j \mu_i}{(1+\delta_i)} < v \leq \frac{\rho_i^{j+1} \mu_i}{(1+\delta_i)}, \\
    0, & \text{if } v = 0 \\
    -\breve{Q}(-v), & \text{if } v < 0
\end{cases}
\]

(7)

which is illustrated in Fig. 2.

\[\theta = (1+\delta)v \quad \theta = v \quad \theta = (1-\delta)v \]

(8)

\[\theta = \breve{Q}(v) \]

\[\delta = \frac{1}{\rho_i} \]

Fig. 2. Logarithmic quantizer with an infinite number of levels.

The closed-loop system of (1)-(3) can be represented by the state-space model as follows:

\[
\begin{align*}
    x(k+1) &= Ax(k) + BQ_2(C_c \xi(k)) \\
    \xi(k+1) &= A_c \xi(k) + B_c Q_1(Cx(k))
\end{align*}
\]

(9)

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which can be shortened as
\[ \tilde{x}(k+1) = f(x(k), \xi(k), Q_1, Q_2) \]  \( (9) \)
where
\[ \begin{bmatrix} x \\ \xi \end{bmatrix}, f(x, \xi, Q_1, Q_2) = \begin{bmatrix} Ax + BQ_2 (C_c \xi) \\ A_c \xi + B_c Q_1 (C x) \end{bmatrix} \]  \( (10) \)

Consider a Lyapunov function candidate \( V(\tilde{x}) = \tilde{x}' P \tilde{x} \), with \( P > 0 \), for the closed-loop system in (9) and let
\[ \Phi = f(x, \xi, Q_1, Q_2)' P f(x, \xi, Q_1, Q_2) - (1 - \varepsilon) \tilde{x}' P \tilde{x} \]  \( (11) \)
where \( \Phi = \Phi(x(\xi, Q_1, Q_2, \varepsilon)) \) and \( \varepsilon \) is a positive scalar.

Since along the trajectory of system (9) we have
\[ V(\tilde{x}(k+1)) - V(\tilde{x}(k)) < \Phi(x(\xi, Q_1, Q_2, \varepsilon)) \]  \( (12) \)
then, system (9) is quadratically stable if and only if there exist a matrix \( P > 0 \) and a scalar \( \varepsilon > 0 \) such that
\[ \Phi(x(\xi, Q_1, Q_2, \varepsilon)) \leq 0, \forall x \in \mathbb{R}^n, \xi \in \mathbb{R}^{n_c}. \]  \( (13) \)

Next, let
\[ \bar{\Delta}(\Delta_1, \Delta_2) = \begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} B(1+\Delta_1) & [0 & C_c] \\ B_c(1+\Delta_1) & [0 & C] \end{bmatrix} \]  \( (14) \)
and
\[ \Omega(\Delta_1, \Delta_2) = \bar{\Delta}(\Delta_1, \Delta_2)' P \bar{\Delta}(\Delta_1, \Delta_2) - P. \]  \( (15) \)

Then, the following result holds:

**Theorem 1 (11):** Consider the closed-loop system in (9) and some given \( P > 0 \). Then, (13) holds for some \( \varepsilon > 0 \), i.e. system (9) is quadratically stable, if and only if
\[ \Omega(\Delta_1, \Delta_2) < 0, \forall \Delta_1, \Delta_2 : |\Delta_1| \leq \delta_1, |\Delta_2| \leq \delta_2. \]  \( (16) \)

Theorem 1 implies that the quadratic stabilization problem for the input-output quantized feedback system with static infinite-level logarithmic quantizers can be transformed, with no conservatism, into a standard robust control problem. This result is strong in the sense that the quadratic stability of the uncertain system \( \tilde{x}(k+1) = \bar{\Delta}(\Delta_1, \Delta_2) \tilde{x}(k) \), with norm-bounded uncertainty \( |\Delta_i| \leq \delta_i, i = 1, 2 \), is a necessary and sufficient condition for the quadratic stability of the quantized closed-loop system of (9). In other words, the sector bound condition:
\[ [Q_i(v) - (1-\delta_i)v] [Q_i(v) - (1+\delta_i)v] \leq 0, i = 1, 2 \]  \( (17) \)
is non-conservative to model infinite-level logarithmic input and output quantizers in the sense of quadratic stability.

**IV. STABILITY ANALYSIS**

The result given in Section III applies to input and output quantized feedback systems where the quantizers follow a logarithmic law and have an infinite number of levels. For finite level quantizers, the convergence of the state trajectory to the system origin (the equilibrium point under analysis) cannot be guaranteed in general. In such scenario, LMI based conditions are derived in the sequel to ascertain the state trajectory convergence, in finite time, to a small invariant region in the neighborhood of the system origin.

### A. General Setup

Firstly, consider the augmented system which represents the closed-loop system of (1)-(3):
\[
\begin{aligned}
\zeta(k+1) &= A_a \zeta(k) + B_a p(k) \\
g(k) &= C_a \zeta(k) \\
p(k) &= Q_a(q(k))
\end{aligned}
\]  \( (18) \)
where
\[
\zeta = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} y \\ w \end{bmatrix},
\]
\[
A_a = \begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 & B \\ B_c & 0 \end{bmatrix}, \quad C_a = \begin{bmatrix} C & 0 \\ 0 & C_c \end{bmatrix}, \quad Q_a(q) = \begin{bmatrix} Q_1(q_1) & 0 \\ 0 & Q_2(q_2) \end{bmatrix}.
\]

Accordingly to the closed-loop system in (18) and the quantizers in (5), we introduce the following sets:
\[
B_i = \{ \zeta \in \mathbb{R}^{n_c} : |C_a \zeta| \leq \mu_i/(1-\delta_i) \}, \quad i = 1, 2
\]
\[
C_i = \{ \zeta \in \mathbb{R}^{n_c} : |C_a \zeta| \leq \epsilon_i \}, \quad \epsilon_i = \rho_i^{N_i-1} \mu_i/(1+\delta_i)
\]
where \( n_c = n+n_c \) and \( C_a \) is the \( i \)-th row of the matrix \( C_a \), namely:
\[
C_{a_i} = \begin{bmatrix} 0 \\ C_c \end{bmatrix}, \quad C_{a_2} = \begin{bmatrix} [0 & C_c] \end{bmatrix}.
\]

The sets \( B_i \) and \( C_i, i = 1, 2 \), are related to respectively the largest and smallest quantization levels of the quantizer \( Q_i \). Notice that \( B_i \) and \( C_i \) are symmetric sets with respect to the origin, which are unbounded in the directions defined by the vectors of an orthogonal basis of the null space of \( C_a \) and are bounded by two hyperplanes orthogonal to \( C_a^{T} \). The distance between these hyperplanes is \( 2\mu_i/(1-\delta_i)^{-1}/\sqrt{C_a^{T} C_a} \) for \( B_i \) and \( 2\epsilon_i/\sqrt{C_a^{T} C_a} \) for \( C_i \).

It turns out that whenever the state \( \zeta \) of system (18) belongs to \( C_i \), one has \( Q_i(\zeta_i) = 0 \), which leads to a zero input signal \( p_i \) for the system (18). Hence, the trajectory of \( \zeta \) will not converge to the system origin, implying that asymptotic stability of the closed-loop system is not ensured. To tackle this behavior, in the sequel we introduce a notion of stability, which was inspired by the concept of practical stability proposed in [4].

Let the quadratic functions
\[ V(\zeta) = \zeta' P \zeta, \quad V_a(\zeta) = \zeta' P_a \zeta, \quad P > 0, \quad P_a > P \]
where \( \zeta \) is as in (19), and the following sets:
\[
D = \{ \zeta \in \mathbb{R}^{n_c} : V(\zeta) \leq 1 \}
\]
\[
A = \{ \zeta \in \mathbb{R}^{n_c} : V_a(\zeta) \leq 1 \}
\]
\[
C_p = \{ \zeta \in C_1 \cup C_2 : D V_a(\zeta) \geq 0 \}
\]
where the notation \( D g(\zeta(\zeta)) \), for a real function \( g(\cdot) \), is defined by \( D g(\zeta(\zeta)) := g(\zeta(\zeta+1)) - g(\zeta(\zeta)) \).

**Definition 1:** The quantized closed-loop system in (18) is said to be widely quadratically stable if there exists quadratic
functions \(V(\zeta)\) and \(V_a(\zeta)\) as in (22) satisfying the following conditions:

\[
A \subset D, \quad D \subset B_i, \quad i = 1, 2 \quad (26)
\]

\[
DV(\zeta) < 0, \quad \forall \zeta \in D \setminus (C_1 \cup C_2) \quad (27)
\]

\[
DV_a(\zeta) < 0, \quad \forall \zeta \in A \setminus C_p \quad (28)
\]

\[\zeta(k+1) \in A \quad \text{whenever} \quad \zeta(k) \in C_p. \quad (29)\]

The above stability notion ensures that for any \(\zeta(0) \in D\), the trajectory of \(\zeta\) will converge to the set \(A\) in finite time. Moreover, \(\zeta(k) \in A, \forall k \geq \bar{k}\), for some finite integer \(\bar{k} > 0\).

In view of that, \(A\) is said to be an attractor of \(D\) and the set \(D\) will be referred to as a set of admissible initial conditions.

For that, we partition \(C_p\) into three complementary sets:

Theorem 2: Consider the system (1), a given controller (2) and the feedback laws in (3) with finite-level quantizers \(Q_1(\cdot)\) and \(Q_2(\cdot)\) as defined in (5), where \(\mu_i, \rho_i\) and \(N_i\) are given. The closed-loop system (18) is widely quadratically stable if there exist symmetric matrices \(P\) and \(P_a\), diagonal matrices \(T > 0\) and \(T_a > 0\), and positive scalars \(\tau_1, \tau_2, \tau_3, \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_1\) and \(\bar{\tau}_2\) satisfying the following inequalities:

\[
P > 0, \quad P_a - P > 0, \quad (34)
\]

\[
P - (1 - \delta_i)^2 \mu_i^{-2} C_{a_i}^2 C_{a_i} > 0, \quad i = 1, 2, \quad (35)
\]

\[
\begin{bmatrix}
\Upsilon_1 & \Upsilon_2' \\
\Upsilon_2 & \Upsilon_3
\end{bmatrix} < 0, \quad (36)
\]

\[
\begin{bmatrix}
\Upsilon_{a1} & \Upsilon_{a2}' \\
\Upsilon_{a2} & \Upsilon_{a3}
\end{bmatrix} < 0, \quad (37)
\]

\[
\tau_3 - (\tau_1 + \tau_2) \geq 0, \quad (38)
\]

\[
P_a + \sum_{i=1}^{2} \tau_i \epsilon_i^{-2} C_{a_i}^2 C_{a_i} - (1 + \tau_3) A_{a_i}^T P_a A_{a_i} \geq 0, \quad (39)
\]

\[
\bar{\tau}_1 - \bar{\tau}_i \geq 0, \quad i = 1, 2, \quad (40)
\]

\[
\begin{bmatrix}
U_1(i,j) & U_2(i,j) \\
U_2(i,j) & U_3(i,j)
\end{bmatrix} \geq 0, \quad i,j = 1, 2, i \neq j, \quad (41)
\]

where \(\delta_i\) is related to \(\rho_i\) by (6), \(\epsilon_i\) is as in (21) and

\[
U_1(i,j) = P_a + \tau_i \epsilon_i^{-2} C_{a_i}^2 C_{a_i} + \bar{\tau}_j (1 - \delta_j^2) C_{a_j}^2 C_{a_j}
\]

\[- (1 + \bar{\tau}_i) A_{a_i}^T P_a A_{a_i}, \quad(i,j = 1, 2, i \neq j, \quad (41)
\]

\[
U_2(i,j) = -\bar{\tau}_j C_{a_j} - (1 + \bar{\tau}_i) B_{a_j}^T P_a A_{a_j}, \quad (42)
\]

\[
U_3(i,j) = \bar{\tau}_j - (1 + \bar{\tau}_i) B_{a_j}^T P_a A_{a_j}, \quad (43)
\]

\[
B_{a_i} = \begin{bmatrix} 0 & B_{a_i}' \end{bmatrix}, \quad B_{a_2} = \begin{bmatrix} B' & 0 \end{bmatrix}. \quad (44)
\]

Moreover, the set \(D\) of admissible initial conditions and its attractor \(A\) are given by (23) and (24), respectively.

**Proof.** Firstly, in view of (20), (23) and (24), the second inequality of (34) together with (35) ensure that \(A \subset D\) and \(D \subset B_i\), \(i = 1, 2\), respectively.

Next, (36) ensures the feasibility of (32), implying that condition (27) is satisfied. Similarly, (37) guarantees that (33) holds which together with (26) and the definition of \(C_p\) imply that (28) holds.

In the sequel it will be shown that (38)-(41) ensure (29). For that, we partition \(C_p\) into three complementary sets:
and consider two cases:
(i) \( \zeta(k) \in C_{p_1} \); Letting \( \phi \in \mathbb{R}^{n_c} \) and adding (38) to (39) post-multiplied by \( \phi \) and pre-multiplied by \( \phi' \), we get
\[
(1 - \phi'_a P_a A_a \phi - \tau_3^{-1} \phi' (A'_a P_a A_a - P_a) \phi) \nonumber
\]
\[
- \tau_3^{-1} \sum_{i=1}^{2} \tau_i (1 - \epsilon_i^{-2} \phi' C_i A_i \phi) \geq 0, \forall \phi \in \mathbb{R}^{n_c}.
\]
Applying the S-procedure, the above inequality yields
\[
(\phi'_a P_a A_a \phi \leq 1, \forall \phi \in \mathbb{R}^{n_c} : \epsilon_i^{-2} \phi' C_i A_i \phi \leq 1, \quad i = 1, 2, \phi' (A'_a P_a A_a - P_a) \phi \geq 0).
\]
Note that (42), for \( i = 1, 2 \), is equivalent to \( \phi \in C_1 \cap C_2 \). Let \( \phi = \zeta(k) \), where \( \zeta(k) \) is as in (18). Since \( \zeta(k) \in C_1 \cap C_2 \), the input signal \( p(k) \) of system (18) is zero, which leads to
\[
\zeta(k+1) P_a \zeta(k+1) \leq 1, \quad \forall \zeta(k) \in C_1 \cap C_2 : \zeta(k+1) P_a \zeta(k+1) = \zeta(k)' P_a \zeta(k) \geq 0.
\]
This guarantees that \( \zeta(k+1) \in A \) whenever \( \zeta(k) \in C_{p_1} \).

(ii) \( \zeta(k) \in C_{p_i}, i=1,2 \); Let \( \phi = \zeta(k) \in \mathbb{R}^{n_c} \) and \( \psi \in \mathbb{R} \). Adding (40) to (41) post-multiplied by \( [\phi' \ \psi'] \) and pre-multiplied by \( [\phi' \ \psi'] \), we obtain:
\[
1 - (A_a \phi + B_{a_j} \psi)' P_a (A_a \phi + B_{a_j} \psi) + \tilde{\gamma}_j^{-1} \tilde{\gamma}_i (\epsilon_i^{-2} \phi' C_i A_i \phi - 1) - \tilde{\gamma}_i^{-1} \tilde{\gamma}_j [\psi - (1 - \delta_j) C_{a_j} \phi][\psi - (1 + \delta_j) C_{a_j} \phi] \geq 0
\]
for all \( \phi \in \mathbb{R}^{n_c} \) and \( \psi \in \mathbb{R} \). Applying the S-procedure, the latter inequality implies for all \( i, j = 1, 2 \) and \( i \neq j \)
\[
(A_a \phi + B_{a_j} \psi)' P_a (A_a \phi + B_{a_j} \psi) \leq 1, \forall \phi \in \mathbb{R}^{n_c}, \psi \in \mathbb{R} : \epsilon_i^{-2} \phi' C_i A_i \phi \leq 1
\]
\[
\left[ \begin{array}{c}
\phi' \\
A'_a P_a A_a - P_a \\
B'_{a_j} P_a A_a - B_{a_j} A_a
\end{array} \right] \begin{bmatrix}
\phi \\
\psi
\end{bmatrix} \geq 0
\]
\[
\left[ \begin{array}{c}
\psi - (1 - \delta_j) C_{a_j} \phi
\end{array} \right] \begin{bmatrix}
\psi - (1 + \delta_j) C_{a_j} \phi
\end{bmatrix} \leq 0
\]

Note that \( \epsilon_i^{-2} \phi' C_i A_i \phi \leq 1 \) is equivalent to \( \phi \in C_i \). Let \( \phi = \zeta(k) \) and \( \psi = p_{a_j}(k) \), where \( \zeta(k) \) and \( p_{a_j}(k) \) are as in (18). Since for \( \zeta(k) \in C_i \), the input signal \( p_{a_j}(k) \) of system (18) is zero and \( p_{a_j}(k) \) satisfies the sector bound inequality in (17), the conditions in (43) guarantee that
\[
\zeta(k+1)' P_a \zeta(k+1) \leq 1, \forall \zeta(k) \in C_i \cup C_j : \zeta(k+1)' P_a \zeta(k+1) - \zeta(k)' P_a \zeta(k) \geq 0
\]
which ensures \( \zeta(k+1) \in A \) whenever \( \zeta(k) \in C_{p_i}, i=1,2 \).

In the light of the above, it follows that the system (18) is widely quadratically stable.

C. Computational Issues

Observe that (39) and (41) are not jointly convex in \( \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3 \) and \( P_a \). However, the conditions in (34)-(41) turn out to be LMIs when the scalars \( \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3 \) are given \emph{a priori}. Thus, applying a gridding procedure, we can perform a search on the latter scalars to obtain a feasible solution to the inequalities in (34)-(41). In addition, we may desire to obtain a maximized set \( \mathcal{D} \) (in the sense of its volume), or a minimized set \( \mathcal{A} \). As the set \( \mathcal{D} \) is an ellipsoid, one way to approximately maximize its size is to minimize \( \text{trace}(P) \). The reason for this is that, for \( P \in \mathbb{R}^{n_c \times n_c} \), we have \( n_c \text{trace}(P) \leq \text{trace}(P^{-1}) \) and \( \text{trace}(P^{-1}) \) is the sum of the squared semi-axis lengths of the ellipsoid \( \mathcal{D} \). Similarly, we can approximately minimize the size of \( \mathcal{A} \) by maximizing \( \text{trace}(P_a) \).

In view of the above discussion, the size of \( \mathcal{D} \) in Theorem 2 can be approximately maximized by means of the following optimization problem:
\[
\min_{\gamma_1, P_a, \tilde{\tau}_1, \tilde{\tau}_2} \gamma_1, \quad \text{subject to (34)-(41)}
\]
\[
\gamma_1 - \text{trace}(P) \geq 0, \quad \tilde{\tau}_1, \tilde{\tau}_2 > 0.
\]
In a similar way, the size of \( \mathcal{A} \) can be approximately minimized via the following optimization problem:
\[
\max_{\gamma_2, P_a, \tilde{\tau}_1, \tilde{\tau}_2} \gamma_2, \quad \text{subject to (34)-(41)}
\]
\[
\text{trace}(P_a) - \gamma_2 \geq 0, \quad \gamma_1, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_2 > 0.
\]

In many situations we desire to jointly optimize the size of \( \mathcal{D} \) and \( \mathcal{A} \), which is generally difficult to solve. A possible solution to jointly maximize \( \mathcal{D} \) and minimize \( \mathcal{A} \) is obtained by minimizing a scalar \( \gamma := \gamma_1 / \gamma_2 \), where \( \gamma_1 \) and \( \gamma_2 \) are the parameters in (44) and (45). This optimization problem can be formulated following the same steps as in [13] in which a single quantizer is considered.

V. NUMERICAL EXAMPLE

To illustrate the results of this papers, we consider the academic system of Example 3.1 in [5] of a non-minimum phase open-loop unstable discrete-time system as given below:
\[
\begin{align*}
\begin{cases}
\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= 2x_2(x) + u(k)
\end{aligned}
\end{cases}
\end{align*}
\]
\[
y(k) = -3x_1(k) + x_2(k)
\]
that has a transfer function \( G(z) = \frac{z^3 - z^{-2}}{z + 2} \).

Firstly, an optimal output feedback \( H_{\infty} \) control law is designed for system (46) leading to the following controller:
\[
\begin{align*}
\begin{cases}
\xi(k+1) &= \begin{bmatrix} 0 & 1 \\ -10 & -1.667 \end{bmatrix} \xi(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k) \\
v(k) &= \begin{bmatrix} 0 & 6.667 \end{bmatrix} \xi(k)
\end{cases}
\end{align*}
\]

We assume that the finite level quantizers \( Q_1(\cdot) \) and \( Q_2(\cdot) \) in (3) are equal and with the parameters as below:
\[
\delta_i = 10^{-2}, \quad \epsilon_i = 10^{-3}, \quad \mu_i = 10, \quad i = 1, 2
\]
By applying Theorem 2 and considering the minimization of $\gamma = \gamma_1/\gamma_2$ as detailed in Section IV-C, we obtain:

$$P = \begin{bmatrix}
42.0766 & -20.9740 & 139.9761 & 0.1328 \\
-20.9740 & 10.8040 & -69.8186 & 1.0008 \\
139.9761 & -69.8186 & 466.6369 & 0.4443 \\
0.1328 & 1.0008 & 0.4443 & 4.3304
\end{bmatrix},$$

$$P_0 = 10^5 \begin{bmatrix}
0.2241 & -0.1119 & 0.7471 & 0.0006 \\
-0.1119 & 0.0559 & -0.3729 & -0.0000 \\
0.7471 & -0.3729 & 2.4904 & 0.0021 \\
0.0006 & -0.0000 & 0.0021 & 0.0012
\end{bmatrix},$$

for $\tau_1 = \tilde{\tau}_1 = \tilde{\tau}_2 = 0.005756$.

Figures 3 and 4 show slices of the sets $\mathcal{D}$ and $\mathcal{A}$ in the plane $\zeta = [x_1 \ x_2 \ 0 \ 0]^T$. For illustrative purposes, stable and unstable trajectories are also plotted in both figures. To ascertain the conservatism of the proposed stability analysis method, the initial conditions (stable and unstable cases) and the terminal part of the stable trajectory are zoomed in the corner of both figures. It turns out that the initial condition of the unstable trajectory, $\zeta(0) = [0.64 \ 0.85 \ 0 \ 0]^T$, is very close to the boundary of $\mathcal{D}$.

**VI. CONCLUDING REMARKS**

This paper has addressed the stability analysis problem of SISO discrete-time linear time-invariant control systems with input and output static finite-level logarithmic quantizers. Based on a relaxed stability notion, an LMI based approach has been devised to estimate a set of admissible initial states and an associated invariant attractor set in a neighborhood of the origin, such that all state trajectories starting in the first set will converge to the attractor in finite time. A numerical example has demonstrated the potentials of the proposed approach.

**REFERENCES**


