Robust Stability of Discrete-Time Linear Descriptor Systems with Time-Varying Uncertainties via Parametric Lyapunov Function

Karina A. Barbosa, Carlos E. de Souza and Daniel Coutinho

Abstract—This paper deals with the problem of admissibility analysis (i.e., regularity, causality and exponential stability) of discrete-time linear descriptor systems with uncertain time-varying parameters. The parameters enter affinely into the state matrix of the system state-space model, and their admissible values and variations are assumed to belong to given intervals. First, necessary and sufficient admissibility conditions for uncertainty-free discrete linear time-varying descriptor systems are presented. Next, strict LMI conditions based on parameter-dependent Lyapunov functions are proposed to ensure robust admissibility of uncertain descriptor systems. Both the cases of Lyapunov functions with affine and quadratic dependence on the system uncertain parameters are considered. The robust admissibility analysis methods incorporate information on available bounds on both the admissible values and variation of the uncertain parameters. Numerical examples are presented to demonstrate the potentials of the proposed methods.

I. INTRODUCTION

In the last decade, increasing attention has been given to the development of control and estimation methods for linear dynamical systems with algebraic constraints on the state variables, known in the literature as descriptor systems or singular systems. This class of dynamic systems is an extension of the standard linear state-space model and can be found in a number of applications, such as in biologic systems, robotics, and power systems. A distinct feature of descriptor systems is that in addition to the stability property, it is also required to ascertain the existence and uniqueness of solution, and causality (in the case of discrete-time systems) or the non-existence of impulsive modes (for continuous-time systems). The latter three properties are commonly referred to in the literature as system admissibility ([1], [2]).

In the case of uncertainty-free discrete-time linear descriptor systems, necessary and sufficient conditions of admissibility have been proposed, as for instance, in [2], [3] and [4], whereas LMI based approaches of admissibility of descriptor systems with polytopic-type uncertain constant parameters have been presented in [5] and [6]. As far as discrete-time linear descriptor system with uncertain time-varying parameters is concerned, few results can be found in the literature. Recently, an admissibility condition has been proposed in [7] considering the system in a difference-algebraic representation and using an affine parameter-dependent Lyapunov function (PDLF). Note that in the latter work it has also been shown that results developed in [8] and [9] are erroneous. To the best of the authors’ knowledge, to-date the problem of admissibility analysis for general descriptor systems with time-varying parameter uncertainty based on more complex PDLFs has not been yet fully addressed.

This paper considers discrete-time linear descriptor systems subject to convex-bounded uncertain time-varying parameters and addresses the problem of admissibility analysis. The uncertain parameters appear affinely in the state matrix of the system state-space model and their values and variations are unknown, however they are assumed to be constrained to given intervals. Initially, we consider uncertainty-free discrete linear time-varying (DLTV) descriptor systems and develop two equivalent necessary and sufficient conditions of admissibility. The first condition is in terms of a generalized Lyapunov difference inequality and the second one is tailored via a strict linear matrix difference inequality. Based on the latter condition, we derive methods of robust admissibility analysis of uncertain descriptor systems (i.e. the system admissibility holds for all allowed trajectories of the uncertain parameters) in terms of PDLFs. The cases of Lyapunov functions with either affine or quadratic dependence on the system uncertain parameters are considered. The proposed methods are given via strict LMIs and, in contrast with the work of [7], they allow for incorporating known bounds on the variation of the uncertain parameters and do not require the transformation of the system model to a difference-algebraic form. The potentials of the robust admissibility analysis methods presented in the paper are illustrated via two numerical examples analysed in the literature.

Notation. $\mathbb{Z}^+$ is the set of nonnegative integers, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices, $I_n$ is the $n\times n$ identity matrix, $0_n$ and $0_{m\times n}$ are the $n\times n$ and $m\times n$ matrices of zeros, respectively, and $\text{diag}\{\cdot\cdot\cdot\}$ stands for a block-diagonal matrix. For a real matrix $S$, $S^T$ denotes its transpose, $\text{He}\{S\}$ stands for $S+S^T$ and $S>0$ means that $S$ is symmetric and positive-definite. For a symmetric block matrix, the symbol $*$ denotes the transpose of the blocks outside the main diagonal block and $\otimes$ is the Kronecker product. For a given convex bounded polyhedral domain $\mathcal{B}$, $\mathcal{V}_B$ denotes the set of all the vertices of $\mathcal{B}$. 
II. ADMISSION OF DLTV DESCRIPTOR SYSTEMS

Consider the following DLTV descriptor system:

$$Ex(k + 1) = A(k)x(k), \quad Ex_0 = x_0,$$

(1)

where $$x(k) \in \mathbb{R}^n$$ is the state, $$A(k) \in \mathbb{R}^{n \times n}$$ is a bounded matrix function, and $$E \in \mathbb{R}^{n \times n}$$ is a constant matrix, which is allowed to be singular with $$\text{rank}\{E\} = r \leq n$$.

This section addresses the issues of regularity, causality and exponential stability of the system (1). Firstly, we will introduce some basic definitions and a Lyapunov stability result for the system (1). Recall that when the state matrix $$A(k)$$ is time-invariant, i.e. $$A(k) = A$$, the descriptor system $$Ex(k + 1) = Ax(k)$$, or the pair $$(E, A)$$, is said to be regular if $$\det(\lambda E - A)$$ is not identically zero, and a regular descriptor system is said to be causal if $$\text{rank}\{E\} = \text{deg}(\det(\lambda E - A))$$ (see, for instance, [10]). In the case of the time-varying system (1), the latter definitions can be extended as follow:

**Definition 1:**

(a) System (1), or the pair $$(E, A(k))$$, is regular if the pair $$(E, A(k))$$ is regular for all $$k \geq 0$$;

(b) System (1) is causal if $$\text{rank}\{E\} = \text{deg}(\det(\lambda E - A(k)))$$, $$\forall k \geq 0$$.

From Definition 1, one can infer that causality implies that the pair $$(E, A(k))$$ is regular.

**Definition 2:** System (1) is said to be admissible if it is causal and exponentially stable.

The next lemma presents an exponential stability result for system (1) based on the Lyapunov stability theory.

**Lemma 1:** Consider a regular and causal DLTV descriptor system as in (1). Let $$V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}^+$$ be a continuous function and $$\Delta V(x(k)) := V(x(k + 1)) - V(x(k))$$ be the forward-difference of $$V(x(k))$$ along the trajectory of system (1). Then, system (1) is exponentially stable if there exist positive scalars $$\varepsilon_1$$ and $$\varepsilon_2$$ such that

$$\varepsilon_1 \|Ex(k)\|^2 \leq V(x(k)) \leq \varepsilon_2 \|Ex(k)\|^2, \quad \forall k \in \mathbb{Z}^+,$$

$$\Delta V(x(k)) < 0, \quad \forall k \in \mathbb{Z}^+.$$

Note that since the matrix $$E$$ is constant, there exist a time-invariant coordinate transformation, which is based on a singular value decomposition (SVD) of $$E$$, such that the system (1) can be represented in the following equivalent form, referred to as SVD coordinate system ([11]):

$$\tilde{E}\xi(k + 1) = \tilde{A}(k)\xi(k)$$

(2)

with

$$\xi(k) = N^{-1}x(k), \quad \tilde{E} := M^TEN = \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix},$$

$$\tilde{A}(k) := M^TA(k)N = \begin{bmatrix} A_1(k) & A_2(k) \\ A_3(k) & A_4(k) \end{bmatrix},$$

(3)

(4)

where $$M$$ and $$N$$ are nonsingular $$n \times n$$ real matrices and $$\tilde{A}(k)$$ is partitioned accordingly to the matrix $$\tilde{E}$$.

Similarly to the continuous-time case, as the matrix $$E$$ is constant, a number of results for discrete linear time-invariant descriptor systems also holds for the system (1), such as that presented in the sequel (the proof is omitted because it is similar to that in the time-invariant context).

**Lemma 2:** The system (1) is causal if and only if the matrix $$A_4(k)$$ is nonsingular for all $$k \in \mathbb{Z}^+$$.

In view of the SVD coordinate system in (2)-(4) we have the following lemma.

**Lemma 3:** If the matrix $$A_4(k)$$ in (4) is nonsingular for all $$k \in \mathbb{Z}^+$$, then there exist a time-varying coordinate transformation such that the system (1) can be represented in an equivalent Weierstrass form given by:

$$\tilde{E}\zeta(k + 1) = \tilde{A}(k)\zeta(k)$$

(5)

where $$\zeta(k) = \tilde{N}^{-1}(k)x(k)$$, with $$x(k)$$ as in system (1),

$$\tilde{E} := \tilde{M}^T(k)E\tilde{N}(k + 1) = \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix},$$

$$\tilde{A}(k) := \tilde{M}^T(k)A(k)\tilde{N}(k) = \begin{bmatrix} A_f(k) & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

(6)

(7)

(8)

where $$A_f(k) = A_1(k) - A_2(k)A_4^{-1}(k)A_3(k)$$.

Hence, pre-multiplying both sides of (2) by $$\tilde{M}^T(k)$$ and letting $$\zeta(k) = N^{-1}(k)\xi(k)$$, $$\tilde{M}(k) = MMM(k)$$ and $$\tilde{N}(k) = NNN(k)$$, we readily obtain the Weierstrass form in (5)-(8).

Observe that by combining Lemmas 2 and 3, the following result can be readily derived.

**Lemma 4:** The system (1) is admissible if and only if the matrix $$A_4(k)$$ is nonsingular for all $$k \in \mathbb{Z}^+$$ and the time-varying system

$$x(k + 1) = A_f(k)x(k)$$

(9)

is exponentially stable, where $$A_f(k)$$ is as in (8).

The next result presents necessary and sufficient conditions of admissibility of system (1).

**Theorem 1:** Consider the DLTV descriptor system (1) and let $$E_0 \in \mathbb{R}^{n \times (n-r)}$$ be any constant full rank-column matrix such that $$E^TE_0 = 0$$. Then, system (1) is admissible if only if any of the following equivalent conditions holds:

(a) There exist bounded matrices $$X(k) > 0$$ and $$R(k)$$, $$\forall k \in \mathbb{Z}^+$$ such that:
\[ A^T(k)X(k + 1)A(k) + He\{ A^T(k)E_0R(k) \} \]
\[ - E^T(k)E(k) < 0, \ \forall \ k \in \mathbb{Z}^+. \quad (10) \]
(b) There exist bounded matrices \( P(k) > 0, Q(k), F(k) \) and \( G(k), \ \forall \ k \in \mathbb{Z}^+ \) such that:
\[
\begin{bmatrix}
-ET(k)P(k)E + \text{He}\{ F(k)A(k) \} & \ast \\
E_0Q(k) + G(k)A(k) - F^T(k)P(k+1) - \text{He}\{ G(k) \}
\end{bmatrix} < 0,
\quad \forall \ k \in \mathbb{Z}^+.
\]
\[ (11) \]

**Proof.** (a) Sufficiency: Firstly, it will be proved that the system (1) is causal. In view of Lemma 2 this will be established by showing that the matrix \( A(k) \) is nonsingular for all \( k \in \mathbb{Z}^+ \). To this end, consider the SVD coordinate system as defined in (2)-(4) and let
\[
\tilde{X}(k) := M^{-1}X(k)M^{-T} = \begin{bmatrix} X_1(k) & X_2(k) \\ X^T_2(k) & X_4(k) \end{bmatrix}.
\]  
(12)
where the dimensions of the partitions are compatible with those of \( \tilde{E} \). Pre- and post-multiplying (10) by \( N^T \) and \( N \), respectively, and considering (3)-(12), it can be readily verified that (10) is equivalent to
\[
\tilde{A}^T(k)\tilde{X}(k + 1)\tilde{A}(k) + \text{He}\{ \tilde{A}^T(k)M^{-1}E_0R(k)N \} \]
\[ - E^T(k)E(k) < 0, \ \forall \ k \in \mathbb{Z}^+. \quad (13) \]

Similarly as in [3], in view of (3) the matrix \( E_0 \) can be assumed to be of the form
\[
E_0 = M \begin{bmatrix} 0_{n \times (n-r)} & I_{n-r} \end{bmatrix} W, \quad (14)
\]
where \( W \in \mathbb{R}^{(n-r) \times (n-r)} \) is a nonsingular matrix and let
\[
WR(k)N = \begin{bmatrix} R_1(k) & R_2(k) \end{bmatrix}, \quad (15)
\]
where \( R_1(k) \in \mathbb{R}^{(n-r) \times r} \) and \( R_2(k) \in \mathbb{R}^{(n-r) \times (n-r)} \).

Denote the left-hand side of (13) by \( \Psi(k) \) and partition \( \Psi(k) \) accordingly to \( E \), namely
\[
\Psi(k) = \begin{bmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{bmatrix}.
\]
Considering (3), (4) and (13)-(15), it can be easily established that
\[
\Psi_{22} = A^T(k)X_1X_2 + A^T(k)X_4A_4 + \text{He}\{ A^T(k)(X^T_2A_2 + R_2) \}
\]
where the argument \( k \) of the matrices has been omitted. Since \( X_1(k) > 0, X_4(k) > 0 \) and \( \Psi_{22}(k) < 0 \) for all \( k \in \mathbb{Z}^+ \), it follows that \( \text{He}\{ A^T(k)(X^T_2A_2 + R_2) \} < 0, \ \forall \ k \in \mathbb{Z}^+ \). This implies that the matrix \( A(k) \) is nonsingular for all \( k \in \mathbb{Z}^+ \), which ensures that the system (1) is regular and causal.

In order to prove the exponential stability of the system (1), consider the following Lyapunov function candidate
\[
V(x(k)) = x^T(k)E^TX(k)Ex(k), \ \forall \ k \in \mathbb{Z}^+. \quad (16)
\]
with \( X(k) > 0, \ \forall \ k \in \mathbb{Z}^+ \) to be determined.

Letting \( \Delta \) and \( \bar{A} \) be respectively the lower-bound and upper-bound of the minimum and maximum eigenvalues of \( X(k) \), we obtain from (16) that
\[
\Delta \| Ex(k) \|^2 \leq V(x(k)) \leq \bar{A} \| Ex(k) \|^2, \ \forall \ k \in \mathbb{Z}^+. \quad (17)
\]
The forward-difference \( \Delta V(x(k)) \) of \( V(x(k)) \) along the trajectory of (1) is given by:
\[
\Delta V(x(k)) = x^T(k)\left[ A^T(k)X(k+1)A(k) - E^TX(k)E \right] x(k).
\]

Following [3], since \( E^TE_0 = 0 \) it turns out that \( \Delta V(x(k)) \) can be rewritten as
\[
\Delta V(x(k)) = x^T(k)\left[ A^T(k)X(k+1)A(k) - E^TX(k)E \right] x(k)
+ 2x^T(k+1)E^TE_0R(k)x(k)
\]
(18)
for any bounded matrix \( R(k) \) of appropriate dimension. In view of (1) and considering (10), we obtain from (18) that \( \Delta V(x(k)) < 0, \ \forall \ k \in \mathbb{Z}^+ \). Thus, by Lemma 1 it follows that the system (1) is exponentially stable.

**Necessity:** Since the system (1) is admissible, by Lemmas 3 and 4, (1) can be represented in the Weierstrass form given in (5)-(8) with an exponentially stable matrix \( A_f(k) \). Similarly as in the sufficiency proof, (14) and (15) also apply here with the matrices \( M, N, W, R_1 \) and \( R_2 \) replaced by \( M(k), N(k), W(k), R_1(k) \) and \( R_2(k) \), respectively, where \( \tilde{M}(k) \) and \( \tilde{N}(k) \) are the transformation matrices of the Weierstrass form and \( \tilde{W}(k) \) is a nonsingular matrix for all \( k \in \mathbb{Z}^+ \). Moreover, let
\[
\tilde{X}(k) = \begin{bmatrix} \tilde{X}_1(k) & \tilde{X}_2(k) \\ \tilde{X}^T_2(k) & \tilde{X}_4(k) \end{bmatrix} = \tilde{M}^{-1}(k-1)X(k)\tilde{M}^{-T}(k-1)
\]
(19)
where the partitions of \( \tilde{X}(k) \) conform with those of \( \tilde{E} \).

Considering (6), (7), (14), (15) and (19), and applying straightforward matrix manipulations it follows that the left-hand side of (10), denoted by \( \tilde{\Psi}(k) \), can be written as
\[
\tilde{\Psi}(k) = \begin{bmatrix} A^T(k)\tilde{X}_1 + A_f(k)\tilde{X}_1(k) & \ast \\ \tilde{X}_2^T & A_f(k) + \tilde{R}_1(k) & \tilde{X}_4^+ & \text{He}\{ \tilde{R}_2(k) \} \end{bmatrix}
\]
where the matrix \( A_f(k) \) is exponentially stable, by the Lyapunov lemma for DLTV systems (112) there exists a bounded matrix \( \tilde{X}_1(k) > 0, \ \forall \ k \in \mathbb{Z}^+ \) such that \( A^T_f(k)\tilde{X}_1(k+1)A_f(k) - \tilde{X}_1(k) < 0, \ \forall \ k \in \mathbb{Z}^+ \). This implies that we will have \( \tilde{\Psi}(k) < 0, \ \forall \ k \in \mathbb{Z}^+ \) by using the latter matrix \( \tilde{X}_1(k) \) and setting \( \tilde{X}_2(k) = 0, \tilde{X}_4(k) > 0 \) and \( \tilde{R}_1(k) = 0, \ \forall \ k \in \mathbb{Z}^+ \), and with \( \tilde{R}_2(k) \) such that \( \tilde{X}_4(k+1) + \text{He}\{ \tilde{R}_2(k) \} < 0, \ \forall \ k \in \mathbb{Z}^+ \). Therefore, it turns out that (10) holds with the matrices
\[
X(k) = \tilde{M}(k-1)\text{diag}\{ \tilde{X}_1(k), \tilde{X}_4(k) \}\tilde{M}^T(k-1),
\]
\[ R(k) = \tilde{W}^{-1}(k) | 0 \ \tilde{R}_2(k) | \tilde{N}^{-1}(k), \]
where \( \tilde{X}_1(k), \tilde{X}_4(k) \) and \( \tilde{R}_2(k) \) are as defined above.

(a) \( \Rightarrow \) (b): Suppose there exist bounded matrices \( X(k) \) and \( R(k), \ \forall \ k \in \mathbb{Z}^+ \) satisfying (10). Letting \( P(k) = X(k), Q(k) = R(k), F(k) = E_0Q(k) \) and \( G(k) = F(k+1), \ \forall \ k \in \mathbb{Z}^+ \), and applying Schur’s complements, it follows that (10) implies the feasibility of (11).
(b) ⇒ (a): Pre- and post-multiplying (11) by \([ I_n \ A^T(k) ]\) and its transpose, respectively, leads to
\[
A^T(k)P(k+1)A(k) + \text{He}\{A^T(k)E_0Q(k)\} - E^TP(k)E < 0, \ \forall \ k \in \mathbb{Z}^+.
\]
Thus (10) holds with \(X(k) = P(k)\) and \(R(k) = Q(k)\). \(\square\)

Theorem 1 (a) extends a result proposed in [3] for linear discrete linear time-invariant descriptor systems to the time-varying context. On the other hand, Theorem 1 (b) presents a novel necessary and sufficient condition of admissibility of the system (1) in terms of a strict matrix difference inequality. Note that although the conditions of Theorem 1 (a) and (b) are equivalent, the latter condition has the advantage of not involving the product of the Lyapunov function matrix \(P(k)\) and the system state matrix \(A(k)\). This feature allows us deriving robust admissibility conditions for discrete-time linear descriptor systems subject to convex-bounded uncertainty parameters using a parameter-dependent Lyapunov function as it will be presented in the next section.

III. ROBUST ADMISSIBILITY ANALYSIS

This section presents the main results of this paper, namely admissibility conditions of discrete-time linear descriptor systems with time-varying uncertainties in terms of strict LMIs and based on a parameter-dependent Lyapunov function. Specifically, we consider the following class of uncertain discrete-time descriptor systems:

\[
Ex(k+1) = A(\theta(k))x(k), \quad Ex_0 = x_0, \quad (20)
\]

where \(x(k) \in \mathbb{R}^n\) is the state, \(\theta(k) = [\theta_1(k), \ldots, \theta_p(k)]^T \in \mathbb{R}^p\) is a vector of uncertain time-varying parameters, \(A(\theta(k)) \in \mathbb{R}^{n \times n}\) is an affine matrix function of \(\theta(k)\), and \(E \in \mathbb{R}^{n \times n}\) is a constant matrix, which is allowed to be singular with \(\text{rank}\{E\} = r \leq n\). Moreover, let \(\Delta \theta(k) := \theta(k+1) - \theta(k)\).

Similarly as in [13], it is assumed that the parameters \(\theta_i(k)\) and their variations \(\Delta \theta_i(k)\) for \(i = 1, \ldots, p\) and at each time instant, are such that \(\underline{\theta}_i \leq \theta_i(k) \leq \bar{\theta}_i\) and \(\underline{\Delta \theta}_i \leq \Delta \theta_i(k) \leq \bar{\Delta \theta}_i\), where \(\underline{\theta}_i, \bar{\theta}_i, \underline{\Delta \theta}_i\), and \(\bar{\Delta \theta}_i\) are known extremum values of \(\theta_i(k)\) and \(\Delta \theta_i(k)\), respectively, and let \(B\) be the polytope representing the set of admissible values of \((\theta, \Delta \theta)\). Moreover, it is assumed that the admissible values of \(\theta(k)\) and \(\Delta \theta(k)\) are such that if \((\theta(k), \Delta \theta(k)) \in B\), then \(\theta(k+1)\) also belongs to \(B\). Such a polytope \(B\) will be referred to as a consistent polytope. Observe that a consistent polytope must include \(\Delta \theta = 0\) as an admissible value of \(\Delta \theta\). In the sequel, the notation \(\theta \in B\) means that \((\theta, 0) \in B\).

In the sequel we address the problem of robust admissibility analysis for the uncertain system (20) in the sense as defined below.

Definition 3: The system (20) is said to be robustly admissible if the pair \((E, A(\theta(k)))\) is admissible for all \((\theta(k), \Delta \theta(k)) \in B\).

Although this paper focuses on the robust admissibility analysis of uncertain descriptor systems with a constant matrix \(E\), it turns out that the results derived here also apply to the more general class of descriptor systems with a matrix \(E\) that is affine in the uncertain parameters. This can be achieved by transforming the latter system into an equivalent uncertain descriptor system with a constant matrix \(E\) as established in the following result derived in [14].

Proposition 1 ([14]): The uncertain descriptor system represented by

\[
J(\theta) \zeta(k+1) = A(\theta) \zeta(k) \quad (21)
\]

where \(\zeta \in \mathbb{R}^{n_{\zeta}}\) is the state, \(\theta \in \mathbb{R}^p\) is a vector of uncertain parameters belonging to a given polytope, and \(J(\theta)\) and \(A(\theta)\) are \(n_{\zeta} \times n_{\zeta}\) real affine matrix functions of \(\theta\) with \(\text{min}_\theta \text{rank}\{J(\theta)\} = r < n_{\zeta}\), is equivalent to an uncertain descriptor system with a constant matrix \(E\) as in (20) with

\[
x = \begin{bmatrix} \zeta & \theta \otimes \zeta \end{bmatrix}, \quad E = \begin{bmatrix} J_0 & 0 \\ 0 & J \end{bmatrix}, \quad A(\theta) = \begin{bmatrix} A(\theta) & 0 \\ \theta \otimes I_{n_{\zeta}} & -I \end{bmatrix},
\]

\(J(\theta) = J_0 + J(\theta \otimes I_{n_{\zeta}})\), where \(J_0\) and \(J\) are known matrices.

The next lemma presents a necessary and sufficient condition of robust admissibility of the system (20). This result is readily obtained by considering a parametric version of Theorem 1 (b).

Lemma 5: Consider the uncertain descriptor system in (20) and let \(B\) be a polytope of admissible \((\theta, \Delta \theta)\) and \(E_0 \in \mathbb{R}^{n \times (n-r)}\) be a full rank-column matrix such that \(E^T E_0 = 0\). Then, system (20) is robustly admissible if and only if there exist bounded matrix functions \(P(\theta) > 0, \forall \theta \in B\), \(Q(\theta), F(\theta)\), and \(G(\theta)\) satisfying the following inequality:

\[
\begin{bmatrix}
-E^T P(\theta)E + \text{He}\{F(\theta)A(\theta)\} & * \\
E_0 Q(\theta) + G(\theta)A(\theta) - F^T(\theta) & \text{P}(\theta + \Delta \theta) - \text{He}\{G(\theta)\}
\end{bmatrix} < 0,
\]

\(\forall (\theta, \Delta \theta) \in B\). \(\quad (22)\)

Lemma 5 presents a necessary and sufficient condition of robust admissibility for the uncertain discrete-time descriptor in (20) based on a parameter-dependent Lyapunov function \(V(x, \theta) = x^T E^T P(\theta) E x\). Note that, due to the nonlinearity and dependence on \(\theta\), the condition in (22) corresponds to an infinite-dimensional problem which is difficult to be solved.

Based on Lemma 5, in this paper we will derive convex sufficient conditions of robust admissibility in terms of strict LMIs by using matrices \(P(\theta)\) and \(Q(\theta)\) either affine or quadratic in \(\theta\). The first result is obtained by considering the Lyapunov function matrix \(P(\theta)\) and \(Q(\theta)\) affine in \(\theta\) and setting \(F(\theta)\) and \(G(\theta)\) independent of \(\theta\).

Theorem 2: Consider the uncertain descriptor system in (20) and let \(B\) be a polytope of admissible \((\theta, \Delta \theta)\) and \(E_0 \in \mathbb{R}^{n \times (n-r)}\) is a full rank-column matrix such that \(E^T E_0 = 0\). Then, system (20) is robustly admissible if there exist matrices \(P_i = P^T_i, Q_i, i = 0, 1, \ldots, p, F\) and \(G\) satisfying the following LMIs:

\[
P(\theta) > 0, \quad \forall \theta \in \mathbb{V}_B, \quad (23)
\]

\[
\begin{bmatrix}
-E^T P(\theta)E + \text{He}\{F A(\theta)\} & * \\
E_0 Q(\theta) + G A(\theta) - F^T & \text{P}(\theta + \Delta \theta) - \text{He}\{G\}
\end{bmatrix} < 0,
\]

\(\forall (\theta, \Delta \theta) \in \mathbb{V}_B\), \(\quad (24)\)

5137
where
\[ P(\theta) = P_0 + \sum_{i=1}^{p} \theta_i P_i, \quad Q(\theta) = Q_0 + \sum_{i=1}^{p} \theta_i Q_i. \]

Motivated by the well known fact that the use of higher order PDLFs allows for deriving less conservative robust stability results, inspired by [15] in the sequel we will derive a method of robust admissibility analysis based on Lemma 5 with a quadratic PDLF and matrices \( F(\theta), G(\theta) \) and \( Q(\theta) \) that are quadratic in \( \theta \).

In order to obtain an LMI based method of robust admissibility analysis, without loss of generality, consider the following representation of the system matrix \( A(\theta) \):
\[
\begin{align*}
A(\theta(k)) &= A \Omega(\theta(k)), \\
\Omega(\theta(k)) &= [I_n \quad \Theta^{T}(k)]^T, \quad \Theta(\theta) = \theta(k) \otimes I_n,
\end{align*}
\]
where \( A \in \mathbb{R}^{n \times (p+1)n} \) is a known constant matrix. Secondly, without loss of generality, we introduce the following decompositions of the quadratic parameter-dependent matrices \( F(\theta), G(\theta), P(\theta) \) and \( Q(\theta) \):
\[
\begin{align*}
F(\theta) &= \Omega^{T}(\theta) F \Omega(\theta), \\
G(\theta) &= \Omega^{T}(\theta) G \Omega(\theta), \\
P(\theta) &= \Omega^{T}(\theta) P \Omega(\theta), \\
Q(\theta) &= \Omega^{T}(\theta) Q \Omega(\theta),
\end{align*}
\]
where
\[
\begin{align*}
\Omega_{q}(\theta) &= [I_{n-r} \quad \Theta^{T}_{q}], \\
\Theta_{q} &= \theta \otimes I_{n-r},
\end{align*}
\]
and \( F, G, P \) and \( Q \) are constant matrices with appropriate dimensions to be determined and with \( P \) symmetric. Furthermore, we shall apply the following version of Finsler’s lemma to handle constrained inequalities:

**Lemma 6:** Given matrices \( M(\varsigma) = M^{T}(\varsigma) \in \mathbb{R}^{n \times n} \), \( N(\varsigma) \in \mathbb{R}^{n \times n} \), and \( \varsigma(\varsigma) \in \mathbb{R}^{n} \) with \( \varsigma \in D \subseteq \mathbb{R}^{n} \), if there exists a matrix \( L \) such that
\[
M(\varsigma) + H e(LN(\varsigma)) < 0, \quad \forall \varsigma \in D.
\]

The second robust admissibility analysis result is presented in next theorem; the proof is omitted due to space limit.

**Theorem 3:** Consider the descriptor system (20) and let \( B \) be a polytope of admissible \( (\theta, \Delta \theta) \) and \( E_0 \in \mathbb{R}^{n \times (n-r)} \) is a full rank-column matrix such that \( E^{T}E_0 = 0 \). Then, the system (20) is robustly admissible if there exist matrices \( P = P^{T}, F, G, Q \),
\[
\begin{align*}
P + H e(L_1 H_1(\theta)) &> 0, \quad \forall \theta \in \mathcal{V}_{B} \\
\Lambda(\theta) + H e(L_2 H_2(\theta, \Delta \theta)) &< 0, \quad \forall \theta, \Delta \theta \in \mathcal{V}_{B}
\end{align*}
\]
where
\[
\begin{align*}
\Lambda(\theta) &= \begin{bmatrix} -E_d^{T}P E_d + H e\{F \Omega(\theta) A\} & \ast \\ \Lambda_1(\theta) & -\Lambda_2(\theta) \end{bmatrix}, \\
\Lambda_1(\theta) &= I_1 E_0 \Omega_{q}^{T}(\theta) Q + I_2(\hat{G}_T(\theta) A - F)^T, \\
\Lambda_2(\theta) &= H e\{I_2 \hat{G}_T I_2^T\} - I_3 P I_3^T.
\end{align*}
\]

Theorems 2 and 3 provide strict LMI methods of robust admissibility analysis for the uncertain descriptor system (20) based on respectively an affine and a quadratic PDLFs. An advantage of the proposed methods as compared to the approach in [7], which uses an affine PDLF, is that they apply directly to the system (20) without requiring any coordinate transformation, whereas [7] considers the system in an SVD coordinate form. Also, note that in [7] it has been implicitly assumed that the parameters are allowed to change from one extreme of their admissible range to the other in one time-step, which corresponds to the case of the largest possible parameter variation. In contrast, Theorems 2 and 3 have the feature of allowing for incorporating information on available bounds on the variation of the uncertain parameters, which may be smaller than those in the case of the largest possible variation. Moreover, since Theorem 3 is based on a quadratic PDLF, it has the potential of offering less conservative results, as it is shown in Example 1 in the next section.

Note that Theorems 2 and 3 can be also applied to descriptor systems with constant uncertain parameters by considering (22) and (29) with \( \Delta \theta = 0 \).

**IV. Numerical Examples**

In the sequel, Theorems 2 and 3 are applied to two examples analysed in [7].

**Example 1.** This example aims to study the admissibility of the time-invariant descriptor system \( E x(k+1) = Ax(k) \), where \( A \) is the convex hull \( \text{Co}(\rho(A_1, A_2)) \), for different values of \( \rho \), where \( E = \text{diag}\{I_3, 0_2\} \) and
\[
A_1 = \begin{bmatrix} 0.2594 & 0.0018 & 0.0590 & -0.1570 & 0.0102 \\
0.2595 & -0.0292 & 0.1574 & 0.1396 & 0.1095 \\
0.0121 & -0.1374 & 0.0562 & -0.0293 & 0.0628 \\
0.0926 & -0.1718 & -0.1722 & 0.1146 & 0.0722 \\
0.0027 & -0.1101 & -0.0296 & 0.0729 & 0.0542 \\
\end{bmatrix}
\]
\[
A_2 = \begin{bmatrix} -0.3612 & -0.2867 & 0.3450 & 0.3837 & 0.5281 \\
0.3114 & -0.4251 & -0.1279 & -0.2867 & -0.1527 \\
0.4565 & -0.1678 & -0.1794 & 0.2067 & -0.0681 \\
-0.3353 & 0.4063 & 0.0141 & 0.6519 & 0.1313 \\
-0.0218 & 0.7138 & -0.1451 & 0.0992 & 0.4533 \\
\end{bmatrix}
\]

Note that this system can be readily represented in the parametric form in (20).

As in [7], we are interested in determining the largest value of \( \rho \) that ensures the robust admissibility of the above system. To this end, Theorems 2 and 3 with \( \Delta \theta = 0 \) have been applied to the underlying system and the largest values
of $\rho$ that have been achieved are presented in Table I along with the results reported in [7] for the largest $\rho$ given by the methods of [5], [6] and [7], and the actual largest admissible value of $\rho$ obtained via the root-locus technique.

Note that Theorem 2, which is based on an affine PDLF gives better result than the methods of [5], [6] and [7] that use either an affine parameter-dependent or a parameter-independent Lyapunov function. On the other hand, the value of $\rho$ obtained with Theorem 3 coincides with the actual largest possible value of $\rho$ for the robust admissibility of the system. These results show the advantages of the proposed methods, in particular the potentials of the quadratic PDLF in achieving less conservative results.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>1.999</td>
<td>1.068</td>
<td>1.703</td>
<td>1.827</td>
<td>1.801</td>
</tr>
</tbody>
</table>

Example 2. Consider the uncertain descriptor system (20) with matrices as below:

$$E = \text{diag}\{I_3, 0\}, \quad A(\theta) = \begin{bmatrix} 0.5 & -0.1\theta & 0.7 & 1 \\ 0 & -0.2 & 0 & -2 \\ 0 & -0.5 - 0.5\theta & 0 & 0 \\ 0.3 & 0 & 0 & 1 \end{bmatrix}$$

where $\theta(k)$ is a bounded uncertain time-varying parameter. The aim is to analyse the admissibility of the above system for different bounds on $\theta(k)$ and its variation $\Delta\theta(k)$. To this end, it is assumed that $|\theta(k)| < \gamma$ for all $k \in \mathbb{Z}^+$. First, similarly as in [7], we consider the case of largest possible parameter variation, which corresponds assuming that $(\theta(k), \Delta\theta(k))$ belongs to a consistent polytope where $|\Delta\theta| \leq 2\gamma$. In this situation, the largest possible value of $\gamma$ achieved with Theorems 2 and 3 was $\gamma = 3.307$, which coincides with the results obtained in [7]. Note that when $\theta$ is assumed to be constant, the three latter approaches also give the same value of $\gamma$, namely $\gamma = 3.3601$.

Now, it is assumed that $(\theta(k), \Delta\theta(k))$ belongs to a consistent polytope where $|\Delta\theta| \leq \gamma$. Note that in this case it is not possible to apply the results proposed in [7] as they do not allow for incorporating information on bounds on the variation of the uncertain parameters. On the other hand, using Theorems 2 and 3 the largest possible values of $\gamma$ one obtains are identical, namely $\gamma = 3.347$. Note that since now the bound on the parameter variation is smaller than in the first case, it is possible to ensure the robust admissibility of the system for a larger bound on $|\theta(k)|$.

V. Conclusion

This paper has proposed two novel methods based on strict LMIs for robust admissibility analysis of discrete-time linear descriptor systems subject to uncertain time-varying parameters that appear affinely in the state matrix of the system state-space model. The parameters and their variations are considered to lie in given intervals. The first method is based on an affine parameter-dependent Lyapunov function and applies constant multiplier matrices, whereas the second one builds on a Lyapunov function and multiplier matrices with quadratic dependence on the uncertain parameters. Two numerical examples treated in the literature have been used to illustrate the advantages of the robust admissibility analysis methods. Another contribution of this paper is to derive necessary and sufficient conditions of admissibility of general discrete linear time-varying descriptor systems.

VI. Acknowledgment

The work of K. A. Barbosa has been supported by “Fondo Nacional de Desarrollo Científico y Tecnológico” - Fondecyt, Chile, under grant 1100568, and C. E. de Souza and D. Coutinho have been supported by CNPq, Brazil, under grants 30.6270/2011-0/PQ and 30.2136/2011-8/PQ, respectively.

References