Abstract—Very often, the control of multivariable industrial processes is performed using multi-loop architectures, in which several SISO PI or PID controllers are committed to control different channels of the plant. A difficulty with such a strategy arises due to the interaction between the different control loops, which may cause the control action in a loop to give rise to significant disturbances in other loops. This paper presents a new robust PI tuning procedure for multi-loop control systems which aims to decouple the different channels of the multivariable system and to guarantee the tracking response performance and measurement noise attenuation, using a reference model approximation scheme. The control problem is formulated as a non-convex multi-objective optimization problem which is formulated directly in the space of PI controller parameters. Polytopic models represent the system uncertainty. An application example is developed for the control of a quadruple-tank process.

Index Terms—Robust PI control, multi-loop control, decoupling control, polytopic uncertainty.

I. INTRODUCTION

The usage of PID controllers for the implementation of decentralized control for multivariable processes is quite popular in industry because such controllers are often effective, easy to understand, implement and tune by operators, and decentralized structures are failure-tolerant [1]. However, when such control schemes are employed for controlling multivariable systems with highly coupled channels, the task of control design may become difficult due to the existence of interactions among the loops. Due to this, the task of developing a satisfactory tuning procedure for PI/PID controllers in multi-loop control architectures constitutes a problem that has received great interest in the last decades (see [1] and references therein). Most of the tuning procedures for multi-loop PID controllers have been developed in the frequency domain. The different approaches include, for instance, static decoupler combined with decentralized PI controller with set-point weighting [2], generalized inverted decoupling technique [3], and an analytical decoupling control method [4].

The contribution of this paper is to present a robust multi-loop PI tuning procedure that aims to decouple the control channels in multivariable systems assuring the tracking response performance and measurement noise attenuation. In this paper, the general tuning procedure presented in [5] is extended for multivariable systems. A reference model approximation scheme is employed in order to perform the channel decoupling and to attain a tracking response performance [6]. Analytical solutions for single-loop PID tuning based on model matching have been presented in [7]. Differently from the former works presented in literature, the procedure proposed here is based on state space and employ polytopic models to represent the uncertainty.

The proposed procedure is illustrated using a quadruple-tank process, presented in Fig. 1. This is a laboratory process with an adjustable zero, that has been used to illustrate many issues in multivariable control [8], [9]. The objective is to control the two lower tank levels by means of two pumps. The three-way valve settings establish the interaction between the two control loops.

Fig. 1. Quadruple-tank process.
The notation in this paper is standard. The compact notation:

\[ G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

is applied to denote the transfer matrix \( G(s) = C(sI - A)^{-1}B + D \).

II. Problem Formulation

Consider a continuous-time linear time-invariant system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Bu_w(t), \\
z(t) &= C_x x(t) + D_{zu} u(t) + D_{zw} w(t), \\
y(t) &= C_y x(t) + D_{yw} w(t),
\end{align*}
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^n_u \) is the control signal vector (manipulated variables), \( w(t) \in \mathbb{R}^n_w \) is the exogenous input vector, \( z(t) \in \mathbb{R}^n_z \) is the controlled output vector, and \( y(t) \in \mathbb{R}^n_y \) is the measured output vector (inputs to the dynamic output-feedback controller).

To simplify the notation, the system matrices in Eq. (1) are gathered in the matrix:

\[ S \triangleq \begin{bmatrix} A & B_u & B_{w_u} \\ C_z & D_{zu} & D_{zw} \\ C_y & 0 & D_{yw} \end{bmatrix}, \]

(2)

that can include uncertain parameters belonging to a known convex compact set, or polytope, defined by its vertices:

\[ \mathcal{P}(\alpha) \triangleq \left\{ S : S = \sum_{i=1}^{N} \alpha_i S_i; \ \alpha \in \Omega \right\}, \]

(3)

\[ \Omega \triangleq \left\{ \alpha : \alpha_i \geq 0, \ \sum_{i=1}^{N} \alpha_i = 1 \right\}, \]

(4)

with \( S_i, i = 1, \ldots, N \), the polytope vertices and \( \alpha = [\alpha_1 \ldots \alpha_N] \) the vector that parameterizes the polytope. The dependence of the system matrices with \( \alpha \) will be omitted.

Consider a dynamic output-feedback control, \( U(s) = K(s)Y(s) \), with

\[ K(s) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}. \]

(5)

The closed-loop transfer matrix relating the controlled variables, \( z(t) \), and the exogenous inputs, \( w(t) \),

\[ T_{zw}(s) = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}, \]

(6)

can be computed by

\[
\begin{align*}
A_f &= \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix}, \\
B_f &= \begin{bmatrix} B_w + B_u D_c y & B_u D_c y_w \\ B_c y & B_c y_w \end{bmatrix}, \\
C_f &= \begin{bmatrix} C_z + D_{zu} D_c C_y & D_{zu} C_c \end{bmatrix}, \\
D_f &= \begin{bmatrix} D_{zu} + D_{zw} D_c D_y \end{bmatrix}.
\end{align*}
\]

(7)

Considering deviations around an operation point and that all inputs and outputs are voltage signals, the quadruple-tank process that will be used here in order to illustrate the proposed design procedure, which is presented in Fig. 1, can be represented by the linearized state space model [8]:

\[
\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_1}{A_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_3}{A_5} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 & 0 \\ 0 & \frac{\gamma_2 k_2}{A_3} & 0 \\ (1-\gamma_1) k_3 & 0 & 0 \end{bmatrix} u, \]

(8)

where the state variables are the four tank level deviations, \( x_i = h_i - h^0_i, \ i = 1, 4; \) the control signals are the two pump voltage deviations, \( u_j = v_j - v^0_j, \ j = 1, 2; \) the exogenous inputs are the reference signals and measurement noises, \( w = [r_1 \ r_2 \ n_1 \ n_2]^T; \) the controlled variables are the measured level signals of tanks 1 and 2, \( z_j = k_c (h_j - h^0_j), \ j = 1, 2, \) \( k_c \) the sensor gain; the measured variables are the reference signals and the measured level signals plus the measurement noises, \( y = [r_1 \ r_2 \ z_1 + n_1 \ z_2 + n_2]^T. \) The time constants are

\[ T_i \triangleq \frac{A_i}{a_i \sqrt{2\beta_i \gamma_i g}} \quad i = 1, \ldots, 4. \]

(9)

The quadruple-tank linearized model has the following parameter values [8]: tank cross-sections \( A_1 = A_3 = 28\text{cm}^2, A_2 = A_4 = 32\text{cm}^2; \) cross-section of the outlet holes \( a_1 = a_3 = 0.071\text{cm}^2, a_2 = a_4 = 0.057\text{cm}^2; \) sensor gain \( k_c = 0.50\text{V/cm}; \) and acceleration of gravity \( g = 981\text{cm/s}^2. \) In [8], it is chosen two operation points, \( P_- \) and \( P_+, \) that present minimum phase and non-minimum phase characteristics, respectively. The corresponding parameter values of the two operation points are presented in Table I. The tank inlet flows are function of the pump coefficients, \( k_1 \) and \( k_2, \) and the three-way valve coefficients, \( \gamma_1 \) and \( \gamma_2. \)

To deal with the decoupling of the quadruple-tank process, it is verified that the PI controller with the proportional action applied only to the measured plant output, also referred as the I-P controller, is the best option [2]. Considering a decentralized I-P controller, the dynamic output-feedback controller can be represented
TABLE I
Parameter values of the two operation points.

<table>
<thead>
<tr>
<th>Parameter values of the two operation points.</th>
<th>Parameter values of the two operation points.</th>
</tr>
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<tbody>
<tr>
<td>((b_1', b_2')) [cm]</td>
<td>((b_1', b_2')) [cm]</td>
</tr>
<tr>
<td>((12, 12.7))</td>
<td>((12, 12.6; 13.0))</td>
</tr>
<tr>
<td>((b_3', b_4')) [cm]</td>
<td>((1.8; 1.4))</td>
</tr>
<tr>
<td>((12.6; 13.0))</td>
<td>((4.8; 4.9))</td>
</tr>
<tr>
<td>((\theta^0_1, \theta^0_2)) [(\gamma)]</td>
<td>((3.00; 3.00))</td>
</tr>
<tr>
<td>((3.15; 3.15))</td>
<td>((3.14; 3.29))</td>
</tr>
<tr>
<td>((\theta_1, \theta_2)) [cm(^2)/Vs]</td>
<td>((0.70; 0.60))</td>
</tr>
<tr>
<td>((T_1, T_2))</td>
<td>((62; 90))</td>
</tr>
<tr>
<td>((T_3, T_4))</td>
<td>((23; 30))</td>
</tr>
</tbody>
</table>

with \(F\) the set of controllers such as the closed-loop system is robustly stable.

The multi-objective robust control problem can be stated as: given a polytope-bounded uncertain, continuous-time, linear time-invariant system, \(P(\alpha), \alpha \in \Omega\), and a reference model, \(T_m(s)\), find the PI controller, \(K(s)\), that minimizes the maximum \(H_\infty\)-norm of the error between the reference model and the closed-loop transfer function, \(E(s)\), and the maximum \(H_2\)-norm of the transfer function \(T_{zn}(s)\) related with the measurement noise attenuation, in the uncertainty domain:

\[
K^* = \arg \min_K \left[ \max_{\alpha \in \Omega} \|E(s, \alpha, K)\|_\infty \right] \max_{\alpha \in \Omega} \|T_{zn}(s, \alpha, K)\|_2
\]

subject to: \(K \in F\).

To apply the proposed controller synthesis presented in the next section, the multi-objective optimization problem is transformed into a scalar optimization problem as:

\[
K^* = \arg \min_K \max_{\alpha \in \Omega} \|T_{zn}\|_2
\]

subject to: \(\max_{\alpha \in \Omega} \|E\|_\infty \leq \epsilon_m\).  \(K \in F\),

where \(\epsilon_m\) can be selected to result different solutions to the multi-objective problem. This choice of the scalar optimization problem considers that there is an upper limit of \(\epsilon_m\) that guarantees the tracking transient response will be closer to the one specified by the reference model and that assures satisfactory decoupling.

III. PROPOSED MULTI-OBJECTIVE ROBUST CONTROL SYNTHESIS PROCEDURE

The proposed procedure to tackle the non-convex optimization problems (16) or (17) directly in the space of controller parameters is based on two steps: synthesis and analysis. In the synthesis step, it is applied an non-linear optimization algorithm to solve the scalar optimization problems (16) or (18) with the infinite set \(\Omega\) replaced by a finite set of points \(\Omega \subset \Omega\). This finite set is initially the set of vertices of the polytope as considered in convex formulations. To consider only the polytope vertices is not sufficient to guarantee the robust stability of the closed-loop system and the minimization of \(\|E\|_\infty\) and \(\|T_{zn}\|_2\) for all \(\alpha \in \Omega\). To verify the controller computed in the first step, in the second step, it is applied an analysis procedure based on a combination of a branch-and-bound algorithm and LMI formulations [10]. If the analysis procedure finds an instance of an unstable system in the uncertain domain or if it is verified that the maximum value of \(\|E\|_\infty\) or \(\|T_{zn}\|_2\) does not occur in a point belonging to \(\Omega\), then this point is included in \(\Omega\) and it is necessary to execute the two steps of the procedure again. The procedure ends when it is verified that the closed-loop system is robustly stable and the maximum
values of the objective functions are in points belonging to $\bar{\Omega}$ (or near them accordingly to a specified accuracy).

In the synthesis step, the scalar optimization problem can be solved by means of the cone-ellipsoidal algorithm [11]. Let $\chi \in \mathbb{R}^d$ be the vector of optimization parameters (in this case the controller parameters), $f(\chi) : \mathbb{R}^d \to \mathbb{R}$ be the objective function to be minimized, and $g_i(\chi) : \mathbb{R}^d \to \mathbb{R}, i = 1, \ldots, s$, be the set of constraint functions. Let $\chi_k$ be the ellipsoid center and $Q_k = Q_k^T > 0$ the matrix that determines the direction and dimension of the ellipsoid axes. Given the initial values $\chi_0$ and $Q_0$, the ellipsoidal algorithm is described by the following recursive equations:

\[
\begin{align*}
\chi_{k+1} &= \chi_k - \frac{1}{d+1} Q_k \hat{m}, \\
Q_{k+1} &= \frac{d^2}{d^2 - 1} \left( Q_k - \frac{2}{d+1} Q_k \hat{m} \hat{m}^T Q_k \right),
\end{align*}
\]

(19)

with

\[
\hat{m} = m_k / \sqrt{m_k^T Q_k m_k},
\]

where $m_k$ is the sum of the normalized gradients (or sub-gradients) of the violated constraint functions, $g_i(\chi) > 0$, when $\chi_k$ is not a feasible solution, or the gradient (or sub-gradient) of the objective function, $f(\chi)$, when $\chi_k$ is a feasible solution. The gradients (or sub-gradients) are computed numerically by means of the finite difference method.

In the analysis step, it is required to compute the $\alpha \in \Omega$ corresponding to the maximum of each objective and constraint functions in (16) or (17) or to find an $\alpha \in \Omega$ that corresponds to an unstable system, if $K \notin F$. The basic strategy of the branch-and-bound algorithm is to partition the uncertainty domain, $\Omega$, such as lower and upper bound functions converge to the maximum value of the norm. This algorithm ends when the difference between the bound functions is lower than the prescribed relative accuracy. The algorithm is implemented considering as lower bound function the $H_\infty$ (or $H_2$) norm computed in the vertices and as upper bound function the $H_\infty$ (or $H_2$) guaranteed cost computed by means of linear matrix inequality (LMI) formulations, both functions calculated for the original polytope and its subdivisions [10]. If the system is not robustly stable, the algorithm finds an unstable system in the polytope while searching for the maximum norm value. A partition technique based on simplicial meshes [12] is applied to allow this procedure to be applied to polytopic models with improved efficiency.

IV. RESULTS

A. Precisely known minimum phase system

For sake of comparison with the results presented in [8], [2], consider firstly the precisely known minimum phase system without measurement noise. The control objectives are to decouple the system with a satisfactory tracking response. A remarkable decoupling is achieved with a slower transient response to the level $h_2$. Adopting $\omega_{n1} = 0.799$, $\zeta_1 = 0.95$, $\omega_{n2} = 0.449$, and $\zeta_2 = 8$, solving the single-objective optimization problem (16), it is achieved $k_{p1} = 36.0272$, $T_{i1} = 2.3532$, $k_{p2} = 272.9343$, and $T_{i2} = 35.5851$. This controller results in $\|E\|_\infty = 0.0016$. The transient responses of the measured tank levels, $c_j(t) = k_i h_i(t), j = 1, 2$, for step changes in the reference signals, $r_1(t) = 1(t)$ and $r_2(t) = 1(t - 50)$, are presented in Fig. 2. With this decentralized PI tuning, the step changes in a reference signal do not affect the other process output.

Fig. 2. Transient responses of process outputs (solid), $T_m(s)$ outputs (doted), and reference signals (dashed) for the precisely known minimum phase system with better decoupling.

The problem with this design is the slow response of $h_2(t)$. A better trade-off between decoupling and settling time can be achieved changing $\zeta_2 = 8$ to $\zeta_2 = 9$. In this case, it is achieved $k_{p1} = 36.0975$, $T_{i1} = 2.3544$, $k_{p2} = 25.3311$, and $T_{i2} = 3.9511$. This controller results in $\|E\|_\infty = 0.0128$. The transient responses are presented in the Fig. 3. Comparing these transient responses with the results in [8], [2], they present similar settling time and much better decoupling.

B. Precisely known non-minimum phase system

The second operation point, where the process presents non-minimum phase characteristics, is more difficult to
control because of the time-domain limitations due to right half-plane zero [9]. It is necessary to choose a reference model that will result in slower transient responses to the system. As observed in [8], in this case, it is better to use pump 2 to control the tank 1 level and pump 1 to control the tank 2 level. Considering $\omega_n = 0.02$, $\zeta_1 = 4$, $\omega_n = 0.01$, and $\zeta_2 = 3$, solving the optimization problem (16), it is achieved $k_{p1} = 0.3458$, $T_{11} = 164.8119$, $k_{p2} = 0.5133$, and $T_{22} = 291.6067$. This controller results in $\|E\|_\infty = 0.3025$. The greater error results in a worse decoupling of the system and in transient responses not matching exactly the reference model responses, as shown in Fig. 4. The decoupling is worse than the case of the minimum phase operation point but it is considerably better than the results presented in [8].

![Fig. 4. Transient responses of the controlled outputs (solid), reference model outputs (dotted), and reference signals (dashed) for the precisely known non-minimum phase system.](image)

C. Uncertain minimum phase system

Let’s consider now the case of the uncertain system with measurement noise. The uncertain parameters are the three-way valve coefficients $\gamma_1$ and $\gamma_2$. It is considered a polytope with three vertices: $(\gamma_1, \gamma_2) = \{(0.84, 0.58); (0.84, 0.72); (0.56, 0.72)\}$. The robust decentralized PI controllers are achieved solving the multi-objective optimization problem (18). Different trade-offs between the tracking response performance (and decoupling) and measurement noise attenuation (and control effort) can be achieved changing the value of the constraint $\epsilon_m$ as presented in Table II. Lower values of $\max \|T_{zn}\|_2$ correspond to less effect of the measurement noise over the system signals and, indirectly, to lower control effort.

For the sake of comparison, solving the single-objective problem, it is achieved $\max \|E\|_\infty = 0.0571$ for $k_{p1} = 38.7620$, $T_{11} = 2.4071$, $k_{p2} = 23.4793$, and $T_{22} = 3.9004$. The simulation considers the same step changes in the reference signals and measurement noises, $n_i = 0.01 \times \text{randn}(t)$, $i = 1, 2$, with $\text{randn}(t)$ a function that generates pseudo-random values drawn from the standard normal distribution. The transient responses of the measured tank levels and pump voltages for the three polytope vertices are presented in Fig. 5 and Fig. 6.

![Fig. 5. Transient responses of the controlled outputs (solid), reference model outputs (dotted), and reference signals (dashed) for the uncertain minimum phase system and single-objective problem.](image)

![Fig. 6. Transient responses of the manipulated variables for the uncertain minimum phase system and single-objective problem.](image)

With this small value for $\max \|E\|_\infty$, the response for the three polytope vertices are similar to the model reference outputs but $v_1$ exceeds the limit of 10V.

<table>
<thead>
<tr>
<th>$\epsilon_m$</th>
<th>$\max |E|_\infty$</th>
<th>$\max |T_{zn}|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0987</td>
<td>1.2333</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1986</td>
<td>1.1366</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2934</td>
<td>1.0484</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3930</td>
<td>0.9880</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4937</td>
<td>0.9558</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5863</td>
<td>0.8707</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6812</td>
<td>0.8120</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7995</td>
<td>0.7454</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9174</td>
<td>0.6845</td>
</tr>
</tbody>
</table>

For $\epsilon_m = 0.5$, it is achieved $k_{p1} = 15.9757$, $T_{11} = 2.7851$, $k_{p2} = 9.3313$, and $T_{22} = 4.1024$. The transient responses of the measured tank levels for the three polytope vertices are presented in Fig. 7. Allowing a higher value to the approximation error, the transient responses become no longer close to the the model reference outputs but they are still satisfactory considering decoupling, overshoot, and settling time. On the other hand, it is achieved a better measurement noise attenuation and lower control effort as presented in Fig. 8. It can be
observed that the pump voltages do not exceed the limit of 10V, what happens also for lower values of $\epsilon_m$.

![Fig. 7. Transient responses of the controlled outputs (solid), reference model outputs (dotted), and reference signals (dashed) for the uncertain minimum phase system and $\epsilon_m = 0.5$.](image1)

![Fig. 8. Transient responses of the pump voltages for the uncertain minimum phase system and $\epsilon_m = 0.5$.](image2)

**D. Uncertain non-minimum phase system**

In the case of the non-minimum phase operation point, consider a polytope with three vertices: $(\gamma_1, \gamma_2) = \{(0.344, 0.272); (0.344, 0.408); (0.504, 0.272)\}$. In this case, it is not necessary to optimize the measurement noise and control effort. Solving the single-objective problem, it is achieved $\max \|E\|_\infty = 0.3412$ for $k_{p1} = 0.3702$, $T_{11} = 183.5767$, $k_{p2} = 0.4426$, and $T_{12} = 243.9774$. The transient responses of the measured tank levels for the three polytope vertices are presented in Fig. 9. The transient responses are not equal to the model reference responses but they are satisfactory.

![Fig. 9. Transient responses of the controlled outputs (solid), reference model outputs (dotted), and reference signals (dashed) for the uncertain non-minimum phase system and single-objective problem.](image3)

**V. Conclusions**

A new robust tuning procedure for PI controllers that are used in multi-loop control of multivariable processes was proposed here. The proposed procedure was illustrated by an application to the quadruple-tank process. Decoupling of the multi-variable system, tracking response performance and measurement noise attenuation are considered as control objectives. It was verified that the reference model approximation can assure satisfactory decoupling and tracking response performance for both minimum phase and non-minimum phase operation points.

**References**


