Piecewise Quadratic Functions for Finite-Time Stability Analysis

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Abstract—In this paper we consider the finite-time stability (FTS) problem for linear time-varying systems. In most of the previous literature, the definition of FTS exploits the standard weighted quadratic norm to define the initial and trajectory domains, which, therefore, turn out to be ellipsoidal; this is consistent with the fact that quadratic Lyapunov functions are used to derive the FTS conditions. Conversely, the recent paper [1], considers the case where the above domains are polytopic and, consequently, the analysis is performed with the aid of polyhedral Lyapunov functions. In the current work, the class of Piecewise Quadratic Lyapunov functions is considered. First, it is shown that such class of functions recovers as particular cases both quadratic and polyhedral Lyapunov functions; then a novel sufficient condition for FTS of linear time-varying systems is provided. A procedure is proposed to convert such condition into a computationally tractable problem. The examples illustrated at the end of the paper show the benefits of the proposed technique with respect to the methodologies available in the literature.

I. INTRODUCTION

The origin of the concept of finite-time stability can be traced back to 1953, when it was first introduced by Kamenkov [2]. Roughly speaking, a system is said to be finite-time stable if, given a bound on the initial state of the system (called initial domain), its state trajectory does not exceed a certain threshold (called trajectories domain) during a specified time interval. According to this definition, the FTS concept differs from classical stability concepts (e.g. Lyapunov stability, asymptotic stability, bounded-input-bounded-output (BIBO) stability) for two important reasons. First of all, it deals with systems whose operation is limited to a fixed finite interval of time. Moreover, FTS requires prescribed bounds on system variables.

The concept of FTS appeared for the first time in the western control literature at the beginning of the Sixties [3], [4] with the name of short-time stability. In the following years, numerous computationally cumbersome results were proposed for the FTS analysis and control design problems [5], [6]. During the Nineties, the concept of FTS has been revisited in the light of recent results coming from Linear Matrix Inequalities (LMIs) theory [7] and Differential Linear Matrix Inequality (DLMI) theory [8], which enabled to find less conservative conditions guaranteeing FTS and finite time stabilization. Finite-time stability and stabilization have been firstly investigated in the linear systems context (see for example [9], [10]). Then, an effort has been spent to extend such results to the context of hybrid systems [11], [12], [13], [14] and nonlinear quadratic systems [15]. In the papers cited above the definition of FTS exploits the standard weighted quadratic norm to define both the initial domain and the trajectories domain; therefore such domains turn out to be ellipsoidal. The definition of the above domains is consistent with the fact that quadratic Lyapunov functions are used to derive the main results both for the analysis and control design. In [16] some necessary and sufficient conditions for FTS have been proposed for the case of polytopic trajectories domain while the initial domain is still assumed ellipsoidal. The use of polytopic domains rather that ellipsoids is important to tackle many practical problems where, for instance, the constraints on the state variables are in the form $a_i \leq x_i \leq b_i$ for $i = 1, \ldots, n$. Although polytopic domains can be always approximated by ellipsoids, the FTS conditions for ellipsoidal domains result to be conservative in this case (see [1]). In [1] ad hoc conditions for the FTS with polytopic domains, based on the use of polyhedral Lyapunov functions, have been proposed. However, these conditions were implemented via a procedure which, due to the nonconvexity of the optimization function, requires an high computational burden.

In this paper we consider the class of Piecewise Quadratic Lyapunov Functions (PQLF) for the FTS analysis of continuous-time linear time-varying systems. This class of functions has been widely investigated for the stability analysis and control design of piecewise linear systems [17]–[18]. In this paper, a preliminary partition of the state space into conic regions is realized; then a quadratic form is associated to each cone. While in [17]–[18] the number and the shapes of the partitions were related only to the system dynamics, in our approach it is a degree on freedom used to define the level curves of the Lyapunov function.

The proposed formulation allows us to find computationally efficient conditions for the FTS of linear systems when the initial and trajectories domains are piecewise quadratic, i.e. their boundaries are the level curves of piecewise quadratic functions. This class of domains, obviously, includes the class of ellipsoids and, at the same time, also represents a meaningful generalization of polytopic domains. Numerical examples show that the proposed formulation obtains an effective improvement with respect to the actual state of the art for the FTS analysis both when polytopic domains are considered and in the more general case of piecewise quadratic domains. In the last case the approaches

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available in the literature can be only used with added conservativeness. The paper is organized as follows: in Section II the FTS problem is stated and the class of piecewise quadratic functions and domains over conical partitions is defined. In Section III the convex optimization conditions needed to check FTS with piecewise quadratic domains are derived. In Section IV some examples are reported to illustrate the effectiveness of the proposed approach. Finally, in Section V some conclusions end the paper.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem Statement

Let us consider the system
\[
\dot{x}(t) = f(t, x), \quad x(t_0) = x_0, \tag{1}
\]
where \( x(t) \in \mathbb{R}^n \). Roughly speaking, system (1) is said to be finite-time stable (FTS) if, given a certain initial domain, its state remains, over a finite-time interval, within a prescribed trajectories domain.

**Definition 1 (Finite Time Stability):** Given an initial time \( t_0 \), a positive scalar \( T \), two sets \( \mathcal{X}_0 \) and \( \mathcal{X}(t) \), \( 0 \in \mathcal{X}_0 \), system (1) is said to be finite-time stable wrt \( (t_0, T, \mathcal{X}_0, \mathcal{X}(t)) \) if
\[
x_0 \in \mathcal{X}_0 \Rightarrow x(t) \in \mathcal{X}(t), \quad t \in [t_0, t_0 + T]. \tag{2}
\]

**Remark 1:** Note that the trajectories domain is allowed to vary in time. For well posedness of Definition 1, it is required that \( \mathcal{X}_0 \subseteq \mathcal{X}(t_0) \). However, in principle, it is not required that \( \mathcal{X}_0 \) is included into \( \mathcal{X}(t) \) for \( t > t_0 \).

In this paper we consider the FTS problem for the class of Continuous Time Linear Time Varying (CT-LTV) systems
\[
\dot{x}(t) = A(t)x(t), \quad t \in [t_0, t_0 + T], \tag{3}
\]
where \( A(\cdot) \in \mathbb{R}^{n \times n} \) is a bounded piecewise continuous matrix-valued function. Moreover, we assume that both the initial and the trajectories domains are *piecewise quadratic domains*, i.e., their boundaries are the unitary level curves of continuous positive definite piecewise quadratic functions.

**Remark 2:** Generally, the level curves of a positive definite piecewise quadratic function are piecewise ellipsoidal, i.e., they are the union of portion of ellipsoids. However, the positiveness of the function doesn’t require that each quadratic form is positive definite, as it will be shown in the following section. Hence, the level curves of a piecewise quadratic function can degenerate into the union of portions of hyperplanes or hyperboloids. Therefore, for the sake of generality, we refer to the unitary level curve of a piecewise quadratic function as a piecewise quadratic curve and to the set bounded from such a curve as a piecewise quadratic domain.

In the following section we propose a new mathematical formulation for the latter class of functions and for the related domains. Due to this formulation, the considered class of piecewise quadratic domains, not only generalizes the class of ellipsoidal domains, but it also contains the class of polytopic domains. Moreover, it provides an interesting framework to find a sufficient condition for the FTS of system (3) in terms of a feasibility problem subject to a set of differential linear matrix inequalities (DLMI).

B. Cones and conical partitions

In this subsection we introduce some geometric concepts necessary to define the class of piecewise quadratic domains.

**Definition 2 (Conical Hull [19], p. 28):** Given a set of points \( D = \{x_1, x_2, \ldots, x_p\} \subseteq \mathbb{R}^n \), the set
\[
cone(D) := \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{p} \lambda_i x_i, \lambda_i \geq 0 \right\}
\]
is said to be the conical hull of \( D \) [19].

A set \( S \subseteq \mathbb{R}^n \) is a convex polyhedral cone in \( \mathbb{R}^n \) (hereafter denoted as con for simplicity) if there exists a generating set of finite cardinality \( D = \{x_1, x_2, \ldots, x_p\} \subseteq \mathbb{R}^n \) such that \( S = cone(D) \). The dimension of a cone having \( D \) as generator set is the column rank of the matrix \( M_D = \begin{bmatrix} x_1 & x_2 & \ldots & x_p \end{bmatrix} \) whose columns are the elements of \( D \), that is
\[
\text{dim}(cone(D)) = \text{rank}(M_D). \tag{4}
\]

Moreover, let us define the set of normalized extremal rays generating \( S \)
\[
\text{extr}(S) = \{\hat{x}_1, \ldots, \hat{x}_q\},
\]
with \( ||\hat{x}_i|| = 1, i = 1, \ldots, q \leq p \), as the minimal set of unit vectors such that \( S = cone(\{\hat{x}_1, \ldots, \hat{x}_q\}) \).

Now we are ready to introduce the concept of conical partition of the space \( \mathbb{R}^n \).

**Definition 3 (Conical partition of \( \mathbb{R}^n \)):** A conical partition \( \mathcal{P} = \{S_1, \ldots, S_r\} \) of \( \mathbb{R}^n \) is a collection of cones \( S_i \), with \( i = 1, 2, \ldots, r \) such that:
- each cone \( S_i \) has dimension \( n \), i.e. \( \text{dim}(S_i) = n \);
- the union of the cones \( S_i \), with \( i = 1, \ldots, r \), covers the whole state space \( \mathbb{R}^n \), that is \( \bigcup_{i=1}^{r} S_i = \mathbb{R}^n \);
- for all \( i \neq j \), the intersection between the interior of \( S_i \) and \( S_j \) is empty:
\[
\text{int}(S_i) \cap \text{int}(S_j) = \emptyset. \tag{5}
\]

The set of generating rays of a conical partition \( \mathcal{P} \) is defined as the union of the normalized extremal rays of each cone in \( \mathcal{P} \), i.e. \( R_\mathcal{P} = \bigcup_{S_i \in \mathcal{P}} \text{extr}(S_i) \).

For the sake of notational simplicity, we shall denote the intersection of two cones as
\[
H(S_i, S_j) := S_i \cap S_j.
\]

Note that for two distinct cones in a same partition \( \mathcal{P} \), \( S_i, S_j \in \mathcal{P} \), then
\[
H(S_i, S_j) = \text{cone}(\text{extr}(S_i)) \cap \text{cone}(\text{extr}(S_j)) = \text{cone}(\text{extr}(S_i) \cap \text{extr}(S_j))
\]
with \( \text{dim}(H(S_i, S_j)) < n \) and \( \text{extr}(H(S_i, S_j)) = \text{extr}(S_i) \cap \text{extr}(S_j) \).
C. Piecewise quadratic functions and domains

In this section we introduce the class of piecewise quadratic functions and the corresponding piecewise quadratic domains defined over a partition $\mathcal{P}$.

Definition 4 (Piecewise quadratic Functions (PQFs)): A time-varying piecewise quadratic function defined over a conical partition $\mathcal{P} = \{S_1, S_2, \ldots, S_r\}$ of $\mathbb{R}^n$ is a space–continuous positive definite function in the form

$$ F_P(x, t) = x^T F_i(t)x \quad \forall x \in S_i \text{ with } i = 1, \ldots, r $$

(6)

where $F_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, r$, are symmetric matrix-valued functions positive definite inside the cone $S_i$, that is

$$ x^T F_i(t)x > 0 \quad \forall x \in S_i - \{0\} ,$$

(7)

with $i = 1, \ldots, r$, and $t > 0$.

In order to ensure the space–continuity of the function $F_P(x, t)$, the matrix–valued functions $F_i(t)$, $i = 1, \ldots, r$, need to be chosen so as to satisfy

$$ x^T F_i(t)x = x^T F_j(t)x, \quad \forall x \in H(S_i, S_j) $$

(8)

for any $i = 1, \ldots, r - 1$, for any $j = i + 1, \ldots, r$ and for $t > 0$.

According to the previous definition it is possible to introduce the concept of piecewise quadratic domain over a partition $\mathcal{P} = \{S_1, S_2, \ldots, S_r\}$.

Definition 5 (Piecewise Quadratic Domains (PQDs)): A time-varying piecewise quadratic domain defined over a conical partition $\mathcal{P} = \{S_1, S_2, \ldots, S_r\}$ of $\mathbb{R}^n$ is a compact domain whose boundary is the unitary level curve of a piecewise quadratic function $F_P(x, t)$, that is

$$ X_{F_P}(t) : = \{ x : F_P(x, t) \leq 1 \} , $$

$$ = \{ x : x^T F_i(t)x \leq 1, \ x \in S_i, \ i = 1, 2, \ldots r \} . $$

(9)

In the following we will denote a time-invariant piecewise quadratic function and a time-invariant piecewise quadratic domain by $F_P(x)$ and $X_{F_P}$, respectively.

The set of piecewise quadratic domains defined in (9) represents a generalization of the set of ellipsoidal domains (an ellipsoidal domain can be expressed as a piecewise quadratic domain with $F_i = F > 0$, $i = 1, \ldots, r$). Moreover it includes the set of polytopic domains whose boundary is the unitary level curve of a polyhedral function [20], as proved in the following theorem.

Theorem 1: Let us consider a polyhedral function $F_{pol}(x)$ defined over a conical partition $\mathcal{P} = \{S_1, \ldots, S_r\}$ such that

$$ F_{pol}(x) = c_i^T x, \quad \text{if } x \in S_i $$

(10)

with $c_i \in \mathbb{R}^n$, and the corresponding polytope

$$ X_{F_{pol}} : = \{ x : F_{pol}(x) \leq 1 \} . $$

Then, there always exists a piecewise quadratic function $F_P(x)$ defined over the same partition $\mathcal{P}$ such that

$$ X_{F_P} = X_{F_{pol}} . $$

(11)

Proof: The proof is constructive. Let us consider a generic cone $S_i \in \mathcal{P}$. From equation (10), it follows that for any $x \in S_i$, $F_{pol}(x) = c_i^T x$. If we consider its square, we can manipulate it in order to obtain

$$ F_{pol}^2(x) = (c_i^T x)^2 = x^T (c_i^T c_i^T) x . $$

It is straightforward to verify that the functions $F_{pol}(x)$ and $F_{pol}^2(x)$ have the same unitary level curve and hence

$$ X_{F_{pol}} = X_{F_{pol}^2} . $$

(12)

Exploiting equation (12), it is possible to build a piecewise quadratic Lyapunov function $F_P(x)$ satisfying (11) by choosing the following set of diadic matrices

$$ F_i = c_i c_i^T, \ i = 1, \ldots, r . $$

To conclude this section, we restrict Definition 1 of FTS to the case of autonomous CT-LTV systems assuming that both the initial domain and the trajectories domain are PQDs.

Definition 6 (Finite Time Stability with PQDs): Given an initial time $t_0$, a positive scalar $T$, a time-invariant piecewise quadratic initial domain $X_{R_0}$, and time-varying piecewise quadratic trajectories domain $X_{R_T}(t)$, with $t \in [t_0, t_0 + T]$, defined over the partition $\mathcal{P} = \{S_1, S_2, \ldots, S_r\}$ of $\mathbb{R}^n$, system (3) is said to be finite-time stable wrt $(t_0, T, X_{R_0}, X_{R_T}(t))$ if

$$ x_0 \in X_{R_0} \Rightarrow x(t) \in X_{R_T}(t), \ t \in [t_0, t_0 + T] . $$

(13)

III. FINITE TIME STABILITY WITH PIECEWISE QUADRATIC DOMAINS

In this section we propose some sufficient conditions for the FTS of CT-LTV systems (3) with piecewise quadratic domains. The following result is further exploited to introduce conditions that can be checked numerically in a more efficient way by means of LMIs.

Theorem 2: Let us consider the CT-LTV system (3), a time interval $\Omega = [t_0, t_0 + T]$ and two PQDs $X_{R_0}$ and $X_{R_T}(t)$, with $t \in \Omega$, defined over the partition $\mathcal{P} = \{S_1, S_2, \ldots, S_r\}$ of $\mathbb{R}^n$. System (3) is FTS wrt $(t_0, T, X_{R_0}, X_{R_T}(t))$ if there exist a piecewise continuously time–differentiable function $P_P(x, t)$ with $t \in \Omega$ verifying conditions (8) and such that:

$$ x^T \left( \dot{P}_i(t) + A(t)P_i(t) + P_i(t)A(t) \right) x < 0 , $$

(14a)

$$ x^T (P_i(t) - \Gamma_i(t)) x \geq 0 , $$

(14b)

$$ x^T (P_i(t_0) - R_i)x < 0 , $$

(14c)

for $t \in \Omega$ and $\forall x \in S_i$, with $i = 1, \ldots, r$.

Proof: Let $t \in [t_0, t_0 + T]$, $\tau \in [t_0, t]$, and consider a candidate piecewise continuously time–differentiable Lyapunov function $V(\tau, x) = P_P(x, \tau)$ as in (6). Now let $x_0 : R_{P}(x_0) \leq 1$ and denote by $x(t, x_0)$ the state evolution of system (3) with $\tau \in [t_0, t]$. In view of (14b), we have

$$ \Gamma_P(x(t, x_0), t) \leq P_P(x(t, x_0), t) . $$

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Since condition (14a) implies that $\dot{V}(\tau, x)$ is negative definite along the trajectories of system (3) we have
\[ P_P(x(t, x_0), t) \leq P_P(x_0, t_0). \]
Finally, in view of (14c), $P_P(x_0, t_0) < R_P(x_0) < 1$. □

Although important from a theoretical viewpoint, the conditions of Theorem 2 cannot be easily applied because they require to check infinitely many DLMIs and equality conditions, one for each $x \in \mathbb{R}^n$. The remainder of the section will be devoted to overcome the above limitations and to resort to a set of computationally tractable conditions. To this end, we first report some numerically tractable conditions able to ensure the continuity conditions (8) for any possible candidate piecewise quadratic Lyapunov function. It is worth noting that these conditions will not introduce any conservativeness with respect to equation (8).

**Theorem 3 ([21]):** Let a piecewise quadratic function $P_P(x, t)$ be defined over a conical partition $P = \{ S_1, S_2, \ldots, S_r \}$ of $\mathbb{R}^n$ and let us denote with $r_P = \{ \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_q \}$ the set of generating rays. A necessary and sufficient condition for the continuity of $P_P(x, t)$ is that
\[
\begin{align*}
\dot{x}_h^T P_i(t) \hat{x}_h &= \hat{x}_h^T P_i(t) \hat{x}_h, \quad (15a) \\
\dot{x}_k^T P_i(t) \hat{x}_k &= \hat{x}_k^T P_i(t) \hat{x}_k, \quad (15b)
\end{align*}
\]
for all $\hat{x}_h, \hat{x}_k \in \text{extr}\{H(S_i, S_j)\}$. □

At this point, taking advantage of the above conditions and making use of $S$–Procedure arguments [22], computationally tractable conditions to verify the FTS of system (3) with piecewise quadratic domains are stated.

**Theorem 4:** Given a piecewise continuously differentiable Lyapunov function $V(x, t) = P_P(x, t)$ as in (6), system (3) is FTS with respect to $(t_0, T, \mathcal{X}_{R_P}, \mathcal{X}_{Gamma}(t))$ if, given a set of matrices $Q_{i,k} \in \mathbb{R}^{n \times n}$, with $i = 1, \ldots, r$ and $k = 1 \ldots s$, such that
\[ x^T Q_{i,k} x \leq 0 \quad \forall x \in S_i, \]
there exist positive scalar $b_{i,k}$, some positive scalar functions $c_{i,k}(t), v_{i,k}(t), z_{i,k}(t)$ and piecewise continuously differentiable valued functions $P_i(t) \in \mathbb{R}^{n \times n}$, with $i = 1 \ldots r$, $k = 1 \ldots s$ and $t \in [t_0, t_0 + T]$, such that the following differential linear matrix inequalities and equalities are verified
\[
\begin{align*}
&\hat{P}_i(t) + A(t)^T P_i(t) + P_i(t) A(t) - \sum_{k=1}^{s} c_{i,k}(t) Q_{i,k}, \quad (16a) \\
&\hat{P}_i(t) - \Gamma_i(t) + \sum_{k=1}^{s} v_{i,k}(t) Q_{i,k} \geq 0, \quad (16b) \\
&\hat{P}_i(t_0) - R_i - \sum_{k=1}^{s} b_{i,k} Q_{i,k} < 0, \quad (16c) \\
&\hat{x}_h^T \hat{x}_h P_i(t) \hat{x}_h = \hat{x}_h^T P_i(t) \hat{x}_h \quad \forall \hat{x}_h \in \text{extr}\{H(S_i, S_j)\}, \quad (16d) \\
&\hat{x}_h^T \hat{x}_k P_i(t) \hat{x}_k = \hat{x}_k^T P_i(t) \hat{x}_k, \quad \forall \hat{x}_h, \hat{x}_k \in \text{extr}\{H(S_i, S_j)\}, \quad (16e)
\end{align*}
\] for $i = 1, \ldots, r$ , for all $j = i + 1, \ldots, r$ and for $t \in [t_0, t_0 + T]$.

**Proof:** By exploiting $S$–Procedure it readily follows that conditions (14a)–(14c) are ensured by (16a)–(16c), as shown in [11]. Finally, according to Theorem 3, condition (16d)–(16e) ensure the continuity of the piecewise quadratic Lyapunov function $V(x, t)$.

**Remark 3:** The use of the $S$–Procedure may introduce conservatism in Theorem 4 respect to Theorem 2. However, in [11] it has been shown that for the case of conical sets in $\mathbb{R}^2$ the $S$–Procedure can be applied with just one matrix $Q_i$ and no conservatism is added. □

Please note that Theorem 4 recasts the FTS problem using piecewise quadratic Lyapunov functions into a computationally tractable convex optimization problem. Nevertheless the latter formulation may lead to numerical conditioning problems due to the presence of the equality constraints (16d) and (16e). To overcome such problem in the following section we propose a reparameterization of matrices $P_i(t)$, $i = 1, \ldots, r$, that allows us to implicitly verify the continuity of the piecewise quadratic Lyapunov function.

**A. Reparameterization of the quadratic forms**

The first step to resort to a more efficient formulation of Theorem 4 is to rewrite the continuity conditions of Theorem 3 as follows.

**Lemma 1:** Let the piecewise quadratic function $P_P(x, t)$ be defined over a conical partition $P = \{ S_1, S_2, \ldots, S_r \}$ of $\mathbb{R}^n$ and let us denote by $r_P = \{ \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_q \}$ the set of generating rays. A necessary and sufficient condition for the continuity of $P_P(x, t)$ is that there exists a symmetric matrix $\Phi = \{ \phi_{ij}(t) \} \in \mathbb{R}^{q \times q}$ such that
\[
\begin{align*}
\hat{x}_i^T P_i(t) \hat{x}_i &= \phi_{ii}(t) \quad (17) \\
\hat{x}_i^T P_i(t) \hat{x}_j &= \phi_{ij}(t) \quad (18)
\end{align*}
\]
for $i = 1, \ldots, q$, $j = i, \ldots, q$ and for all $k : \hat{x}_i, \hat{x}_j \in \text{extr}(S_k)$. □

**Remark 4:** Note that, if for a certain couple $(i, j)$ there does not exist an integer $k$ such that $\hat{x}_i, \hat{x}_j \in \text{extr}(S_k)$, then $\phi_{ij}(t)$ remains unconstrained and may assume any arbitrary value. □

In order to directly verify the conditions of Theorem 3 the following reparameterization of the quadratic forms $\hat{x}_i^T P_i(t) x, i = 1, \ldots, r$, is proposed. As a first step (and without loss of generality) let us restrict our attention to functions defined over conical partition $P = \{ S_1, \ldots, S_r \}$ such that each cone $S_i$ has exactly $n$ extremal vector, i.e. $\text{extr}(S_i) = \{ \hat{x}_1, \ldots, \hat{x}_n \} \subset r_P$. Moreover let $\Gamma_i$ be the selection matrix that extrapolates the matrix with the extremal rays of $S_i$, namely $\mathcal{M}_{\text{extr}(S_i)} = [\hat{x}_1, \ldots, \hat{x}_n]$, from the matrix containing all the generating rays of the conical partition $\mathcal{M}_{\mathcal{R}_P} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_q]$, i.e.
\[
\mathcal{M}_{\text{extr}(S_i)} = \mathcal{M}_{\mathcal{R}_P} \Gamma_i.
\]
Now, given the matrix $\Phi(.)$ introduced in Lemma 1, let us define the matrix containing the parameters associated with
the \( i \)-th cone as
\[
\Phi_i(t) = \begin{bmatrix}
\phi_{i}^{11}(t) & \phi_{i}^{12}(t) & \cdots & \phi_{i}^{1n}(t)
\end{bmatrix}
\cdot \begin{bmatrix}
\phi_{i}^{21}(t) & \phi_{i}^{22}(t) & \cdots & \phi_{i}^{2n}(t)
\end{bmatrix}
\cdots \begin{bmatrix}
\phi_{i}^{n-11}(t) & \phi_{i}^{n-12}(t) & \cdots & \phi_{i}^{n-1n}(t)
\end{bmatrix}
\cdot \begin{bmatrix}
\phi_{i}^{n-11}(t) & \phi_{i}^{n-12}(t) & \cdots & \phi_{i}^{nn}(t)
\end{bmatrix}.
\]
(19)

Clearly, they are related by \( \Phi_i(t) = \Gamma_i^T \Phi(t) \Gamma_i \). According with (19), continuity conditions (17)-(18) for the \( i \)-th cone may be written as \( \mathcal{M}_{\text{ext}}(S_i) P_i(t) \mathcal{M}_{\text{ext}}(S_i) = \Phi_i(t) \). By the definition of conical partition each cone has dimension \( n \) and then \( \mathcal{M}_{\text{ext}}(S_i) \) is invertible. Hence,
\[
P_i(t) = \mathcal{M}_{\text{ext}}(S_i)^{-T} \Phi_i(t) \mathcal{M}_{\text{ext}}(S_i)^{-1}
= (\mathcal{M}_{\mathcal{R}_p} \Gamma_i)^{-T} \Gamma_i^T \Phi_i(t) \Gamma_i (\mathcal{M}_{\mathcal{R}_p} \Gamma_i)^{-1}.
\]
(20)

This last equation reparameterizes \( P_i(t) \) as a function of \( \Phi_i(t) \) without introducing any conservatism. Based on this reparameterization, we are able to state a numerically efficient condition to test the FTS system (3) with piecewise quadratic domains.

**Theorem 5:** Given a piecewise continuously time-differentiable Lyapunov function \( V(x,t) = P_i(x,t) \) as in (6), system (3) is FTS with respect to \((t_0,T,\mathcal{X}_{\mathcal{R}_p},\mathcal{X}_{\mathcal{R}_p})(t)\) if, given a set of matrices \( Q_{i,k} \in \mathbb{R}^{n \times n} \), \( i = 1, \ldots, J+1 \) and \( \mathcal{R}_p = \{S_1, S_2, S_3, S_4\} \) with
\[
S_1 = \text{cone} \left\{ \begin{bmatrix} 0.8 \\ 2.5 \end{bmatrix} : \begin{bmatrix} 0.8 \\ -2.5 \end{bmatrix} \right\},
S_2 = \text{cone} \left\{ \begin{bmatrix} 0.8 \\ -2.5 \end{bmatrix} : \begin{bmatrix} -0.8 \\ -2.5 \end{bmatrix} \right\},
S_3 = \text{cone} \left\{ \begin{bmatrix} -0.8 \\ -2.5 \end{bmatrix} : \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} \right\},
S_4 = \text{cone} \left\{ \begin{bmatrix} -0.8 \\ 2.5 \end{bmatrix} : \begin{bmatrix} 0.8 \\ 2.5 \end{bmatrix} \right\};
\]
(26)

and the piecewise quadratic functions \( \mathcal{R}_p(x) \) and \( \Gamma_p(x) \) are composed by the following diadic matrices
\[
R_1 = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix},
R_2 = \begin{bmatrix} 0 & -0.4 \\ -0.4 & 0 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix},
R_3 = \begin{bmatrix} -1.25 & 0 \\ 0 & -1.25 \end{bmatrix} \quad \Gamma_3 = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix},
R_4 = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix} \quad \Gamma_4 = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}.
\]

In figure 1 the state trajectories of system (23) starting from the vertices of the initial polytopic domain are shown. Note that when \( x_0 \in \mathcal{X}_{\mathcal{R}_p} \), the trajectories of the system remain in \( \mathcal{X}_{\mathcal{R}_p} \) for all \( t > 0 \) and they reach the minimum distance from the external boundary for \( T \approx 1 \). The FTS problem for system (23) with the polytopic constraints in (24) was firstly proposed in [1]. The authors of [1] show that their sufficient condition for FTS with polytopic domain was able to prove the FTS of system (23) for \( T = 0.8s \) while the technique proposed in [9] was not.

We applied the conditions in Theorem 5 to the presented example. In order to recast the DLMIs conditions provided in Theorem 5 in terms of LMIs, the matrix function \( \Phi(\cdot) \) has been assumed piecewise linear time, that is
\[
\Phi(t) = \begin{cases}
\Phi_0 + \Theta(t - t_0) & t \in [t_0, t_0 + T_s], \\
\Phi_0 + \sum_{j=1}^{J} \Theta_i T_s + \Theta_{j+1} (t - j T_s - t_0) & t \in [t_0 + j T_s, t_0 + (j + 1) T_s],
\end{cases}
\]
where \( J = \max\{j \in \mathbb{N} : j < T/T_s\}, T_s \ll T \) and \( \Phi_0, \Theta_i, \Theta_{j+1}, \Theta_{j+1} \) are the optimization variables. In this way, it has been possible to prove that system (23) with the

IV. NUMERICAL EXAMPLES

In this section a comparison with the existing literature is proposed for both the cases of polytopic domains [1] and ellipsoidal domains [16].

**Example 1:** Polytopic domains

Let us consider the mass-spring-friction system
\[
M \ddot{y} + K_f \dot{y} + K_s y = 0,
\]
(23)
where \( y \) is the position of the mass expressed in meters, \( M = 1 \text{ Kg}, K_f = 0.25 \text{ Ns/m}, K_s = 1 \text{ N/m} \) and assume that the following constraints on the state variables are imposed
\[
-0.8 \leq y(0) \leq 0.8, \quad (24a)
-2.5 \leq \dot{y}(0) \leq 2.5, \quad (24b)
-2.4 \leq y(t) \leq 2.4, \quad t \in [0, T], \quad (24c)
-7.5 \leq \dot{y}(t) \leq 7.5, \quad t \in [0, T]. \quad (24d)
\]

System (23) can be rewritten in the form (3) where
\[
\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -0.25 \end{bmatrix}, \quad x(0) := \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix}.
\]
(25)

Moreover, with the aid of Theorem 1, the polytopic bounds on the initial state and the trajectories of the system can be expressed in terms of piecewise initial domain \( \mathcal{X}_{\mathcal{R}_p} \) and trajectories domain \( \mathcal{X}_{\mathcal{R}_p} \), where the conical partition is \( \mathcal{P} = \{S_1, S_2, S_3, S_4\} \) with
polytopic constraints in (24) is FTS for $T > > 1$; in particular we proved that system (23) is FTS for $T = 10s$ with a piecewise linear function $\Phi(t)$ with $T_k = 0.5s$. It is important to remark that the proposed technique improves the results in [1] in terms of FTS analysis and it is also computationally less expensive. Moreover it is able to consider a more general class of dynamical systems (time-varying) and domains (piecewise quadratic).

Remark 5: In order to apply the condition of Theorem 5 it is necessary to fix the matrices $Q_{i,k}$ for $i = 1 \ldots r$ and $k = 1 \ldots s$ verifying condition (21) on the conical sets $S_i$. A constructive geometrical procedure to fix the matrices $Q_{i,k}$ to reduce the conservatism of the S-procedure is proposed in [11].

Example 2: Ellipsoidal domains

In this example we consider a FTS problem presented in [16], where the authors propose a necessary and sufficient condition for the FTS with ellipsoidal domains. Let us consider the CT-LTV system

$$\dot{x} = \begin{pmatrix} 2t & 1 \\ 0 & -1 + t \end{pmatrix} x,$$  

(27)

with $t \in \Omega = [0 \ 1]$ and ellipsoidal initial and trajectories domains

$$\mathcal{X}_0(\gamma) = \{x : x^T x < \gamma^2\}, \quad \mathcal{X}_t = \{x : x^T x < 1\}. \quad (28)$$

The authors in [16] show that the maximum value of $\gamma$ such that (27) is FTS wrt $(0, 1, \mathcal{X}_0(\gamma), \mathcal{X})$ is $\gamma = 0.1616$.

Making use of Theorem 5 we proved that system (27) is FTS wrt $(0, 1, \mathcal{X}_0(\gamma), \mathcal{X})$ for $\gamma$ up to 0.16 that is less that 1% far from the theoretical bound. We achieved this result with a piecewise linear function $\Phi(t)$, with $T_k = 0.02$.

It is important to recognize that the authors in [16] proposed an ad hoc condition for the case of ellipsoidal domains (moreover they extended this result to the case of ellipsoidal initial domain and polyhedral trajectories domain) while the conditions of Theorem 5 are applicable to any possible piecewise domain.

V. CONCLUSIONS

The FTS problem with piecewise quadratic domains has been presented in this paper. The definition of the above domains is consistent with the fact that piecewise quadratic Lyapunov functions are used to derive the FTS condition. It is shown that such class of Lyapunov functions includes both the classes of quadratic and polyhedral Lyapunov functions. Some numerical examples have been presented to illustrate the effectiveness of the proposed methodology.

REFERENCES


