Discrete-to-Continuous Dynamics Reconstruction for Bilinear Systems

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Abstract—In this work, we study the reconstruction of continuous-time models from discrete-time models of bilinear systems. Many dynamical systems contain nonlinearities and evolve continuously in time. However, due to the use of digital sensors for data acquisition, system identification methods typically rely on sampled data and yield a discrete-time model. Linking such a discrete-time model to its equivalent continuous-time dynamics is nontrivial.

This paper proposes a discrete-to-continuous dynamics reconstruction for discrete-time models of a class of bilinear systems obtained by system identification. We show that for bilinear systems we can obtain a discrete-time model that is consistent with the measurement data. Furthermore, we show that one can derive from the discrete-time model, its equivalent continuous-time model. Two examples illustrate the approach.

Keywords: Nonlinear identification, Parameter estimation, Discrete-to-continuous reconstruction, Bilinear system.

I. INTRODUCTION

Rapid development of digital technology creates a high reliance on digital computers in handling simple day-to-day tasks or complex problems. While to a certain extent digital computers are extremely useful to facilitate many new methods, they also impose challenges. The word “analog”, which implies continuity in time, that nowadays gives a connotation of old or outdated, is actually the word that describes continuous-time phenomena mostly observed in nature. However, if digital sensors are employed for taking measurements at distinct time instances, discrete-time models of a system might be easier to identify [8]. This contradiction motivates our study.

In this work, we investigate the reconstruction of continuous-time models from bilinear discrete-time models obtained from system identification. System identification has attracted a lot of attention recently (see, e.g. [8], [10], [4] for an overview). This increasing interest is for instance driven by the increasing industrial requirement for product quality and safety, e.g. in pharmaceutical, medical and automotive industry. Identifying an accurate model from measurement data alone without a priori knowledge on the dynamic structure of the system is in general challenging.

If some prior knowledge of the system structure is available, model identification is reduced to parameter estimation, e.g. [12], [7], [4], [23]. Typically, parameter estimation is performed based on numerical optimization which is in general hard for nonlinear systems. Although, there are several reliable tools available for parameter estimation, see for instance [9], verifying additionally that the obtained model guarantees a certain safety requirement is more involved and often depends on finding a global optimum of a nonconvex problem [1]. To this end, set-based parameter estimation can be an alternative approach, as it allows to determine outer-bounds on parameterizations that lead to a model behavior consistent with the measurement data or safety requirements (see e.g. [16] and references therein).

After a discrete-time model of the nonlinear dynamics is obtained, one crucial question is: how can we obtain an equivalent continuous-time model that represents the “real” dynamics of the system? Relating a discrete and a continuous-time model is not straightforward. While discretization of nonlinear dynamics has been well studied [20], most of the results are numerical, thus approximative. It is also well known that for nonlinear systems an explicit expression of an exact discrete-time model does not exist in general [6]. Therefore, simply taking the parameters of the discrete-time model may yield an inaccurate, or even incorrect continuous-time model.

We propose results on the reconstruction of continuous-time dynamics from a discrete-time model obtained from identification (D2C reconstruction) for bilinear systems. Bilinear systems cover a large subset of nonlinear systems and are often found in bio-chemical processes [11], [14]. We concentrate our efforts on bilinear systems for which the bilinear structure is preserved under sampling [3].

While this problem was recognized in [5], only a very particular subclass of bilinear systems was addressed. Furthermore, the continuous-time model has to be known for the reconstruction rendering this approach impractical. In this paper, we extend the result by expanding the system coverage and relaxing the restrictive conditions in our D2C reconstruction. We present furthermore two examples to illustrate our main result and we provide some remarks for possible extensions.

II. PRELIMINARIES

A. Problem Setup

We consider a class of nonlinear systems called bilinear systems with parameters:

\[ \dot{x} = A_p x + B_p u + \sum_{l=1}^{r} D_{p,l} x; \quad y = C_p x, \]

(1)

where \( x \in \mathbb{R}^n \) denotes the states, \( u \in \mathbb{R}^r \) the input, \( y \in \mathbb{R}^m \) the output. Here, \( A_p \in \mathbb{R}^{n \times n} \) is the state matrix, \( B_p \in \mathbb{R}^{n \times r} \).
the input matrix, \( C_p \in \mathbb{R}^{m \times n} \) the output matrix and \( D_{p,l} \in \mathbb{R}^{n \times n} \) is the bilinear weighting matrix for the \( l \)-th input, \( l = 1, \ldots, r \). We denote with \( p \in \mathcal{P} \subseteq \mathbb{R}^{nr} \) the parameter vector consisting of all unknown-but-bounded \((10)\) entries of the system matrices, where \( n_p := n^2 + nr + n^2r + mn \).

Bilinear systems represent a subclass of nonlinear systems commonly used to model dynamical systems, such as biochemical processes [11], [14] and mechanical systems [18], [22]. While this class has the advantage of having a similar structure to linear systems, due to the coupling between the state and the input most linear system properties do not hold anymore. Thus, the analysis and design for this class become challenging.

We consider in particular bilinear systems for which the structure is preserved under sampling [3]. Furthermore, we assume that for the pair \((A_p, C_p)\) an observability condition holds, which is a necessary condition for the parameter identification. Suppose that at certain time instances \( t_i \in \mathbb{R}_{\geq 0} \) measurements of the input \( U_i \) and of the state \( \mathcal{X}_i \) are taken from the system under study. In order to consider measurement uncertainties, we assume \( U_i, \mathcal{X}_i \) are given as unknown-but-bounded compact and convex sets. We denote with \( \mathcal{M} = \{t_i, i \in \mathbb{Z} : a \leq t_i \leq b\} \) the set of all time instances at which a measurement was taken, where \( a, b \in \mathbb{R} \) denote the first and the last measurement time point. \( \mathcal{I} = \{1, \ldots, n_t\} \) denotes the measurement index.

To simplify presentation, we consider the collection of input and state measurements

\[
U = \{U_i \subset \mathbb{R}^r, i \in \mathcal{I}\}, \quad \mathcal{X} = \{\mathcal{X}_i \subset \mathbb{R}^n, i \in \mathcal{I}\},
\]

and write in the remainder simply \( x \in \mathcal{X}, u \in U \) instead of the formally correct \( x(t_i) \in \mathcal{X}_i, u(t_i) \in U_i, \forall t_i \in \mathcal{M}, i \in \mathcal{I} \).

Note that due to the observability of the system, the pair \((A_p, C_p)\) has full rank, we can reconstruct the bounds \( \mathcal{X} \) on the states by accordingly assigning the output vector components \( C_p \).

The discrete-time model of a bilinear system is given by

\[
x_{k+1} = F_p x_k + G_p u_k + \sum_{l=1}^{r} u_{l,k} N_p,l x_k; \quad y = H_p x, \quad (2)
\]

where \( x_k := x(kT) \) with a discretization step size \( T > 0 \) is chosen such that for all \( t_i \in \mathcal{M} \) there exist some \( k \in \mathbb{Z} \) such that \( kT = t_i \). \( \mathcal{M} \) implicitly defines the index-set \( \mathcal{Z} = \{k \in \mathbb{Z} : a \leq kT \leq b\} \). Note that the matrices \( F_p, G_p, N_p \) and \( H_p \) are dependent on the sampling time \( T \).

We address the following problem: Given a set of discrete-time data obtained from measurements with homogeneous sampling period \( T > 0 \), from a nonlinear bilinear system \((1)\) with the parameters in \( A_p, B_p \) and \( D_p \) unknown. We construct the discrete-time model \((2)\) by applying a nonlinear identification to estimate the parameters of the model that are consistent with the measurements. After obtaining \((2)\), how can we reconstruct the original continuous-time dynamics of the system?

### B. Parameter Estimation

We assume that we use a parameter estimation scheme that provides an outer-approximation \( \hat{\mathcal{P}} \subseteq \mathcal{P} \) of the parameters leading to a consistent behavior of the discrete-time model \((2)\) with respect to the given uncertain-but-bounded measurements \( U, \mathcal{X} \). Note that a model is said to be consistent with the measurements, if there exist parameters \( p \in \mathcal{P} \) such that \( x \in \mathcal{X} \) and \( u \in U \). There are several approaches suitable to derive such an approximation, e.g. in [23], [7] interval arithmetic approaches are presented.

In this work, we apply the approach presented in [16] based on a nonconvex feasibility problem formulation. The main idea of this approach is to simplify the feasibility problem using semidefinite relaxations. There are some advantages to such a set-based method, even though it will increase typically the computational burden in contrast to other parameter estimation approaches. First, if there exists no parameterization for \((2)\) that leads to a consistent behavior, this can be proven by a so-called infeasibility certificate. Second, if the resulting parameterization does not fulfill the necessary separability condition for the reconstruction (cf. Definition 1) one can simply modify one or more parameters within the bounds defined by \( \hat{\mathcal{P}} \). In principle for every \( p \in \hat{\mathcal{P}} \) the modified system is still consistent with the measurements. However, as set-based methods typically derive an outer-approximation additional conditions have to be considered, (the chosen parameterization has to be a strictly feasible solution, see e.g. the rank constraint in [15]), which might require an iterative change of the parameters.

### III. DISCRETE-TO-CONTINUOUS (D2C) RECONSTRUCTION

Suppose we have obtained an outer-approximation \( \hat{\mathcal{P}} \) of the parameters for which the discrete-time system \((2)\) is consistent with the available measurements (cf. Section II-B). Suppose furthermore, that a parameter vector \( p \in \hat{\mathcal{P}} \) was chosen. To simplify notation and clarify that the system matrices are no longer parameter dependent, we drop the index \( p \) from the system representation. We will now discuss the D2C reconstruction of the model.

#### A. Separable Bilinear Systems

Suppose the discrete-time system

\[
z_{k+1} = \hat{F} z_k + \hat{G} u_k + \sum_{l=1}^{r} u_{l,k} \hat{N}_l z_k
\]

has been obtained from a parameter identification using a set of data obtained from measurements with homogeneous sampling \( T > 0 \). We introduce the following definition:

**Definition 1 (Separable Bilinear Systems):** A bilinear system \((3)\) is called pairwise separable (or separable) under a constant input \( u_{i,k} \neq 0, u_{i,k} = 0, i \in \{1, \ldots, r\} \setminus \{l\} \), if there exists a coordinate transformation with non singular transformation matrices \( \Phi_l \) that transforms \( \hat{F} + u_{i,k} \hat{N}_l \) into a diagonal form and \( \hat{F} \) is full rank.

It has been shown in [3] that if a continuous-time bilinear system is (pairwise) separable, there exists an exact continuous-time to discrete-time model (C2D) transformation. Furthermore, the bilinear discrete-time model of the system will be separable as well. Thus for this class of
systems, the structure is preserved under sampling. For the reconstruction, we exploit the reverse, which is also true as shown next.

Given system (3), we consider the system with each single input at a time. Suppose we consider the $l$-th input, $u_{i,k}$, thus we can write (3) as

$$z_{k+1} = \tilde{F}z_k + \tilde{G}_l u_{i,k} + u_{i,k} \tilde{N}_l z_k,$$

(4)

with $\tilde{G}_l$ the $l$-th column of $\tilde{G}$. We transform the state vector $z$ into a new state vector $x$ using a coordinate transformation with a nonsingular $n \times n$ transformation matrix $\Phi_l$, such that the system (4) is transformed into a diagonal form

$$x_{k+1} = \mathcal{F}x_k + G_l u_{i,k},$$

(5)

where the matrix $\mathcal{F}$ is diagonal. To keep the bilinear structure of the system, as $u_{i,k}$ is known, we can write (5) as

$$x_{k+1} = (F + u_{i,k}N_l)x_k + G_l u_{i,k},$$

(6)

keeping $F$ as a constant matrix with $n$ different eigenvalues, which are the same as the eigenvalues of $\tilde{F}$ and putting the residue of the diagonal components into $u_{i,k}N_l$. Repeating the same procedure with all inputs $u_{i,k}$, $l = 1, \ldots, r$, using the transformation matrices $\Phi_l$, $l = 1, \ldots, r$, we have

$$x_{k+1} = Fx_k + Gu_k + \sum_{l=1}^r u_{i,k}N_l x_k,$$

(7)

with $G := [G_1, \ldots, G_r] \in \mathbb{R}^{n \times r}$. In this case, the dynamics of each state is decoupled.

Viewing each of the states as a single scalar system, for each $x_j$ we can write

$$x_{j,k+1} = f_j x_j, k + \sum_{l=1}^r g_{j,l} u_k + \sum_{l=1}^r \eta_{j,l} u_{i,k} x_{j,k},$$

(8)

for $j = 1, \ldots, n$ and $l = 1, \ldots, r$. A solution to the problem of exact discretization of separable bilinear systems with single input has been proposed in [3]. Now, based on this result, and with the knowledge of the parameters of the discrete-time model (7) and the sampling time $T$, we solve the inverse problem, to reconstruct the continuous-time dynamics

$$\dot{x}_j = \alpha_j x_j + \sum_{l=1}^r \beta_{j,l} u_l + \sum_{l=1}^r \rho_{j,l} u_{i,k} x_j$$

(9)

of each state separately. Considering each input at a time, we solve for each $x_j$ with each $u_i$.

$$\alpha_j = \frac{1}{T} \ln(f_j), \quad \beta_{j,l} = \frac{g_{j,l} \ln(f_j + u_{i,k} \eta_{j,l})}{(f_j + u_{i,k} \eta_{j,l} - 1)},$$

$$\rho_{j,l} = \frac{\ln(f_j + u_{i,k} \eta_{j,l})}{Tu_l}, \quad j = 1, \ldots, n, \quad l = 1, \ldots, r.$$

(10)

Combining all states yields the continuous-time dynamics

$$\dot{x} = Ax + Bu + \sum_{l=1}^r u_{i,k} D_{\beta,l} x$$

(11)

which is also in a bilinear diagonal form. Hence, with this reconstruction the separable bilinear structure of the system is preserved. We can state the following result to conclude the process.

**Proposition 1 (D2C Reconstruction (Bilinear))**: Consider a discrete-time diagonal separable bilinear system of the form (7) from a model identification based on measurement data obtained with sampling $T > 0$. If all system matrices were identified, the D2C reconstruction yields the continuous-time dynamics (11) and the elements of the system matrices are given by (10).

**Proof of Proposition 1**: Considering each input at a time and borrowing the linear system representation, we write (8) as

$$x_{k+1} = f x_k + gu_k + \eta u_k x_k = (f + \eta u_k) x_k + gu_k.$$

(12)

Given a pair of scalar linear systems

$$\dot{x} = \alpha x + \beta u \quad \text{and} \quad x_{k+1} = f x_k + gu_k.$$

(13)

If $f$, $g$ and $T$ are known, we have the relationships $\alpha = \frac{\ln(f)}{T}$ and $\beta = \frac{g \ln(f)}{T(f - 1)}$. Applying these relationships to a separable bilinear system, given $f$, $g$, $\eta$ and $T$, we obtain

$$\alpha + \rho u = \frac{1}{T} \ln(f + \eta u) = \frac{1}{T} \left[ \ln(f) + \ln(f + \eta u) - \ln(f) \right]$$

$$= \frac{1}{T} \ln(f) + \frac{1}{T} \left[ \ln(f + \eta u) - \ln(f) \right] = \frac{1}{T} \ln(f) + \frac{\ln(f + \eta u) - \ln(f)}{Tu}.\quad \text{(14)}$$

Moreover, substituting $f$ with $f + \eta u$ we obtain $\beta$. Hence,

$$\alpha = \frac{\ln(f)}{T}, \quad \rho = \frac{\ln(f + \eta u) - \ln(f)}{Tu}, \quad \beta = \frac{g \ln(f + \eta u)}{T(f + \eta u - 1)}.$$

Returning the indices, we obtain (10) and therefore (11).

Note that the resulting continuous-time model (11) is equivalent to the continuous-time counterpart of (3), in the sense that the eigenvalues of $\tilde{F}$ in (3) are the discrete-time equivalent of the eigenvalues of $A$ in (11), excluding the effect of measurement noise in the data. However, due to the coordinate transformation, the structure of the models might be different, thus another coordinate transformation is required to return to the original coordinates. Thanks to the bilinear structure, as for linear systems, the transformation matrix for this coordinate transformation is the inverse matrix of the diagonalization transformation matrix, i.e. $\Phi_{d,l} = \Phi_l^{-1}$. The existence of $\Phi_{d,l}$ is guaranteed as $\Phi_l$ is nonsingular. Hence, we can obtain

$$\dot{z} = \tilde{A}z + \tilde{B}u + \sum_{l=1}^r u_{i,k} \tilde{D}_{l} z$$

(15)

the continuous-time model in the same coordinate as (3).

**B. Driftless Bilinear Systems**

A class of driftless bilinear systems have the structure

$$\dot{x} = \sum_{l=1}^r u_{i,k} g_l(x).$$

(16)

This class of systems often arises in models of nonholonomic mechanical systems, due to nonholonomic constraints. It
is well known that this class of systems does not satisfy Brockett’s necessary condition for smooth stabilization using pure state feedback [2], which makes it necessary to use either controls depending on time (time-varying controls) or discontinuous controls. Without loss of generality, we assume \( u_1 \) to be piecewise constant.

From the result of [13], if the inputs \( u_1 \) are constant on the time interval \( T \), we can explicitly compute the exact discretization of this class of systems.

**Theorem 1** ([22], [13]): Consider the driftless system (16) driven by piecewise constant inputs of the form

\[
u_l(t) = u_{l,k}, \quad t \in [kT, (k+1)T), k \geq 0, \quad l = 1, \ldots, r.
\]

Then the exact discrete-time model of (16) is given by

\[
x_{k+1} = e^{T \sum_{i=1}^l u_{i,k}^2 L_{ii}(I_d)} x_k
\]

where \( L_{ii}(\cdot) \) is the Lie derivative along the vector field \( g_i, I_d \) is an identity function and \( T > 0 \). For the same initialization \( x_k(0) = x(0) \) and \( x_k = x_{[k-T]} \), the discrete-time model (17) reproduces at the sampling instants the input-state behavior of the continuous-time system.

The proof of Theorem 1 is available in [13].

In the following, we consider the bilinear subclass of driftless systems with \( r = 2 \) and

\[
g_1 = \begin{pmatrix} 1 & 0 \\ d_3 x_2 & 0 \\ \vdots & \vdots \\ d_n x_{n-1} & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\]

We can rewrite the system into a bilinear matrix form as

\[
\dot{x} = Bu + u_1 Dx := Bu + ((Dx)^T u_1)^T \tag{19}
\]

with

\[
B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{n-1} \end{pmatrix}.
\]

Note that for this class of nonlinear systems, due to the nondiagonal form of the matrix \( D \), we need to solve the reconstruction as an iterative procedure, as will become clear from the following proposition.

**Proposition 2** (**D2C Reconstruction (Driftless)**): Consider a discrete-time driftless bilinear system of the form (17), with \( r = 2 \) and \( g_i, \quad l = 1, 2 \), given by (18). If all parameters (in this case coefficients) of \( g_i \) have been obtained from the identification based on a set of measurement data taken with a sampling period \( T > 0 \), the D2C reconstruction yields (19).

**Proof of Proposition 2:** The proof is based on Theorem 1, by applying directly the inverse exact discretization of the system. From Theorem 1, the discrete-time bilinear driftless system (17) is the exact discretization of (19) with sampling time \( T \). Expanding the model, we obtain

\[
\begin{align*}
x_1(k+1) &= x_1(k) + Tu_1(k) \\
x_2(k+1) &= x_2(k) + Tu_2(k) \\
x_3(k+1) &= x_3(k) + Td_1 u_1(k) (x_2(k) + Tu_2(k)) \\
x_4(k+1) &= x_4(k) + Td_1 u_1(k) (x_3(k) + Td_3 u_1(k) (x_2(k) + Tu_2(k))) \\
&& \vdots \\
x_n(k+1) &= x_n(k) + Td_{n-1} u_1(k) ((x_2(k) + Tu_2(k)) \cdots (x_n-1(k) + Tu_{n-1}(k)) + Tu_1(k) (x_{n-1}(k) + Tu_{n-2}(k)) \cdots ))
\end{align*}
\]

It is obvious that the parameters \( d_i, \quad i = 3, \ldots, n \), of the discrete-time model (20) can be uncoupled. Although the identification technique proposed in [16], that we apply in this paper, requires the system to be in polynomial form, which means creating cross terms and coupling of parameters, carrying out the D2C reconstruction recursively from the lower (top) subsystem to the higher (bottom) subsystem will allow separating the coupled parameters. Hence, obtaining the continuous-time model is immediate.

**Remark 1:** A special case of such systems are systems in power form, which are bilinear driftless systems with \( g_1 \) and \( g_2 \) given by (18) with \( d_i = 1, \quad i = 3, \ldots, n \). This class of systems is often obtained as a result of a conversion from a kinematic model of \( n \)-trailer vehicles [18]. For these systems the exact discrete-time model has all coefficients one or multiplications of \( T \), as \( T \) is known, the D2C reconstruction is straightforward. This result can further be extended to more general classes of systems in power form with \( r > 2 \) inputs (see for instance [22]).

**C. Nonlinear Systems in Byrnes-Isidori Normal Form**

While extending the result to cover all nonlinear systems is unlikely, it is possible to apply the same principle of D2C reconstruction to wider classes of nonlinear systems. Basically the technique can be applied to any class of nonlinear systems whose exact discrete-time model exists. Diffeomorphisms or immersions might be needed, similar to the diagonalization of separable bilinear systems, before the technique can be applied. In this subsection, we will consider one of these cases. We consider systems in Byrnes-Isidori normal form with relative degree equal to \( n \), which is not bilinear, but allows the application of our result under some additional assumptions. Given a nonlinear system in Byrnes-Isidori normal form:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= f(x, u)
\end{align*}
\]

This class of systems consists of a chain of integrators with the last subsystem containing the nonlinearity.
Assumption 1: The nonlinear dynamic (the $n$-th subsystem) of (21) is transformable into a bilinear system. We introduce the so-called phase variable canonical form:

$$\dot{x} = Ax + Bu + uDx$$

with $y = Cx$.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_0 & \cdots & -a_{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad C = (c_1 \ c_2 \ \cdots \ c_n \ \cdots \ 0)$$

with $n > n_c$. It is always possible to transform a separable bilinear system into this form. Using this form we can state the following result.

Proposition 3 (D2C Reconstruction (Byrnes-Isidori)): Given a system of form (21) for which the $n$-th subsystem satisfies Assumption 1. Then the discrete-to-continuous reconstruction problem is solvable using Proposition 1. ■

Proof of Proposition 3: Given the system (21) and notice that the first $n-1$ subsystems are a chain of integrators, hence it is linear. To proof this proposition, we consider two cases:

Case 1: The $n$-th subsystem is transformable (by diffeomorphism) into a first order bilinear system. In this case the whole system forms a phase variable canonical form. Hence the problem reduces to the separable bilinear system case.

Case 2: The $n$-th subsystem is transformable (through immersion) into an $m$-th order linear system $m > 1$ or $m$-th order separable bilinear system. Then the whole system becomes a linear system (which solution is trivial) or a separable bilinear systems which follows Proposition 1. ■

Remark 2: Bilinear systems cover a large sub-class of nonlinear systems found in practice. It is also shown that transformations are very useful to reduce one seemingly complex nonlinear system into a more structured bilinear system, that allows the D2C reconstruction. These results are clearly not exhaustive, and extensions to more general classes of nonlinear systems, in particular those whose exact discrete-time model exists, is possible and will be subject of future research. ■

IV. EXAMPLES

In this section, we present two examples to illustrate the usefulness of the results proposed in the previous sections.

Example 1 (Separable Bilinear Systems): We consider a compartment model of a cell cycle as proposed in [21] for optimal chemotherapy. The continuous-time model is described by the following set of differential equations:

$$\dot{x} + u_1 \begin{pmatrix} 0 & 0 & -0.214 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + u_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.295 & 0 \\ 0 & -0.295 & 0 \end{pmatrix} x.$$
To evaluate how well the reconstructed continuous-time model is able to reproduce the nominal model, we compared the eigenvalues of $A$ of the original system \{-0.843, -0.308, -0.022\} and of $\tilde{A}$ of the reconstructed model \{-0.837, -0.326, -0.009\}. While the two sets of eigenvalues show a close match, we also notice a slight discrepancy. Therefore, we have additionally simulated both the nominal and the reconstructed model from the same initial condition, applying a different input scheme from what was used in the identification to quantify this difference. The results are depicted in Fig. 2. One can see that the measurement data was sufficient to estimate the dynamics of $x_1$ and $x_2$ closely, however, in the dynamics of $x_3$ there are some discrepancies. This discrepancy derives from $\tilde{D}_1$ as the estimation of $\tilde{D}_1$ (cf. (10)) also depends on the estimation of $\tilde{A}$, thus imprecisions can be propagated from $\tilde{A}$ to $\tilde{D}_1$.

Example 2 (Systems in Byrnes-Isidori Normal Form): To exemplify Proposition 3, we consider the following simple model in Byrnes-Isidori normal form

\[
\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= -x_1 - x_2 + \exp(-x_1) - x_2 u. 
\end{align*}
\] (23)

By adding a third state $x_3 := \exp(-x_1)$ and taking the Lie derivative $\frac{\partial \exp(-x_1)}{\partial x} f(x, u) = -x_2 x_3$ we obtain the bilinear system

\[
\begin{align*}
\dot{x}_1 &= x_2; \\
\dot{x}_2 &= -x_1 - x_2 + x_3 - x_2 u; \\
\dot{x}_3 &= -x_2 x_3. 
\end{align*}
\] (24)

This simple example shows that we can apply Proposition 3 for systems that can be transformed into a bilinear system by immersion.

V. CONCLUSIONS

We have addressed two important problems in identification and modeling of bilinear systems. First we have applied a nonlinear parameter identification that can be used to obtain a parameterization of a discrete-time model that leads to a consistent behavior with the available measurement data. Second, we showed that for the considered system class we can reconstruct a continuous-time model that is equivalent to the discrete-time model obtained from the parameter estimation. These results extend the results of [5, 16, 17] and allow us to address more general classes of systems and allow a systematic procedure to integrate the two processes to obtain accurate results.

The presented examples underline that the proposed methods work effectively for the considered systems. This study provides a good insight into the problem and opens ways to address identification problems for more general classes of nonlinear systems. In particular, we have only used one parameter in the consistent region providing a reconstruction for all solutions of the set-based parameter estimation approach presented will be further research.

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