Kernel-based Non-Asymptotic Parameter Estimation of Continuous-time Systems

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Abstract—This work introduces a framework devoted to the design of parametric estimators with very fast convergence properties for continuous-time dynamic systems characterized by bounded relative degree and possibly affected by structured perturbations. More specifically, the design of suitable kernels of non-anticipative linear integral operators gives rise to estimators that are ideally not influenced by the transient effects due to the unknown initial conditions of the hidden states of the system under concern. The analysis of the properties of the kernels guaranteeing such a fast convergence is addressed and two classes of admissible kernel functions are introduced. The operators induced by the proposed kernels admit implementable (i.e., finite-dimensional and internally stable) state-space realizations. Numerical examples are reported to show the effectiveness of the proposed methodology; comparisons with some existing algebraic estimators are addressed as well.

I. INTRODUCTION

In many engineering applications the direct estimation of the parameters of a continuous-time (CT) model from sampled input-output data is an important problem for which several methods and tools are available.

Among the various techniques proposed in the literature for CT parameter identification of linear dynamical systems (see [1], [2], [3], [4], [5], [6], the contributed volume [7] and the recent special issue [8]), two main categories can be defined depending on the approach used to overcome the impossibility to measure the time-derivatives of the input-output signals of the system dealt with: i) State Variable Filtering (SVF) and ii) integral methods.

The SVF approach - not dealt with in this paper - consists in filtering the system’s inputs and outputs in order to obtain prefiltered time-derivatives in the bandwidth of interest that may be exploited, in place of the unmeasured derivatives of the signals, to estimate the model parameters. Instead, integral methods are related to the proposed methodology and they have quite a long history in the field of continuous-time identification. Among integral techniques, we recall i) the Modulating Function (MF) method, which relies on the repeated integration of input-output signals over finite-length intervals to minimize the effect of unknown initial conditions on the estimates; ii) the linear integral filter method, in which the initial conditions must be considered explicitly, by augmenting the dimension of the decision space with the unknown initialization variables; iii) the reinitialized partial moments method, that consists in integrating the input-output signals over finite-length time windows, in sampling the integrals, and finally in performing the regression over a discrete time-series, making the overall estimator an inherently hybrid dynamical system.

Typically, in the context of CT identification, asymptotic convergence properties can be proved and several algorithms are available characterized by a good performance in terms of transient behavior of the estimates (see, for example, [4] and the references cited therein). However, in order to achieve estimates’ modes of behavior characterized by very fast convergence properties, it is typically necessary to augment the vector of decision variables with the unknown initial conditions of the unmeasured states. The main drawback of this technique is related to numerical issues in estimating the initial hidden states as time goes on.

In the present paper, it is shown that the design of an internally stable dynamic estimator characterized by very fast convergence properties can be carried out by devising a suitable kernel of a non-anticipative linear integral operator, yielding a stable nonlinear dynamic system implementation. Namely, by transforming the measurable input-output signals of an unknown linear system through suitably designed Volterra operators, it is possible to obtain auxiliary signals that can be used in place of the unmeasurable derivatives to obtain in a very short time the estimates of the system’s parameters. Such “surrogate” signal derivatives can be made independent from initial conditions by exploiting the so-called Non-asymptotic Kernel (NK) functions. The use of Volterra operators induced by NKs, together with a suitable augmentation strategy, allows to form a linear algebraic system that can be solved for the unknown parameters under suitable excitation conditions on the input-output signals.

Two different classes of admissible NK functions are addressed. First, a class of univariate kernel functions is considered, called Univariate - Non-asymptotic Kernels (U-NKs). A method to enforce the internal stability of estimators based on U-NKs is proposed; however, due to the inherent algebraic properties of U-NKs, the internal stability is obtained at the price of a practical “freezing” of the estimator as time proceeds. To avoid this latter issue, the main contribution of the paper consists in the definition and the characterization of the class of Bivariate Causal Non-Asymptotic Kernels (BC-NK). The operators induced by the proposed BC-NKs yield a “non-asymptotic” estimator that admits a finite-dimensional time-varying linear state-space realization with internal stability guarantees. Moreover, input and output injections are never suppressed.

Finally, some simulation results are reported showing the effectiveness of the proposed methodology and carrying out a comparison with a fast algebraic identification technique recently proposed in [9].

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following SISO CT system $S_{u \rightarrow y}$:

$$
y^{(n)}(t) = \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \sum_{k=0}^{m-1} b_k u^{(k)}(t) + \sum_{r=0}^{p-1} c_r t^r, \forall t \in \mathbb{R}_{\geq 0};$

$$
y^{(0)}(0) = y_0^{(0)}, i \in \{0, \ldots, n-1\};$

$$
u^{(k)}(0) = \nu_0^{(k)}, k \in \{0, \ldots, m-1\}$$

(1)

with $m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{\geq 0}, m \leq n$ and $p \in \mathbb{Z}_{\geq 0}$. The values of the constant parameters $a_i \in \mathbb{R}, i \in \{0, \ldots, n-1\}$, $b_k \in \mathbb{R}, k \in \{0, \ldots, m-1\}$, and $c_r \in \mathbb{R}, r \in \{0, \ldots, p-1\}$ are unknown. The only measurable signals are $y(t)$ and
$u(t)$, while their time-derivatives are not assumed to be available. The term $\sum_{r=0}^{p-1} c_r t^r$ represents a polynomial-in-time parametrization for non-measurable time-varying perturbations. Such a parametrization allows us to incorporate in the model both biases (constant offsets) and drifts in the measurements. Our objective consists in estimating the system’s parameters $a_i$ and $b_i$ by suitably processing the input and output signals $u(\cdot)$ and $y(\cdot)$.

In the following, for the reader’s convenience, some basic concepts of linear integral operators’ algebra (see [10] and the references therein) and realization theory (for example, refer to [11] and [12]) are recalled. More specifically, we consider transformations acting on the Hilbert space $L_{loc}^2(\mathbb{R}_0)\subset L^2(\mathbb{R})$ of locally square-integrable functions with domain $\mathbb{R}_0$ and range $\mathbb{R}$ (i.e., $x(\cdot) \in L_{loc}^2(\mathbb{R}_0)$ if $x(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\int_0^\infty |x(t)|^2 dt < \infty$).

The notation $B(L_{loc}^2(\mathbb{R}_0))$ will be used to denote the set of all bounded linear operators $T : L_{loc}^2(\mathbb{R}_0) \to L_{loc}^2(\mathbb{R}_0)$. Given a function $u(\cdot) : \mathbb{R}_0 \to \mathbb{R}$, with $u \in L_{loc}^2(\mathbb{R}_0)$, the image function through a linear operator $T \in B(L_{loc}^2(\mathbb{R}_0))$ is denoted as $Tu$, and its value at time $t \in \mathbb{R}_0$ is denoted as $[Tu](t)$.

In the paper, we resort to Volterra linear integral operators $V_K \in B(L_{loc}^2(\mathbb{R}_0), L_{loc}^2(\mathbb{R}_0))$, defined as

$$[V_K u](t) = \int_0^t K(t, \tau) u(\tau) d\tau, \quad t \in \mathbb{R}_0,$$  

(2)

where $u(\cdot) \in L_{loc}^1(\mathbb{R}_0)$ and the function $K(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is taken as a Hilbert-Schmidt Kernel Function. Moreover, a signal is defined as a generic function of time $u(t) : t \to u(t), u(t) \in \mathbb{R}$, such that $u(\cdot) \in L_{loc}^1(\mathbb{R}_0)$. Furthermore, given two scalars $a, b \in \mathbb{R}_0$, with $a < b$, let us denote by $u_{[a,b]}(\cdot)$ and $u_{(a,b)}(\cdot)$ the restriction of a signal $u(\cdot)$ to the closed interval $[a, b]$ and to the left-open interval $(a, b]$, respectively. Indeed, we have the following:

**Definition 2.1 (Weak (generalized) Derivative):** Let $u(\cdot) \in L_{loc}^1(\mathbb{R}_0)$. We say that $u^{(i)}(\cdot)$ is a weak derivative of $u(\cdot)$ if

$$\int_0^t u(\tau) \left( \frac{d}{d\tau} \phi(\tau) \right) d\tau = - \int_0^t u^{(i)}(\tau) \phi(\tau) d\tau, \quad \forall t \in \mathbb{R}_0$$  

for all $\phi \in C^\infty$, with $\phi(0) = \phi(t) = 0$. □

We denote the $i$-th order generalized derivative as $u^{(i)}(\cdot), i \in \mathbb{Z}_0$. Moreover, given a kernel function $K(\cdot, \cdot) \in \mathcal{HS}$ in two variables, the $i$-th order weak derivative of $K$ with respect to the second argument will be denoted as $K^{(i)}(\cdot), i \in \mathbb{Z}_0$.

Finally, the notion of BIBO stability for an integral operator is introduced.

**Definition 2.2 (BIBO Stability):** A bounded linear operator $T \in B(L_{loc}^2(\mathbb{R}_0), L_{loc}^2(\mathbb{R}_0))$ is said BIBO-stable iff:

$$\|Tx(t)\|_{L_{loc}^2(\mathbb{R}_0)} < \infty, \forall t \in \mathbb{R}_0; \|x(\cdot)\|_{L_{loc}^2(\mathbb{R}_0)} < \infty, \forall x(\cdot) \in L_{loc}^2(\mathbb{R}_0); \|x(\cdot)\|_{L_{loc}^2(\mathbb{R}_0)} < \infty, \forall x(\cdot) \in L_{loc}^2(\mathbb{R}_0).$$ □

In the case of a Volterra operator $V_K$, BIBO stability is equivalent to the following property of the kernel:

$$\sup_{t \in \mathbb{R}_0} \left( \int_0^t |K(t, \tau)| d\tau \right) < \infty.$$  

(3)

Condition (3) will be used in the sequel to assess the stability of the operators in our setting. A kernel fulfilling (3) will be called a **BIBO stable kernel**. In this respect, it is worth noting that BIBO stability *per se* is not sufficient to establish the existence of a finite-dimensional state-space realization for an operator, that is, its implementability. The order of the realization can be determined only when an analytical expression for the kernel is available.

### III. NON-ANTICIPATIVE AND NON-ASYMPTOTIC VOLterra OPERATORS

#### A. Non-anticipativity of the Volterra operator

The notions of causality and non-anticipativity play a key role in characterizing the implementability (existence of a stable finite-dimensional state-space realization for the integral operators) of the proposed methodology. In rough (nonformal) terms, an operator $T \in B(L_{loc}^2(\mathbb{R}_0), L_{loc}^2(\mathbb{R}_0))$ is said to be causal (non-anticipative) if at any time $t > 0$ (respectively, $t \geq 0$) the image of a signal $x(\cdot)$ at time $t$, $[Tx](t)$, depends only on the restriction $x_{[0,t]}(\cdot)$ (respectively, $x_{[0,t]}(\cdot)$). Being the Volterra operator inherently non-anticipative, the signal $[V_K x](t)$, for $t > 0$, can be obtained as the output of a dynamic system described by the following scalar integro-differential equation:

$$\begin{cases}
\xi^{(i)}(t) = \left( K(t, t) x(t) + \int_0^t \left( \frac{d}{dt} K(t, \tau) \right) x(\tau) d\tau, t \in \mathbb{R}_0; \\
0,
\end{cases} \quad t = 0;

\begin{cases}
\xi(0) = 0 = \int_0^0 K(0, \tau) x(\tau) d\tau; \\
[V_K x](t) = \xi(t), \quad \forall t \in \mathbb{R}_0.
\end{cases}$$  

(4)

where $\xi^{(i)}(t) = \frac{d}{dt}[V_K x](t)$ has been obtained by applying the Leibnitz rule in deriving the integral.

Now, we introduce some useful results dealing with the application of Volterra operators to the derivatives of a signal.\footnote{The proof of the following results is omitted due to space constraints.}

**Lemma 3.1 (Proof in [13]):** For a given $i \geq 0$, consider a signal $x(\cdot) \in L^2(\mathbb{R}_0)$ that admits an $i$-th weak derivative in $\mathbb{R}_0$ and a kernel function $K(\cdot, \cdot) \in \mathcal{HS}$ that admits the $i$-th derivative (in the conventional sense) with respect to the second argument, $\forall t \in \mathbb{R}_0$. Then, it holds that:

$$[V_K x^{(i)}](t) = \sum_{j=0}^{i-1} (-1)^i j x^{(j)}(t) K((i-j-1)(t), t) + \sum_{j=0}^{i-1} (-1)^i j x^{(j)}(0) K((i-j-1)(t), 0) - (-1)^i [V_K x](t), \quad \forall t \in \mathbb{R}_0,$$  

(5)

that is, the function $[V_K x^{(i)}](\cdot)$ is non-anticipative with respect to the lower-order derivatives $x(\cdot), \ldots, x^{(i-1)}(\cdot)$. □

Lemma 3.1 is a key result for the whole successive discussions. Indeed, in view of (4) and (5), the signal $[V_K x^{(i)}](t)$, for $t > 0$, can be obtained as the output of the dynamic system described by the following integro-differential equation:

$$\begin{cases}
\xi^{(i)}(t) = \left( (-1)^i K^{(i)}(t, t) x(t) + \int_0^t \left( \frac{d}{dt} K^{(i)}(t, \tau) \right) x(\tau) d\tau, t \in \mathbb{R}_0; \\
0,
\end{cases} \quad t = 0;

\begin{cases}
\xi(0) = 0; \\
[V_K x^{(i)}](t) = \xi(t) + \sum_{j=0}^{i-1} (-1)^i j x^{(j)}(t) K((i-j-1)(t), t) + \sum_{j=0}^{i-1} (-1)^i j x^{(j)}(0) K((i-j-1)(t), 0), \forall t \in \mathbb{R}_0.
\end{cases}$$  

(6)
B. Non-asymptoticity conditions

The integro-differential characterization of the signal \( [K x(i)](\cdot) \) given in (6) allows to identify a class of kernels such that \([K x(i)](t), t > 0 \) are independent from the initial states \( x(0), x^{(1)}(0), \ldots, x^{(i-1)}(0) \). The following definition characterizes the kernels yielding non-asymptotic Volterra operators.

**Definition 3.1 (i-th Order Non-Asymptotic Kernel):** Consider a kernel \( K(\cdot,\cdot) \) satisfying the assumptions posed in the statement of Lemma 3.1; if for a given \( i \geq 1 \), the kernel verifies the supplementary condition

\[
K(j)(t, 0) = 0, \quad \forall t \in \mathbb{R}_{\geq 0}, \forall j \in \{0, \ldots, i-1\},
\]

then, it is called an \( i \)-th order non-asymptotic kernel. \( \square \)

**Lemma 3.2 (Non-asymptoticity Implication):**

If a kernel \( K(\cdot,\cdot) \), is at least \( i-th \) order non-asymptotic, then the image function of \( x(i)(\cdot) \) at time \( t \), \([K x(i)](t)\), depends only on the instantaneous values of the lower-order derivatives \( x^{(1)}(t), \ldots, x^{(i-1)}(t) \) and on the restriction \( x([0,0)(\cdot), \) not on the initial states \( x(0), x^{(1)}(0), \ldots, x^{(i-1)}(0) \).

The proof of Lemma 3.2 follows immediately from (6) and is therefore omitted.

Up to now, we have characterized a candidate class of kernels which allow to remove the influence of the unknown initial states on the derivative from the transformed signal \([K x](t)\). However, beyond depending on the current value \( x(t) \) and its past time-behaviour, such a signal depends also on the unmeasurable instantaneous values of the lower-order derivatives \( x^{(j)}(t) \), with \( j \in \{1, \ldots, n-1\} \). To address this issue, we need to introduce the notion of composed (or nested) Volterra operators and to discuss some relevant properties.

Let us denote by \( [K_{N_1} \cdots K_{N_i}] x(i)(\cdot) \), the image function obtained by composing \( N \) Volterra integral operators to \( x(i)(\cdot) \):

\[
[K_{N_1} \cdots K_{N_i}] x(i)(t) = \left[ K_{N_1} \cdots [K_{N_{i-1}} x(i)] \right](t).
\]

In view of the composition property of Volterra operators, it holds that the composed operator is in turn a Volterra operator with kernel

\[
K_{N_1} \cdots K_{N_{i-1}} \cdots K_1 \cdots K_2 \cdots K_1,
\]

where \( (K_h \ast K_g)(t,\tau) = \int_\tau^t K_h(t,\sigma) K_g(\sigma,\tau) d\sigma \) (8) denotes the kernel composition integral. In [13], the following result is proved.

**Theorem 3.1 (Non-asymptotic Derivative Image):** Let \( x(i)(\cdot) \) be the \( i-th \) derivative of the signal \( x(\cdot) \) and let \( N \geq i \) be an arbitrary integer. Given \( N \) kernels functions \( K_1(\cdot,\cdot), \ldots, K_N(\cdot,\cdot) \), such that \( K_1 \) is \( d\)-th order non-asymptotic, with \( d \geq i-1 \) and \( K_j \in \mathcal{H}_d \), \( \forall j \in \{1, \ldots, N\} \), consider the composed operator \( V_{P_{N}} , \) with kernel

\[
P_N = K_N \cdots K_2 \ast K_1.
\]

The image of \( x(i)(\cdot) \) through \( V_{P_{N}} , \) \([V_{P_{N}} x(i)](\cdot)\), can be obtained as the image of the restriction \( x([0,0](\cdot) \) through a non-anticipative operator. Indeed, there exists an operator \( V_{R_{N,i}} \), induced by the kernel

\[
R_{N,i} = K_N \cdots K_{i+1} \ast T_i,
\]

with \( T_i(\cdot,\cdot) \) defined recursively by

\[
\begin{align*}
T_1 & \equiv K_1, \\
T_j(t,\tau) & \equiv -(K_j \ast T_{j-1})(t,\tau) + K_{j-1}(t,\tau) T_{j-1}(\tau,\tau), \quad j \in \{2, \ldots, i\}, \forall t, \tau \in \mathbb{R}^2,
\end{align*}
\]

such that

\[
[V_{P_{N}} x(i)](t) = R_{N,i}(t, t) x(t) - R_{N,i}(t, 0) x(0) - [V_{R_{N,i}} x](t).
\]

The following Remark sheds some light on significant implications of Theorem 3.1.

**Remark 3.1 (Implications):** In Theorem 3.1, the existence of a composed Volterra integral operator has been shown, namely \( V_{P_{N}} = V_{K_{N_1} \cdots K_{N_i}} \), that, fed by the \( i-th \) derivative \( x(i)(\cdot) \) of a signal, produces an image signal, say \([V_{P_{N}} x(i)](\cdot) \), which, in turn, can be expressed, in the most general case, in terms of the sole restriction \( x([0,0](\cdot) \) under slightly stronger assumptions) and that, in any case, does not depend on the initial conditions of the hidden derivatives. Assume now that \( x(i)(\cdot) \) is not measurable while \( x(\cdot) \) is available; then, thanks to (11), the signal \([V_{P_{N}} x(i)](\cdot) \) can be obtained by applying a non-anticipative operator (see (11)) to the measurable signal \( x(\cdot) \).

IV. NON-ASYMPTOTIC KERNELS FOR PARAMETER ESTIMATION

Consider the parameter estimation problem formulated in Section II and focus on the structural constraint (1) which relates the unknown parameters with the time-derivatives of the signals \( u(\cdot) \) and \( y(\cdot) \). In the sequel, the results presented in the previous section will be exploited to overcome the unavailability of signal derivatives (hidden internal states of the system) in (1), thus obtaining non-asymptotic estimates of the unknown parameters by means of causal filtering.

First, in order to get rid of the structured perturbation term, let us take the \( p-th \) generalized derivative of both sides of the structural constraint, obtaining:

\[
y^{(n+p)}(t) = \sum_{i=0}^{n-1} a_i y^{(i+p)}(t) + \sum_{k=0}^{m-1} b_k u^{(k+p)}(t).
\]

Moreover, after choosing an integer \( N \geq n+p \), let us apply the Volterra operator \( V_{P_N} = V_{K_{N_1} \cdots K_{N_i}} \) (with kernels taken as in Theorem 3.1) to both sides of (12):

\[
[V_{P_N} y^{(n+p)}(\cdot)](\cdot) = \sum_{i=0}^{n-1} a_i [V_{P_N} y^{(i+p)}(\cdot)](\cdot) + \sum_{k=0}^{m-1} b_k [V_{P_N} u^{(k+p)}(\cdot)](\cdot).
\]

In view of (11), we can rewrite (13) as

\[
r_{y,n+p}(t) = \sum_{i=0}^{n-1} a_i r_{y,i+p}(t) + \sum_{k=0}^{m-1} b_k r_{u,k+p}(t), \quad \forall t \in \mathbb{R}_{\geq 0},
\]

where the auxiliary signals in (14) can be obtained as the image of measurable signals \( y(\cdot) \) and \( u(\cdot) \) through non-anticipative operators:

\[
r_{y,j}(t) = R_{N,j}(t, t) y(t) - R_{N,j}(t, 0) y(0) - [V_{R_{N,j}} y](t),
\]

\[
r_{u,j}(t) = R_{N,j}(t, t) u(t) - R_{N,j}(t, 0) u(0) - [V_{R_{N,j}} u](t),
\]

with \( j \in \{p, \ldots, n+p\} \) and \( j \in \{p, \ldots, m+p-1\} \), respectively. Finally, by introducing the true parameter vector

\[
\theta^* = [a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}]^T,
\]

and the vector of auxiliary signals

\[
z(t) = [r_{y,p}(t), r_{y,p+n-1}(t), r_{u,p}(t), \ldots, r_{u,p+m-1}(t)]^T,
\]
equation (14) can be rewritten in compact notation as
\[ z^\top(t)\theta^* = r_{y,n} + p(t), \quad t \in \mathbb{R}_{\geq 0}. \] (16)
Now, assuming that all the operators in our formulation admit a stable realization, we need to collect a suitable number of equations like (14) in order to form a well-posed algebraic system, to be solved in the unknown parameters. Several approaches can be used to obtain the needed set of constraints (see [13]), among which we are going to focus on covariance filtering, that will be used in the numerical example. This method consists in first forming the instantaneous covariance equation by left-multiplying (16) by \( z \):
\[ R(t)\theta^* = S(t), \quad \forall t \in \mathbb{R}_{\geq 0}, \] (17)
where \( R(\cdot) \in \mathcal{L}_{loc}^2([\mathbb{R}_{\geq 0}]^{(n+m)\times(n+m)}) \) and \( S(\cdot) \in \mathcal{L}_{loc}^2([\mathbb{R}_{\geq 0}]^{n+m}) \) are the so-called auto-covariance and cross-covariance matrices defined as
\[ R(t) = z(t)z^\top(t), \quad S(t) = z(t)r_{y,n} + p(t), \quad \forall t \in \mathbb{R}_{\geq 0}. \]
The instantaneous auto-covariance matrix \( R(t) \) is rank-one and therefore it can never be inverted when \( n + m \geq 2 \). In order to get an invertible system for some suitable input-output signal pair, we apply a further Volterra operator \( V_G \), with an arbitrary non-negative kernel \( G(\cdot, \cdot) \in \mathcal{HS} : G(t, \tau) \geq 0, \forall (t, \tau) \in \mathbb{R}_{\geq 0}^2 \), on both sides of (17), obtaining the filtered covariance equation:
\[ [V_G R(t)]\theta^* = [V_G S(t)], \quad \forall t \in \mathbb{R}_{\geq 0}, \] (18)
where the operator \( V_G \) has to be applied element-wise on \( R(\cdot) \) and \( S(\cdot) \). Letting \( Z(t) = [V_G R(t)] \) and \( r(t) = [V_G S(t)] \), an estimate \( \hat{\theta}(t) \) of the parameter vector \( \theta^* \) can be obtained by minimizing the quadratic criterion
\[ \hat{\theta}(t) = \arg \min_{\theta \in \mathbb{R}^{n+m}} \| Z(t)\theta - r(t) \|^2. \] (19)
In this case, the invertibility of the filtered auto-covariance matrix \( Z(t) = [V_G R(t)] \) characterizes a sufficiently informative input-output signal pair at time \( t \).
In order to emphasize the generality of the proposed methodology, we still have not assigned explicit analytic expressions to the kernels \( R_1, \ldots, R_N \) and to \( R_{N+1}, j \in \{ p, \ldots, n + p \} \), which are needed to compute the auxiliary signals. The problem of identifying a class of non-asymptotic kernels yielding stable finite-dimensional state-space realizations will be addressed in the following sections.

V. Univariate Non-Asymptotic Kernels
In this section, we first consider a class of simplified univariate kernels \( W(\tau), \tau \in \mathbb{R}_{\geq 0} \). The Volterra operators induced by this kind of kernels are given by
\[ [V_W y](t) = \int_0^t W(\tau)y(\tau)d\tau, \quad \forall t \in \mathbb{R}_{\geq 0}, \]
and are typically known as weighted integral operators. In our setting, we consider weighting patterns \( W(\cdot) \in \mathcal{L}_{loc}^2(\mathbb{R}_{\geq 0}) \) satisfying the non-asymptoticity conditions up to the \( i \)-th order and we call them Univariate Non-asymptotic Kernels (U-NKs).

Remark 5.1 (U-NKs Vs. Modulating Functions): Within the class of IM methods, the MF approach uses time-dependent univariate kernels to minimize the effect of unknown initial conditions. Univariate kernels, in our setting, need to fulfill weaker assumptions than those that the usual modulating functions need to subsume in the identification context (see [2]): while in the usual MF approach the width of the integration window is a critical parameter for noise sensitivity and the estimates are available only at the end of the integration interval, in our setting, a point-wise estimate (independent from initial conditions) is available at any time instant \( t > 0 \), and the integration process may proceed indefinitely without re-initialization. This is a distinctive feature of the proposed methodology.

In order to exploit the U-NKs for parameter estimation, let us specialise the kernel construction procedure outlined in Theorem 3.1 to produce the needed causal auxiliary signals. To this end, consider \( n + p \) modulating functions
\[ W_1(\tau), W_2(\tau), \ldots, W_{n+p}(\tau) \] (20)
and assume that \( W_1 \) is at least \( (n + p - 1) \)-th order non-asymptotic (see Theorem 3.1). Now, analogously to (13), by applying the composed operator \( V_{P_N} \) with \( P_N = W_{n+p} \star \cdots \star W_p \), we obtain the transformed dynamic constraint:
\[ [V_{P_N} y](t) = \sum_{i=0}^{n-1} a_i[V_{P_N} y^i](t) + \sum_{k=0}^{m-1} b_k[V_{P_N} u^{(k+p)}](t). \]
Thanks to Theorem 3.1, the surrogate signal derivatives in (15), in the U-NK setting, admit the following expressions:
\[ r_{y,i}(t) = [V_{P_N} y^i](\cdot), \quad i \in \{ p, \ldots, n + p \}, \]
\[ r_{u,i}(t) = [V_{P_N} u^i](\cdot), \quad i \in \{ p, \ldots, p + m - 1 \}. \]
These signals can be used in place of the unmeasurable input-output derivatives to estimate the parameters.

Remark 5.2: Admissible instances of U-NKs of the \( i \)-th order are, among many other possible functions:
1) the \( \tau \)-monomial \( W(\tau) = \tau^i \);
2) the damped unitary-step function \( W(\tau) = (1 - e^{-\omega \tau})^i \);
3) the exponential damped \( \tau \)-monomial \( W(\tau) = \tau^i e^{-\omega \tau} \),
where \( \omega \in \mathbb{R}_{\geq 0} \) is an arbitrary constant. According to (3), the first two kernels are locally square-integrable but not BIBO-stable, while the third one is BIBO-stable.

Now, it is worth noting that any BIBO stable univariate kernel \( W(\cdot) \) has to satisfy the asymptotic condition \( \lim_{\tau \to \infty} W(\tau) = 0 \), in order to meet the requirement (3). Indeed, particularizing (3) to univariate kernels, we obtain
\[ \sup_{t \in \mathbb{R}_{\geq 0}} \int_0^t |W(\tau)|d\tau = \int_0^\infty |W(\tau)|d\tau < \infty. \]
In practice, it follows that the rate of update of the estimates generated by any BIBO-stable U-NK estimator decays as time proceeds far from the initial instant \( t = 0 \) because the weighting patterns fade toward zero. In other terms, the input-output injection undergoes an asymptotic suppression. This drawback will be addressed in the next section by using bivariate causal non-asymptotic kernels (still guaranteeing the internal stability of the estimator).

VI. Bivariate Causal Non-Asymptotic Kernels
In this section, the main result is presented. To this end, let us introduce the following definition:

Definition 6.1 (\( i \)-th Order BC-NK): If a kernel \( K(\cdot, \cdot) \in \mathcal{HS} \), in addition to the assumptions posed in the statement of Lemma 3.1, for a given \( i \geq 1 \), verifies the conditions
\[ K^{(j)}(t, 0) = 0 \quad \forall j \in \{ 0, \ldots, i - 1 \}, \]
and...
then, it is called an \( i \)-th Order Bivariate (strict) Causal Non-
Asymptotic kernel.

It is of great importance to emphasize that only by using
bivariate kernels all the conditions (21) can be fulfilled simul-
taneously. While Theorem 3.1 enabled us to construct
auxiliary signals yielding the unavailable derivatives by tak-
ing advantage of non-asymptotic kernels, the following result
can be used to exploit the causality property of BC-NKs
to achieve the same goal in an easier way. Indeed, by the
conditions (21), thanks to Lemma 3.1, the image of a signal
derivative can be expressed as
\[
[V_{\omega,N}x^{(i)}](t) = (-1)^i [V_{\omega,N}x](t).
\]

We show that the following bivariate function is a possible
instance of BC-NK, and it will be used in the sequel to carry
out the design of the stable non-asymptotic estimator:
\[
C_{\omega,N}(t, \tau) \triangleq e^{-\omega(t-\tau)} \left( 1 - e^{-\omega \tau} \right)^N \left( 1 - e^{-\omega (t-\tau)} \right)^N,
\]
where \( \omega \in \mathbb{R}_{>0} \) is an arbitrary scalar parameter. The non-
asymptoticity, causality and BIBO-stability properties of the
devised kernel are illustrated by the following lemma.

**Lemma 3.1 (Kernel Characterization: \( C_{\omega,N}(t, \tau) \)):** The
bivariate kernel \( C_{\omega,N}(t, \tau) \) is BIBO-stable and \( N \)-th order
BC-NK. Moreover, all the kernel derivatives \( C_{\omega,N}^{(i)}(t, \tau) \),
with \( i \in \{0, \ldots, N-1\} \), are BIBO-stable.

**Proof:** First, we prove that the kernel \( C_{\omega,N}(t, \tau) \) is a \( N \)-th order
BC-NK. Indeed, all the non-anticipativity conditions up to the \( N \)-th order are met by the factor \( (1 - e^{-\omega \tau})^N \). The
causality conditions up to the \( N \)-th order are met by the third
factor \( (1 - e^{-\omega (t-\tau)})^N \). The BIBO-stability of
\( C_{\omega,N}^{(i)}(t, \tau) \) is implied by the fact that each \( (e^{-\omega (t-\tau)})^{(i)} \),
with \( i \in \{0, \ldots, N-1\} \), is BIBO-stable and the following
terms are bounded: \( |(1 - e^{-\omega (t-\tau)})^N| < 1 \), \( \forall \tau : 0 \leq \tau \leq t \)
and \( |(1 - e^{-\omega (t-\tau)})^N| < 1, \forall \tau : 0 \leq \tau \leq t \) and their derivatives
up to the \( (N-1) \)-th order are bounded.

Now, we describe how the image of the derivative
\( x^{(i)}(\cdot) \) through the operator \( V_{\omega,N} \), \( i.e., \ [V_{\omega,N}x^{(i)}] = \)
\( (-1)^i[V_{\omega,N}x] \) can be obtained as the output of a BIBO-
stable finite-dimensional time-varying linear system.

First, the \( i \)-th derivative of the BC-NK (22) with respect to
the second argument can be expressed as:
\[
C_{\omega,N}^{(i)}(t, \tau) = \sum_{j=1}^{N+1} e^{-\omega j \tau} f_{\omega,N^{(i,j)}}(\tau)
\]
where \( f_{\omega,N^{(i,j)}}(\cdot) \) are univariate functions of \( \tau \). Let
\( C_{\omega,N^{(i,j)}}(t, \tau) \triangleq \left( -1 \right)^i e^{-\omega j \tau} f_{\omega,N^{(i,j)}}(\tau) \); then, by the linearity
of the Volterra operator, it follows that
\[
[V_{\omega,N}x^{(i)}](t) = (-1)^i [V_{\omega,N}x](t) = \sum_{j=1}^{N+1} [V_{\omega,N^{(i,j)}}x](t).
\]

Moreover, letting \( \xi_{i,j}(t) \triangleq [V_{\omega,N^{(i,j)}}x](t) \), with \( i \in \{p, \ldots, n + p\} \), \( j \in \{1, \ldots, N+1\} \), and considering that,
\[
C_{\omega,N^{(i,j)}}(t, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} C_{\omega,N^{(i,j)}}(t, \tau) = -\omega j e^{-\omega j \tau} f_{\omega,N^{(i,j)}}(\tau),
\]
then \( [V_{\omega,N}x^{(i)}] \) admits the following \( (N+1) \)-th dimensional
state-space realization:
\[
\begin{align*}
\xi_{i,1}(t) &= C_{\omega,N^{(i,j)}}(t, t)x(t) - \omega \xi_{i,1}(t), \\
\vdots \\
\xi_{i,j}(t) &= C_{\omega,N^{(i,j)}}(t, t)x(t) - \omega j \xi_{i,j}(t), \\
\vdots \\
\xi_{i,N+1}(t) &= C_{\omega,N^{(i,j)}}(t, N+1, t)x(t) - \omega (N+1) \xi_{i,N+1}(t),
\end{align*}
\]
\[
[V_{\omega,N}x^{(i)}](t) = \sum_{j=1}^{N+1} \xi_{i,j}(t), \quad \forall t \in \mathbb{R}_{\geq 0}
\]
\( \xi_{i,1}(0) = 0, \ldots, \xi_{i,N+1}(0) = 0 \).

(26)

Being \( |e^{-\omega \tau} f_{\omega,N^{(i,j)}}(\tau)| \to 0, \forall j \in \{1, \ldots, N + 1\}, \) (i.e.,
all the time-varying terms affine to the \( x(t) \)-injection are
bounded), and since the system is diagonal with \( \omega > 0 \), then
(26) is a BIBO-stable time-varying linear system.

Moreover, there exist finite bounded scalars \( \beta_{i,j} \in \mathbb{R}_{>0} \)
such that \( (e^{-\omega \tau} f_{\omega,N^{(i,j)}}(\tau)) \xrightarrow{\tau \to \infty} \beta_{i,j} \). This implies that
the time-varying system (26), for \( t \to \infty \), tends to a stable
linear time-invariant system in which the \( x(t) \)-injection is
never suppressed. Thanks to (26), the extended auxiliary
signal vector \( z_e(t) \), which embeds both the signals \( z(t) \) and
\( r_{y,p+n} \) needed to form the constraint (16):
\[
z_e(t) = [r_{y,p}(t), \ldots, r_{y,p+n}(t), r_{u,p}(t), \ldots, r_{u,p+m-1}(t)],
\]
with
\[
\begin{align*}
gr_{y,i} &= [V_{\omega,N}y^{(i)}], i \in \{p, \ldots, n + p\}, \\
r_{u,i} &= [V_{\omega,N}u^{(i)}], i \in \{p, \ldots, m - 1 + p\},
\end{align*}
\]
can be obtained as the output of an overall \( n_{\xi} = (n + \)
\( m + 1)(N + 1) \)-dimensional linear time-varying dynamical
system:
\[
G_{\xi,t} : \{ \xi^{(1)}(t) = G_{\xi}(t) + E(t)y(t) + F(t)u(t),
\]
\[
z_e(t) = H_{\xi}(t), \quad t \in \mathbb{R}_{\geq 0}
\]
\( \xi(0) = 0 \),
\]
where \( \xi \in \mathbb{R}_{n\xi} \) is the overall state-vector,
\[
G = \begin{bmatrix}
G_p & 0 \\
\vdots & \vdots \\
0 & G_{p+m-1}
\end{bmatrix} \in \mathbb{R}^{n\xi \times n\xi},
\]
\[
G_i = \text{diag} \left( -\omega, \ldots, -\omega(N+1) \right) \in \mathbb{R}^{(N+1) \times (N+1)},
\]
with \( i \in \{p, \ldots, p + n\} \). Moreover, we have
\[
E(t) = \begin{bmatrix}
E_{p}(t) \\
\vdots \\
E_{p+n}(t)
\end{bmatrix} \in \mathbb{R}^{n\xi}, \quad F(t) = \begin{bmatrix}
0 \\
\vdots \\
F_{p}(t) \\
\vdots \\
F_{p+m-1}(t)
\end{bmatrix} \in \mathbb{R}^{n\xi}.
\]
where $1^\top$ denotes a row vector of ones with $(N + 1)$ elements. By choosing the covariance filtering method as augmentation strategy (see Section IV), and by assuming that the $V_G$ operator used for augmentation admits a one-dimensional stable state-space realization (take, for instance a kernel $G(t, \tau) = e^{-\omega(t-\tau)}$) the overall BC-NK estimator can be implemented as an internally stable $(n + m + 1)(N + 1) + (n + m + 1)(n + m - 1)$-th order linear time-varying dynamical system.

Indeed, the augmentation system can be viewed, in turn, as a CT dynamical system $\mathcal{A}_{x \rightarrow \text{vec}[Z, r]}$, where $\text{vec}[Z, r]$ represents the outcome of the augmentation, obtained by stacking the columns of $Z(t)$ in a single vector to which $r(t)$ is finally appended.

The dynamic part of the BC-NK estimator consists of the cascade of the auxiliary-signal-generation system $\mathcal{G}_{u, y \rightarrow z_e}$ and of the augmentation system $\mathcal{A}_{z_e \rightarrow \text{vec}[Z, r]}$. The internal stability of the BC-NK estimator refers to the stability of both these subsystems, but does not guarantee the boundedness of the estimates at time $t$. Indeed, according to Section IV, to obtain the parameter estimates we need a further processing step. The estimation process is completed by the algebraic inversion map (see Figure 1):

$$E(\cdot) : \text{vec}[Z(t), r(t)] \mapsto \hat{\theta}(t)$$

where the estimated parameter vector $\hat{\theta}(t)$ is obtained as

$$\hat{\theta}(t) = E\left(\text{vec}[Z(t), r(t)]\right) \triangleq (Z(t))^\dagger r(t).$$

The estimation map $E(\cdot)$ is not guaranteed to be bounded for all values of its argument, but only when the excitation condition outlined in Section IV is met (the fulfillment of this condition depends on the informative content of the input-output signals restrictions $u_{(0,1)}(\cdot)$ and $y_{(0,1)}(\cdot)$). A supervision scheme can be introduced to check the invertibility of $Z(t)$ in order to avoid singularities.

**VII. Numerical Example**

Let us consider a mass-spring-damper system model consisting of an inertial mass $M = 1$ kg, a spring with elastic constant $k = 3$ N/m, and a linear damping element with $c = 2$ Ns/m. We obtain immediately the following second-order differential equation for mass-displacement $x$:

$$\begin{align*}
M \dddot{x}(t) + c \ddot{x}(t) + kx(t) = u(t), \\
y(t) = x(t) + \eta_1(t), \\
u(t) = v(t),
\end{align*}$$

with $x(0) = x_0, x'(0) = x_1(1)$. Moreover, $v(\cdot)$ represents a measurable external force input for the system, $y(\cdot)$ is the measured position signal, affected by a constant measurement bias $\sigma = 1$ m and by an unstructured perturbation term $\eta_1(\cdot)$ (addressed as output measurement noise), while $u(\cdot)$ is the measured forcing input signal. Neglecting for the moment the influence of $\eta_1(\cdot)$, the following input-output dynamic constraint can be obtained by rearranging (29):

$$y(t) = a_1 y_1(t) + a_0 (y(t) - \sigma) + b_0 u(t),$$

with $a_0 = -k M^{-1} = -3$, $a_1 = -e^{-M^{-1}} = -2$, $b_0 = M^{-1} = 1$. Now, to estimate the parameters in presence of bias on the measurements, let us set $p = 1$ and, being $n = 2$ for the considered system, let us set $N = 3$.

In the following, we address a numeric comparison between the proposed BC-NK estimator and the fast CT estimation algorithm presented in [9]. It is easy to show that the latter estimator can be obtained as a particular case of the U-NK scheme with (non-BIBO) kernels $W_1(\tau) = \tau^{n+p}$, $W_2(\tau) = H(\tau)$, $i \in \{2, \ldots, n\}$ by using a successive integration augmentation technique with a kernel given by the Heaviside unitary-step function $G(\tau) = H(\tau)$. Due to space limitations, we do not illustrate this augmentation strategy in the paper; the interested reader is referred to [13] for a detailed description.

Unlike the algorithm in [9], our operator-oriented approach allows for the use of stable kernels, i.e., yielding BIBO stable realizations. For instance, the exponentially damped kernels $W_1(\tau) = \tau^{n+p} e^{-\sigma \tau}$, $W_2(\tau) = e^{-\sigma \tau}$, $i \in \{2, \ldots, n\}$, together with a successive integration kernel $G(\tau) = e^{-\sigma \tau}$, lead to a stable state-space implementation of the estimator.

The following U-NKs yield the estimator described in [14] for the system considered in the present example:

$$\begin{align*}
W_1(\tau) &= \tau^3; & W_2(\tau) &= H(\tau); \\
W_3(\tau) &= H(\tau); & W_4(\tau) &= H(\tau).
\end{align*}$$

In view of Theorem 3.1, we are able to compute the auxiliary signals $r_{u,1}(\cdot)$ and $r_{u,1}(\cdot)$ as

$$\begin{align*}
r_{x_1}(t) &= R_{3,1}(t, \tau) x(t) - R_{3,1}(t, 0) x(0) - \left[ V_{R_{3,1}} x(0) \right], \\
r_{x_2}(t) &= R_{3,2}(t, \tau) x(t) - R_{3,2}(t, 0) x(0) - \left[ V_{R_{3,2}} x(0) \right], \\
r_{x_3}(t) &= R_{3,3}(t, \tau) x(t) - R_{3,3}(t, 0) x(0) - \left[ V_{R_{3,3}} x(0) \right],
\end{align*}$$

with $x \in \{u, y\}$ and where the R-kernels are obtained from the U-NKs $W_1, W_2$ and $W_3$ by the iterative procedure outlined in the statement of Theorem (3.1). For the chosen U-NKs it holds that

$$\begin{align*}
R_{3,1}(t, \tau) &= -\frac{3t^2 + 4t^3}{2} - \frac{5t^4}{4}, \\
R_{3,2}(t, \tau) &= 3t^2 - 12t^2 + 10t^3, \\
R_{3,3}(t, \tau) &= -3t^2 + 24t^2 - 30t^2
\end{align*}$$
thus yielding the integral forms
\[ r_{x_1}(t) = \frac{3}{2} t^2 \int_0^t \tau^2 x(\tau) d\tau - 4t \int_0^t \tau^2 x(\tau) d\tau + \frac{5}{2} \int_0^t \tau^4 x(\tau) d\tau, \]
\[ r_{x_2}(t) = -3t^2 \int_0^t \tau x(\tau) d\tau + 12t \int_0^t \tau^2 x(\tau) d\tau - 10 \int_0^t \tau^3 x(\tau) d\tau, \]
\[ r_{x_3}(t) = -t^3 x(t) + 3t^2 \int_0^t \tau x(\tau) d\tau - 24t \int_0^t \tau^2 x(\tau) d\tau + 30 \int_0^t \tau^4 x(\tau) d\tau. \]

Note that, while in [14] the auxiliary signals are expressed in terms of nested integrals, here we have reported the equivalent single-integral expressions, producing the same signals. Successive integration has been used in this case as augmentation method. By choosing \( G = H(\tau) \) as kernel of the augmentation operator \( V_G \), the U-NK estimator exactly reproduces the estimator in [14].

To carry out a comparable simulation in the BC-NK framework, the same \( N = 3 \) value has been used in the implementation of the BC-NK kernel (22). The kernel parameter \( \omega \) has been set to \( \omega = 1 \). The procedure for constructing the auxiliary signals generation system by BC-NK kernels consists in taking the derivatives \( C^{(i)}_{\omega,N}(t,\tau), \ i \in \{1,2,3\} \) of the BC-NK (22), then in identifying the terms \( C_{\omega,N_{i,j}} \), with \( j \in \{1,2,3,4\} \) (see (23) and (24)), and finally in computing \( C_{\omega,N_{i,j}}(t,\tau) \) to form the \( E_i(t) \) matrices (see (28)) needed for the implementation of the auxiliary signal generation system \( G_{u,y\rightarrow x} \) (see (27)). Neglecting the intermediate algebraic manipulations, we get:

\[
E_1 = \begin{bmatrix}
-2(e^{-t}+1)(e^{-t}-1)^2 \\
3(e^{-t}-1)^2(e^{-t}+2) \\
-9(e^{-t}-1)^2(e^{-t}-4)
\end{bmatrix},
\]

\[
E_2 = \begin{bmatrix}
-4e^{-3t}+3e^{-2t}+1 \\
3e^{-3t}+9e^{-t}-12 \\
9(e^{-t}-1)(e^{-t}-3)
\end{bmatrix},
\]

\[
E_3 = \begin{bmatrix}
-8e^{-3t}+3e^{-2t}-1 \\
3e^{-3t}-9e^{-t}+24 \\
-9e^{-2t}+72e^{-t}-81 \\
-e^{-3t}+24e^{-2t}-81e^{-t}+64
\end{bmatrix},
\]

and \( G_i = \text{diag}(-1,-2,-3,-4), \ i \in \{1,2,3\} \). Covariance filtering has been used in this case as augmentation method. The kernel \( G \) of the filtering operator \( V_G \) has been chosen as \( G(t,\tau) = e^{-0.1(e^{-t}-\tau)} \), thus yielding a one-dimensional stable linear system. The construction of the augmentation system with the chosen kernel is trivial, so its state-space realization is omitted for brevity.

A. Non-Asymptotic Estimation in the Noise-free Scenario

The first simulation deals with a noise-free scenario, i.e., \( \eta_{sb}(t) = 0, \forall t \in \mathbb{R}_{\geq 0} \). The initial conditions for the mass-spring-damper system have been set to \( x(0) = 1 \) \( m \) and \( x^{(1)}(0) = 10 \) \( m/s \), while the forcing input has been chosen as a sum of sinusoids \( v(t) = 10 \sin(t) + \sin(10t) \).

Fig. 2 shows the measured input and output signals \( u(t) = v(t) \) and \( y(t) = x(t) + o \) in the noise-free case. Although the theoretical instantaneity of the method gets lost in the time-discretization of the estimator’s dynamics, in the digital representation of the signals and in the numerical computation of the pseudo-inverse, the parameters are correctly estimated with negligible duration of the transient by both methods, as shown in Fig. 3. It is worth noting that the proposed BC-NK estimator, beyond fast convergence, is characterized by guaranteed internal stability. In Figs. 4 and 5, the time-behaviors of the singular values \( \Sigma(Z(t)) \) of matrices \( Z(t) \) yielded by the augmentation systems of the two estimators are shown. As can be observed, the BC-NK technique shows a bounded behavior of the singular values \( \Sigma(Z(t)) \), whereas the other estimation technique, though showing a fast convergence behavior toward accurate estimates of the parameters, requires periodic reset in order to cope with the integrator windup issue.

We point out that no high-gain output injection has been performed by the two methods. In this respect, a further simulation has been carried out in noisy conditions.
B. Estimation with Unstructured Measurement Perturbations

In this example, the additive output measurement noise $\eta_y(\cdot)$ has been simulated as a uniformly distributed random signal taking values in the interval $[-0.8, 0.8]$. The perturbed signal used for parameter estimation is depicted in Fig. 6. As can be seen from Fig. 7 the BC-NK estimator shows good robustness against the output noise and the estimated parameters converge to a neighborhood of the true values. Conversely, the U-NK estimator, implemented without further provisions to remove the noise effects, turns out to be more sensitive to unstructured perturbations; in this respect, as suggested in [14], it would be possible to further process the estimates generated by the estimation algorithm by a low-pass filter in order to mitigate the influence of noise.

It is important to notice that, although the BC-NK method has shown in simulation to enjoy robustness properties against measurement perturbations, further improvements can be obtained by tuning the $\omega$ parameter of the non-asymptotic causal kernel (22) and by accurately choosing the filtering operator $V_G$ used for augmentation. Further research is needed on this specific issue.

![Fig. 6](image6.png)

Fig. 6. Trends of the input signal $u(t)$ (gray) and noisy output measurement signals $y(t)$ (black) used for the estimation.

VIII. CONCLUDING REMARKS

In the present work, a theoretical framework arising from the algebra of linear integral operators has been established for the design of non-asymptotic parametric estimators for continuous-time dynamical systems. In particular, a novel class of bivariate kernels has been devised allowing to get rid of the influence of the unknown initial conditions while, at the same time, guaranteeing the internal stability of the estimator. As a consequence, very fast convergence (ideally, non-asymptotic) of the estimates can be achieved. The effectiveness of the proposed algorithm has been shown and some comparisons with a fast CT estimation method available in the recent literature are provided as well.

Future research efforts will be devoted to the analysis of the consistency of the proposed estimation method in a stochastic context as well as to the characterization of its robustness properties.

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REFERENCES